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CONGRUENCE RELATIONS ON DIRECT PRODUCTS OF LATTICES

JOZEF PÓCS

G. A. Fraser and A. Horn [2] found necessary and sufficient conditions under which each congruence relation on a direct product of two algebras is directly factorizable. I. Chajda [1] established a sufficient condition for the direct factorizability of congruence relations on direct products of conditionally complete chains. In this note it will be shown that the condition considered in [1] is sufficient for the direct factorizability also in the more general case when instead of conditionally complete chains we have arbitrary lattices.

A congruence relation \( \Theta \) on the algebra \( A = \prod_{\gamma \in \Gamma} A_{\gamma} \) is said to be directly factorizable if there exist congruence relations \( \Theta_\gamma \) on \( A_\gamma \) (\( \gamma \in \Gamma \)) such that for each \( x, y \in A \) we have \( x \Theta y \) iff \( x(\gamma) \Theta_\gamma y(\gamma) \) is valid for each \( \gamma \in \Gamma \). In this case we write \( \Theta = \prod_{\gamma \in \Gamma} \Theta_\gamma \).

Let \( L_\gamma \) (\( \gamma \in \Gamma \)) be lattices, \( L = \prod_{\gamma \in \Gamma} L_\gamma \). For \( x, y \in L \) we denote by \( f(x, y, \gamma) \) the element of \( L \) fulfilling \( f(x, y, \gamma)(\gamma) = x(\gamma) \) and \( f(x, y, \gamma)(\gamma') = y(\gamma') \) for each \( \gamma' \in \Gamma \), \( \gamma' \neq \gamma \).

**Lemma.** Let \( \Theta \) be a congruence relation on the lattice \( L = \prod_{\gamma \in \Gamma} L_\gamma \). Let \( x, y \in L \) and \( x \Theta y \). Then for each \( z \in L \) and \( \gamma \in \Gamma \) the relation \( f(x, z, \gamma) \Theta f(y, z, \gamma) \) is valid.

**Proof.** For each \( \gamma \in \Gamma \) we have \( x \land y \leq f(y, x, \gamma) \leq x \lor y \). Then from \( x \Theta y \) it follows that \( x \Theta f(y, x, \gamma) \). Hence

\[
x \land f(x, z, \gamma) \Theta f(y, x, \gamma) \land f(x, z, \gamma) = f(x \land y, x \land z, \gamma)
\]

and thus \( f(x, x \land z, \gamma) \Theta f(x \land y, x \land z, \gamma) \). By forming the join of both sides of this relation with the element \( f(y, z, \gamma) \) we obtain

\[
f(x \lor y, z, \gamma) \Theta f(y, z, \gamma).
\]

Analogously we can prove that the relation \( f(x \lor y, z, \gamma) \Theta f(x, z, \gamma) \) is valid. In view of the transitivity of \( \Theta \) we get \( f(x, z, \gamma) \Theta f(y, z, \gamma) \).
A sublattice $A$ of the lattice $L$ is called $\vee$-closed if whenever $\{z_i\}_{i \in T}$ is a nonempty subset of $A$ such that the join $\bigvee_{i \in T} z_i$ does exist in $L$, then $\bigvee_{i \in T} z_i$ belongs to $A$. In a dual way we define the notion of a $\wedge$-closed sublattice.

**Theorem.** Let $\Theta$ be a congruence relation on the lattice $L = \prod_{\gamma \in \Gamma} L_{\gamma}$ such that each class of the corresponding partition of $L$ is a $\vee$-closed sublattice of $L$. Then $\Theta$ is directly factorizable.

**Proof.** For $\gamma \in \Gamma$ we define the relation $\Theta_{\gamma}$ on $L_{\gamma}$ as follows: let $x, y, z \in L_{\gamma}$; we put $x \Theta_{\gamma} y$ if there exist elements $x, y \in L$ such that $x \Theta y$ and $x(\gamma) = y, y(\gamma) = y$. It is obvious that all relations $\Theta_{\gamma}$ are reflexive and symmetric. Let $\gamma \in \Gamma$, $x, z, y, y' \in L_{\gamma}$, $x \Theta_{\gamma} z, y \Theta_{\gamma} y'$, and then according to the definition of $\Theta_{\gamma}$ there are elements $x, y, y', z \in L$ such that $x(\gamma) = x, y(\gamma) = y, z(\gamma) = z, x \Theta y$ and $y' \Theta z$. From this and from the Lemma we infer $f(y', y, \gamma) \Theta f(z, y, \gamma)$. But $y = f(y', y, \gamma)$ and hence $x \Theta f(z, y, \gamma)$. Therefore $x, y \Theta_{\gamma} z$. Hence $\Theta_{\gamma}$ is transitive. The substitution property of $\Theta_{\gamma}$ obviously holds. Now we prove that $\Theta = \prod_{\gamma \in \Gamma} \Theta_{\gamma}$ is valid.

The relation $\Theta \subseteq \prod_{\gamma \in \Gamma} \Theta_{\gamma}$ is obvious. Let $x, y \in L$, $x \Theta \left( \prod_{\gamma \in \Gamma} \Theta_{\gamma} \right) y$. We have to verify that $x \Theta y$ is valid. It suffices to consider the case when $x \leq y$. For each $\gamma \in \Gamma$ we have $x(\gamma) \Theta_{\gamma} y(\gamma)$. Hence for each $\gamma \in \Gamma$ there exist elements $z, u \in L$ such that $f(x, z, \gamma) \Theta f(y, u, \gamma)$. From this and from the Lemma we obtain that

$$x = f(x, x, \gamma) \Theta f(y, x, \gamma)$$

for each $\gamma \in \Gamma$.

is valid. Because of $x \leq y$ we infer $y = \bigvee_{\gamma \in \Gamma} f(y, x, \gamma)$.

Since the classes of the partition corresponding to $\Theta$ are $\vee$-closed, we have $y \Theta x$.

Analogously we can prove the dual assertion (by assuming that the classes of the partition corresponding to $\Theta$ are $\wedge$-closed).

A lattice $L$ is said to be conditionally complete if each nonempty bounded subset of $L$ possesses a supremum and an infimum in $L$. A congruence relation $\Theta$ on the lattice $L$ is called conditionally complete if, whenever $\{a_{\mu} : \mu \in \mathcal{M}\}$ and $\{b_{\mu} : \mu \in \mathcal{M}\}$ are nonempty bounded subsets of $L$ such that $a_{\mu} \Theta b_{\mu}$ is valid for each $\mu \in \mathcal{M}$, then

$$\left( \bigvee_{\mu \in \mathcal{M}} a_{\mu} \right) \Theta \left( \bigvee_{\mu \in \mathcal{M}} b_{\mu} \right) \quad \text{and} \quad \left( \bigwedge_{\mu \in \mathcal{M}} a_{\mu} \right) \Theta \left( \bigwedge_{\mu \in \mathcal{M}} b_{\mu} \right)$$

holds.

From the above Theorem we obtain immediately:
Corollary. ([1], Theorem 1.) Let $A_i$ be conditionally complete chains for $i \in I$ and let $\Theta$ be a conditionally complete congruence on $A = \prod_{i \in I} A_i$. Then there exist congruences $\Theta_i$ on $A_i$ such that $\Theta = \prod_{i \in I} \Theta_i$.

REFERENCES


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КОНГРУЭНЦИИ НА ПРЯМЫХ ПРОИЗВЕДЕНИЯХ СТРУКТУР

Йозеф Поч

Резюме

И. Хайд установил достаточные условия для прямой разложимости конгруэнций на прямом произведении цепей.

В этой статье найдено условие, при котором конгруэнция на прямом произведении любых структур будет разложимой конгруэнцией.