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THE COVERING OF RINGS BY VALUATION RINGS

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For the question dealt with in the present paper it is sufficient to recall the following definition of a valuation ring.

A subring A of the field K is said to be a valuation ring of the field K if and only if for every $x \in K - \{0\}$ at least one of x, x^{-1} belongs to A. (See, e.g., [1], Chapter 3, 16, Theorem 16.3, (6)).

The valuation rings in a field have some properties analogous to those of prime ideals in a ring. It is easy to understand this from the historical origin of these notions. A valuation ring can be defined in a way completely analogous to that of a prime ideal.

As a matter of fact a subring of a field is a valuation ring if and only if its complement is closed under multiplication. (Throughout the whole paper, with the exception of Remark 2, we assume that the ring has a unit element.)

Indeed, if A is a valuation ring of the field K and $x, y \in K - A$, then x^{-1}, y^{-1} belong to the ring A. If there were $x \cdot y \in A$, then $x = (x \cdot y) \cdot y^{-1} \in A$, which is a contradiction with the assumption that $x, y \notin A$. Thus the complement of A is closed under multiplication.

If conversely the complement of a subring A of the field K is closed under multiplication, then from $x \cdot x^{-1} = 1 \in A$ for every $x \in K - \{0\}$ we have x or $x^{-1} \in A$ and A is a valuation ring of the field K.

N. H. McCoy has shown in [3] (see also [1], Chapter 1, § 4, 4.9 Proposition) that if in a commutative ring an ideal A is covered by a finite number of ideals $A_1, \ldots, A_n$, where all A_i, $i = 1, \ldots, n$, with the exception of at most two of them, are prime ideals, then the covered ideal A is contained in some A_i, $i = 1, 2, \ldots, n$.

We now prove the following Theorem. (This Theorem can be viewed also as a generalisation of the Lemma used in [2].)

Theorem. *Let $A_1, A_2, \ldots, A_n$ be subrings of the field K such that all of them except at most two are valuation rings. Then for every subring B of K such that $B \subset \bigcup_{i=1}^{n} A_i$ there exists an $A_j \in \{A_1, \ldots, A_n\}$ such that $B \subset A_j$.*

Proof. We proceed by induction with respect to n. For $n=2$ the assertion is easy to prove. Let $B\subset A_1\cup A_2$ and, e.g., $B\not\subset A_1$. Then there exists an element $a_2\in B\cap A_2-A_1$ and for every element $a_1\in B\cap A_1$ we have $a=a_1+a_2 \in B\subset A_1\cup A_2$. But the element a cannot be contained in A_1, since otherwise $a_2=a-a_1\in A_1$, contrary to hypothesis. And so $a\in A_2$, and we have $a_1=a-a_2\in A_2$. Since a_1 is an arbitrary element from $B\cap A_1$ and $B\subset A_1\cup A_2$, we have $B\subset A_2$.

Let now $B\subset\bigcup_{i=1}^{n} A_i$, where B is a subring of K and $A_1, \ldots, A_n$ are valuation rings with the exception of at most two of them. By the inductive supposition we may assume that $B\not\subset\bigcup_{i\neq j} A_i$ for every $j\in\{1, 2, \ldots, n\}$. Thus we can find the elements $a_i\in B\cap A_i - \bigcup_{j\neq i} A_j$, $1\leqslant i\leqslant n$. Now we assume that the rings $A_3, A_4, \ldots, A_n$ are valuation rings. Further we may assume that the elements $a_3, a_4, \ldots, a_n$ are units in the rings $A_3, \ldots, A_n$, respectively. Since if a_i is not a unit we may replace it by $1+a_i$ which is a unit in A_i and it is contained in $B\cap A_i - \bigcup_{j\neq i} A_j$. (To see that $1+a_i$ is a unit if $a_i\neq 0$ is not a unit in A_i $(i\geqslant 3)$, notice that we have successively, $a_i^{-1}\notin A_i$, $a_i^{-1}+1\notin A_i$, $1-a_i(1+a_i)^{-1} = (1+a_i)^{-1}\in A_i$).

Put $z=a_1a_2 \ldots a_n$. Then $z\notin A_3\cup\ldots\cup A_n$. To prove this suppose for an indirect proof that $z\in A_i$, $(i\geqslant 3)$. This implies $za_i^{-1}\in A_i$, i.e. $a_1a_2 \ldots a_{i-1}a_{i+1} \ldots a_n\in A_i$. Now by the choice of a_k each $a_k(k=1, \ldots, i-1, i+1, \ldots, n)$ is contained in $K-A_i$ and since A_i is a valuation ring their product is in $K-A_i$. This contradiction proves our statement.

Now $z\in B\subset A_1\cup A_2\cup \ldots \cup A_n$ implies $z\in A_1\cup A_2$. Let us put

$$y=\begin{cases} a_3+z & \text{if} \quad z\in A_1\cap A_2 \\ a_2+z & \text{if} \quad z\in A_1-A_2 \\ a_1+z & \text{if} \quad z\in A_2-A_1 \end{cases}$$

Then the element y belongs to B, but it does not belong to any A_i, $i=1, 2, \ldots, n$. Indeed, if $z\in A_1\cap A_2$, then $a_3+z\notin A_1\cup A_2\cup A_3$. For $i\geqslant 4$ we have $a_3+z = a_3(1+a_1a_2a_4 \ldots a_n)$, which does not belong to A_i since neither a_3 nor $1+a_1a_2a_4 \ldots a_n$ belongs to A_i. If $z\in A_1-A_2$, then $a_2+z\notin A_1\cup A_2$ and to show $a_2+z\notin\bigcup_{i\geqslant 3} A_i$ we use the same argument as above. The case $z\in A_2-A_1$ is symmetrical with the case $z\in A_1-A_2$.

And so we have found an element $y\in B-\bigcup_{i=1}^{n} A_i$, a contradiction with the assumtion $B\subset\bigcup_{i=1}^{n} A_i$. Our Theorem is proved.

Remark 1. We give an example to show that if more than two subrings A_i are not valuation rings, our Theorem need not hold.

Example 1. Let T_2 be the field of residue classes mod 2 and $K = T_2(x, y)$ the field of rational functions in two variables x, y. The rings A_1, A_2, A_3 are defined as subrings of $T_2[x, y]$ in the following manner.

A_1 is the set of all polynomials $p(x, y)$

$$p(x, y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2 + \ldots + a_{mn}x^m y^n \in T_2[x, y]$$

such that $a_{01} = 0$

A_2 is the set of all polynomials with $a_{10} = 0$.

A_3 is the set of all polynomials $q(x, y)$

$$q(x, y) = b_{00} + b_{10}x + b_{01}y + b_{11}xy + \ldots + b_{kj}x^k y^j \in T_2[x, y]$$

such that $b_{10} = b_{01} = 0$ or $b_{10} = b_{01} = 1$.

We have $T_2[x, y] = A_1 \cup A_2 \cup A_3$, but $T_2[x, y]$ is contained in none of the rings A_1, A_2, A_3 (which are, of course, not valuation rings of $T_2(x, y)$).

Remark 2. Denote by G_K the family of all subrings R of a given field K having the following property: R does not contain the unit element, and $K - R$ is multiplicative closed.

In [4] we have stressed that G_K and the set of all prime ideals of a ring have some common features.

We show that our Theorem does not hold if valuation rings are replaced by the rings contained in G_K. To be more exact: We construct a field K and its subrings B, B_1, B_2, M without unit such that $M \in G_K$. Here we have $B \subset B_1 \cup B_2 \cup M$, but B is contained in none of the rings B_1, B_2, M. This is the subject of the following example.

Example 2. Denote by $T_2\{y\}$ the field of all formal series in the indeterminate y over T_2. Define $K = T_2\{y\}\{x\}$. (Hence the field of formal series in x over $T_2\{y\}$).

Define first the ring A as the ideal in $T_2[x, y]$ generated by $x(1+y)$, $x(1+x)$, $y(1+y)$.

a) Definition of the rings B_1, B_2, B_3, M.

In the following $A + u$ denotes $\{v + u \mid v \in A\}$.

Define

$B_1 = (1 + x + A) \cup A,$
$B_2 = (1 + y + A) \cup A,$
$B_3 = (x + y + A) \cup A,$

$M = \{b_0(y) + b_1(y)x + \ldots + b_n(y)x^n + \ldots,$ where $b_i(y)$ are formal series in the variable y and in $b_0(y)$ only the positive powers of y occur$\}$.

We prove that these are rings. Since it is clear that the sets B_1, B_2, B_3 are abelian groups, it is sufficient to prove that they are closed under multiplication. This follows from the following inclusions where $a, \acute{a} \in A$.

$$(1+x+a)(1+x+\acute{a})=1+x+x(1+x)+a(1+x)+\acute{a}(1+x+a)\in B_1,$$

$$(x+y+a)(x+y+\acute{a})=x+y+(x+x^2)+(y+y^2)+a(x+y)+\acute{a}(x+y+a)\in B_3,$$

$$(1+y+a)(1+y+\acute{a})=1+y+y(1+y)+a(1+y)+\acute{a}(1+y+a)\in B_2,$$

b) The rings B_1, B_2, B_3, M do not contain the unit element of K.

First of all we prove that the ring A does not contain the unit element of K.

Indeed, if this were not true, then there would exist three polynomials $P_1(x, y)$, $P_2(x, y)$, $P_3(x, y) \in T_2[x, y]$ such that

$$x(1+x)P_1(x, y)+y(1+y)P_2(x, y)+x(1+y)P_3(x, y)=1.$$

If we put $x=0$, we get

$$y(1+y)P_2(0, y)=1,$$

which is impossible, since on the left hand side we have either zero or a non-constant polynomial.

Now we prove that the ring B_1 does not contain $1 \in K$.

If this were not true, then there would exist an element $a \in A$ such that $1+x+a=1$. This means that $x \in A$. But this implies that there exist three polynomials $Q_1(x, y)$, $Q_2(x, y)$, $Q_3(x, y) \in T_2[x, y]$ such that we have

$$x(1+x)Q_1(x, y)+y(1+y)Q_2(x, y)+x(1+y)Q_3(x, y)=x$$

If we put $y=1$, we get

$$x(1+x)Q_1(x, 1)=x$$

which is impossible, since on the left — hand side we have either zero or a polynomial of degree at least 2.

The fact that the ring B_2 does not contain $1 \in K$ follows in an analogous manner.

Finally it is clear from the definition that the rings B_3 and M do not contain unit element $\in K$.

c) We show that none of the inclusions $B_i \subset B_j$ $(i, j=1, 2, 3, i \neq j)$ holds.

If there were, e.g., $B_1 \supset B_2$, we would have $1+y \in B_1$ and there would exist three polynomials $S_1(x, y)$, $S_2(x, y)$, $S_3(x, y) \in T_2[x, y]$ such that

$$1+y=S_1(x, y)x(1+x)+S_2(x, y)y(1+y)+S_3(x, y)x(1+y)+(1+x).$$

If we put $x=0$, we get

$$y=S_2(0, y)y(1+y),$$

which is impossible.

If there were $B_1 \supset B_3$, then there would exist three polynomials $U_1(x, y)$, $U_2(x, y)$, $U_3(x, y) \in T_2[x, y]$ such that

$$x+y=1+x+U_1(x, y)x(1+x)+U_2(x, y)y(1+y)+x(1+y)U_3(x,y).$$

If we put $x=y=0$, we get $0=1$ — a contradiction.

It is clear that B_3 is neither an overring of B_2, *nor of* B_1. From the considerations analogical to those above it follows that B_2 is not an overring of B_1 or B_3. Hence none of the inclusions $B_i \subset B_j$ $(i, j=1, 2, 3, i \neq j)$ holds.

d) Next we prove that $B=B_1 \cup B_2 \cup B_3$ is a ring. Since it is easy to see that B is an abelian group, with respect to the addition we have only to show that B is closed under multiplication.

We have

$$(1+x+a)(1+y+\acute{a})=1+y+x(1+y)+a(1+y+\acute{a})+\acute{a}(1+x) \in B_2,$$

so that $B_1 \cdot B_2 \subset B_2$.

Further

$$(1+x+a)(x+y+\acute{a})=x+y+x(1+x)+x(1+y)+\acute{a}(1+x+a)+a(x+y) \in B_3,$$

so that $B_1 \cdot B_3 \subset B_3$.

Finally, we have

$$(1+y+a)(x+y+\acute{a})=x(1+y)+y(1+y)+\acute{a}(1+y+a)+a(x+y) \in A,$$

so that $B_2 \cdot B_3 \subset A$.

These inclusions imply that B is a ring.

e) We prove that the complement of the ring M is closed under multiplication.

Let C, D be elements of the field K such that $C \cdot D \in M$. Let us consider C, D as formal series in the variable x over the field $T_2\{y\}$. Recall that the elements of the ring M contain only non-negative powers of x.

To satisfy $C \cdot D \in M$ we have only two possibilities.

a) One of the elements C, D, say C, contains negative powers of x. Then D contains necessarily only positive powers of x. Hence $D \in M$. Therefore $C \cdot D \in M$. implies $D \in M$.

b) Both C, D have only non-negative powers of x. Let there be $C=c_0(y)+c_1(y)x+\ldots$, $D=d_0(y)+d_1(y)x+\ldots$. Then $C \cdot D \in M$ implies that $c_0(y)d_0(y)$ contains only positive powers of y. Hence at least one of them, say $d_0(y)$, contains only positive powers of y. But then $D \in M$. Hence $C \cdot D \in M$ implies $D \in M$.

Summarily we have shown that $C \cdot D \in M$ implies that either $C \in M$ or $D \in M$. Otherwise expressed the complement of M is multiplicatively closed and $M \in G_K$.

Thus we may conclude that obviously $B=B_1 \cup B_2 \cup B_3 \subset B_1 \cup B_2 \cup M$, where $M \in G_K$. But none of the inclusions $B \subset B_1$, $B \subset B_2$, $B \subset M$ holds. This proves our statement.

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ПОКРЫТИЕ КОЛЕЦ КОЛЬЦАМИ НОРМИРОВАНИЯ

Ян Минач

Резюме

В работе доказана следующая теорема: Пусть A, $Б_1$, ., $Б_k$ подкольца с единицей поля K, такие, что

$$A \subset \bigcup_{i=1}^{k} B_i$$

и все кольца B_i, $i=1, \ldots, k$, кроме быть может двух, являются кольцами нормирования. Тогда существует такое кольцо B_i, $i \in \{1, \ldots, k\}$, что $A \subset B_i$.

Показано, что аналогичная теорема не верна для колец без единицы.