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CONGRUENCE RELATIONS ON DIRECT PRODUCTS OF LATTICES

JOZEF PÓCS

G. A. Fraser and A. Horn [2] found necessary and sufficient conditions under which each congruence relation on a direct product of two algebras is directly factorizable. I. Chajda [1] established a sufficient condition for the direct factorizability of congruence relations on direct products of conditionally complete chains. In this note it will be shown that the condition considered in [1] is sufficient for the direct factorizability also in the more general case when instead of conditionally complete chains we have arbitrary lattices.

A congruence relation Θ on the algebra $A=\prod_{\gamma\in\Gamma} A_\gamma$ is said to be directly factorizable if there exist congruence relations Θ_γ on A_γ ($\gamma\in\Gamma$) such that for each $x, y\in A$ we have $x\Theta y$ iff $x(\gamma)\Theta_\gamma y(\gamma)$ is valid for each $\gamma\in\Gamma$. In this case we write $\Theta=\prod_{\gamma\in\Gamma}\Theta_\gamma$.

Let L_γ ($\gamma\in\Gamma$) be lattices, $L=\prod_{\gamma\in\Gamma} L_\gamma$. For $x, y\in L$ we denote by $f(x, y, \gamma)$ the element of L fulfilling $f(x, y, \gamma)(\gamma) = x(\gamma)$ and $f(x, y, \gamma)(\gamma') = y(\gamma')$ for each $\gamma'\in\Gamma$, $\gamma'\neq\gamma$.

Lemma. *Let Θ be a congruence relation on the lattice $L=\prod_{\gamma\in\Gamma} L_\gamma$. Let $x, y\in L$ and $x\Theta y$. Then for each $z\in L$ and $\gamma\in\Gamma$ the relation $f(x, z, \gamma)\Theta f(y, z, \gamma)$ is valid.*

Proof. For each $\gamma\in\Gamma$ we have $x\wedge y \leqq f(y, x, \gamma) \leqq x\vee y$. Then from $x\Theta y$ it follows that $x\Theta f(y, x, \gamma)$. Hence

$$x\wedge f(x, z, \gamma)\Theta f(y, x, \gamma)\wedge f(x, z, \gamma)=f(x\wedge y, x\wedge z, \gamma)$$

and thus $f(x, x\wedge z, \gamma)\Theta f(x\wedge y, x\wedge z, \gamma)$. By forming the join of both sides of this relation with the element $f(y, z, \gamma)$ we obtain

$$f(x\vee y, z, \gamma)\Theta f(y, z, \gamma).$$

Analogously we can prove that the relation $f(x\vee y, z, \gamma)\Theta f(x, z, \gamma)$ is valid. In view of the transitivity of Θ we get $f(x, z, \gamma)\Theta f(y, z, \gamma)$.

A sublattice A of the lattice L is called $\bigvee$-closed if whenever $\{z_t\}_{t \in T}$ is a nonempty subset of A such that the join $\bigvee_{t \in T} z_t$ does exist in L, then $\bigvee_{t \in T} z_t$ belongs to A. In a dual way we define the notion of a $\bigwedge$-closed sublattice.

Theorem. *Let Θ be a congruence relation on the lattice $L = \prod_{\gamma \in \Gamma} L_\gamma$ such that each class of the corresponding partition of L is a $\bigvee$-closed sublattice of L. Then Θ is dierectly factorizable.*

Proof. For $\gamma \in \Gamma$ we define the relation Θ_γ on L_γ as follows: let $x_\gamma, y_\gamma \in L_\gamma$; we put $x_\gamma \Theta_\gamma y_\gamma$ if there exist elements $x, y \in L$ such that $x \Theta y$ and $x(\gamma) = x_\gamma$, $y(\gamma) = y_\gamma$. It is obvious that all relations Θ_γ are reflexive and symmetric. Let $\gamma \in \Gamma$, x_γ, y_γ, $z_\gamma \in L_\gamma$, $x_\gamma \Theta_\gamma y_\gamma$ and $y_\gamma \Theta_\gamma z_\gamma$; then according to the definition of Θ_γ there are elements $x, y, y', z \in L$ such that $x(\gamma) = x_\gamma$, $y(\gamma) = y'(\gamma) = y_\gamma$, $z(\gamma) = z_\gamma$, $x \Theta y$ and $y' \Theta z$. From this and from the Lemma we infer $f(y', y, \gamma) \Theta f(z, y, \gamma)$. But $y = f(y', y, \gamma)$ and hence $x \Theta f(z, y, \gamma)$. Therefore $x_\gamma \Theta_\gamma z_\gamma$. Hence Θ_γ is transitive. The substitution property of Θ_γ obviously holds. Now we prove that $\Theta = \prod_{\gamma \in \Gamma} \Theta_\gamma$ is valid. The relation $\Theta \leqq \prod_{\gamma \in \Gamma} \Theta_\gamma$ is obvious. Let $x, y \in L$, $x \left(\prod_{\gamma \in \Gamma} \Theta_\gamma \right) y$. We have to verify that $x \Theta y$ is valid. It suffices to consider the case when $x \leqq y$. For each $\gamma \in \Gamma$ we have $x(\gamma) \Theta_\gamma y(\gamma)$. Hence for each $\gamma \in \Gamma$ there exist elements z^γ, $u^\gamma \in L$ such that $f(x, z^\gamma, \gamma) \Theta f(y, u^\gamma, \gamma)$. From this and from the Lemma we obtain that

$$x = f(x, x, \gamma) \Theta f(y, x, \gamma) \text{ for each } \gamma \in \Gamma$$

is valid. Because of $x \leqq y$ we infer $y = \bigvee_{\gamma \in \Gamma} f(y, x, \gamma)$.

Since the classes of the partition corresponding to Θ are $\bigvee$-closed, we have $y \Theta x$.

Analogously we can prove the dual assertion (by assuming that the classes of the partition corresponding to Θ are $\bigwedge$-closed).

A lattice L is said to be conditionally complete if each nonempty bounded subset of L possesses a supremum and an infimum in L. A congruence relation Θ on the lattice L is called conditionally complete if, whenever $\{a_\mu : \mu \in M\}$ and $\{b_\mu : \mu \in M\}$ are nonempty bounded subsets of L such that $a_\mu \Theta b_\mu$ is valid for each $\mu \in M$, then

$$\left(\bigvee_{\mu \in M} a_\mu \right) \Theta \left(\bigvee_{\mu \in M} b_\mu \right) \quad \text{and} \quad \left(\bigwedge_{\mu \in M} a_\mu \right) \Theta \left(\bigwedge_{\mu \in M} b_\mu \right)$$

holds.

From the above Theorem we obtain immediately:

Corollary. ([1], Theorem 1.) *Let A_i be conditionally complete chains for $i \in I$ and let Θ be a conditionally complete congruence on $A = \prod_{i \in I} A_i$. Then there exist congruences Θ_i on A_i such that $\Theta = \prod_{i \in I} \Theta_i$.*

REFERENCES

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КОНГРУЭНЦИИ НА ПРЯМЫХ ПРОИЗВЕДЕНИЯХ СТРУКТУР

Йозеф Поч

Резюме

И. Хайда установил достаточные условия для прямой разложимости конгруэнций на прямом произведении цепей.

В этой статье найдено условие, при котором конгруэнция на прямом произведении любых структур будет разложимой конгружнцией.