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AN OPTIMAL CONTROL PROBLEM FOR AN ELLIPTIC VARIATIONAL INEQUALITY

IGOR BOCK—JÁN LOVÍŠEK

We shall be dealing with an optimal control for an elliptic variational inequality with controls involved both in the operator of the problem and in the right hand side. A similar problem with controls only in the right hand side has been solved in the book [2].

1. The Existence Theorem

Let $U$ with a norm $\| \cdot \|_U$ be a reflexive Banach space of controls, $U_{ad} \subset U$ a set of admissible controls. We assume $U_{ad}$ be convex, closed and bounded in $U$.

We assume further a reflexive Banach space $V$ with a norm $\| \cdot \|$ and a convex closed subset $K \subset V$. $V^*$ means a dual space of $V$ with a norm $\| \cdot \|^*$ and a duality pairing $[\cdot,\cdot]$ between $V^*$ and $V$.

Let $\{A(e)\}, A(e): K \rightarrow V^*$ for every $e \in U_{ad}$, be a family of operators satisfying the following assumptions:

1. $A(e)$ is for every $e \in U_{ad}$ strongly monotone i.e.
   $[A(e)u - A(e)v, u - v] > 0$ for every $u, v \in K, u \neq v, e \in U_{ad}$

2. $A(e)$ is for every $e \in U_{ad}$ hemicontinuous i.e.
   \[\lim_{t \to 0} [A(e)(u + t(v - u)), w] = [A(e)u, w]\]
   for every $e \in U_{ad}, u, v \in K, w \in V$

3. $\{A(e)\}$ is uniformly bounded i.e.
   $\|A(e)v\|^* \leq C$, if $\|e\|_U \leq C_1$ and $\|v\| \leq C_2$

4. $\{A(e)\}$ is uniformly coercive i.e. there exist such $v_0 \in K$ and a real function
   \[r: [0, \infty) \rightarrow R, \lim_{t \to \infty} r(t) = \infty, \text{ that}\]
   $\|A(e)v, v - v_0\| \geq \|v\|r(\|v\|)$ for every $v \in K$

5. $A(\cdot)v: U_{ad} \rightarrow V^*$ is for every $v \in K$ strengthenly continuous i.e.
   $e_n \rightarrow e_0$ (weakly) in $U$ implies $A(e_n)v \rightarrow A(e_0)v$ (strongly) in $V^*$. 


Let the operator $B: U_{ad} \to V^*$ be strengthenly continuous and $f \in V^*$. Under the assumptions (1), (2), (3) the operator $A(e): K \to V^*$ is pseudomonotone for every $e \in U_{ad}$ (def. in [3]) and then due to the theorem from [3] there exists a unique solution $u(e) \in K$ of a variational inequality

\[(6) \quad [A(e)u(e), v - u(e)] \geq [f + B(e), v - u(e)]\]

for every $v \in K$

Our aim is to solve the following optimal control problem:

**Problem P.** To find a control $e_0 \in U_{ad}$ which fulfills:

\[(7) \quad [A(e_0)u(e_0), v - u(e_0)] \geq [f + B(e_0), v - u(e_0)]\]

for every $v \in K$

\[(8) \quad \|Cu(e_0) - z_d\|^2_H = \min_{e \in U_{ad}} \|Cu(e) - z_d\|^2_H,
\]

where $u(e) \in K$ is a solution of (6), $\mathcal{H}$ is a Hilbert space, $C \in L(V, \mathcal{H})$ is a linear control operator, $z_d \in \mathcal{H}$ is a fixed element.

**Theorem 1.** There exists at least one solution $e_0 \in U_{ad}$ of Problem P.

**Proof.** We have $J(e) = \|Cu(e) - z_d\|^2_H \geq 0$ for every $e \in U_{ad}$. Hence $\inf_{e \in U_{ad}} J(e) \geq 0$. Let $(e_n)_{n=1}^\infty$ be the minimizing sequence for a functional $J(\cdot)$ i.e.

\[(9) \quad \lim_{n \to \infty} J(e_n) = \inf_{e \in U_{ad}} J(e)\]

As the set $U_{ad}$ is convex and closed in the reflexive space $U$ it is weakly closed in $U$. Then there exist such a subsequence of $(e_n)_{n=1}^\infty$ (we denote it again by $(e_n)_{n=1}^\infty$) and the element $e_0 \in U_{ad}$ that

\[(10) \quad e_n \rightharpoonup e_0 \quad \text{(weakly in } U)\]

Denoting $u_n = u(e_n) \in K$, $n = 1, 2, ...$ we have

\[(11) \quad [A(e_n)u_n, v - u_n] \geq [f + b(e_n), v - u_n] \quad \text{for every } v \in K, \ n = 1, 2, ...
\]

Inserting $v_0 \in K$ in (11) we arrive at

\[(12) \quad [A(e_n)u_n, u_n - v_0] \leq [f + B(e_n), u_n - v_0]\]

Using the uniform coerciveness of a system $\{A(e)\}$ and the streghten continuity of $B$ we obtain

\[(13) \quad \|u_n\| r(\|u_n\|) \leq C_1 \|u_n\| + C_2\]

As $\lim_{t \to \infty} r(t) = \infty$ we have

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We can now extract such a subsequence of \((u_n)_{n=1}^\infty\) denoted again by \((u_n)_{n=1}^\infty\) that

\[ u_n \rightharpoonup u \quad \text{(weakly in } V) \]

Moreover \(u \in K\), because \(u_n \in K, \, n = 1, 2, \ldots\) and \(K\) is weakly closed in \(V\).

As a family of operators \(\{A(e)\}\) is uniformly bounded (see (2)) we have 

\[ \|A(e_n)u_n\|_* \leq C \quad \text{for } n = 1, 2, \ldots \]

Then there exists an element \(\chi \in V^*\) such that

\[ A(e_n)u_n \rightharpoonup \chi \quad \text{(weakly in } V^*) \]

Monotonicity of \(A(e_n)\) implies

\[ [A(e_n)u_n - A(e_n)v, \, u_n - v] \geq 0 \]

for every \(v \in K, \, n = 1, 2, \ldots\)

Inserting \(v = u\) in (11) we obtain using (10), (15) and the strengthened continuity of the operator \(B\),

\[ \limsup [A(e_n)u_n, \, u_n - u] = 0 \]

and combining with (16)

\[ \limsup [A(e_n)u_n, \, u_n] \leq [\chi, \, u] \]

Taking into account relations (15), (16), (17), (19) and the strengthened continuity of the operator \(A(\cdot)v: \, U_d \to V^*\) we arrive at

\[ [\chi - A(e_0)v, \, u - v] \geq 0 \quad \text{for every } v \in K \]

Let \(v = u + t(w - u), \, t \in (0, 1), \, w \in K\). Then we have

\[ [\chi - A(e_0)(u + t(w - u)), \, u - w] \geq 0 \]

for every \(w \in K, \, t \in (0, 1)\)

Making use of hemicontinuity of \(A(e_0)\) we obtain after \(t \to 0\) and putting again \(w = v\)

\[ [A(e_0)u, \, u - v] \leq [\chi, \, u - v] \quad \text{for every } v \in K \]

Putting \(v = u\) in (17) we have 

\[ [A(e_n)u_n, \, u_n - u] \geq [A(e_n)u, \, u_n - u] \]

The strengthened continuity of \(A(\cdot)u\) and the weak convergence \(u_n \rightharpoonup u\) imply immediately

\[ \lim_{n \to \infty} [A(e_n)u, \, u_n - u] = 0 \]

and hence \(\liminf_{n \to \infty} [A(e_n)u_n, \, u_n - u] \geq 0\). Comparing with (18) we have

\[ \lim_{n \to \infty} [A(e_n)u_n, \, u_n - u] = 0 \]

Relations (16), (22), (23) enable us to estimate
(24) \[ A(e_0)u, u-v \leq \lim_{n \to \infty} [A(e_n)u_n, u_n-v] \]

for every \( v \in K \)

We are coming now to the conclusion that the element \( u \in K \) is a solution of a variational inequality

(25) \[ A(e_0)u, u-v \leq [f + B(e_0), u-v] \]

for every \( v \in K \),

having used (24), (11), (15) and the strengthened continuity of \( B \).

Hence we have proved

(26) \( u = u(e_0), u(e_n) \rightharpoonup u(e_0) \) (weakly in \( V \))

what implies

(27) \( Cu(e_n) \rightharpoonup Cu(e_0) \) (weakly in \( x \))

A functional \( g: x \to R, g(w) = \|w-z_d\|^2, w \in x \) is weakly lower semicontinuous and therefore

(28) \[ J(e_0) = \|Cu(e_0) - z_d\|^2 \leq \lim \inf \|Cu(e_n) - z_d\|^2 \]

\[ = \lim \inf J(e_n) = \inf J(e_n) \]

which completes the proof of (8) and of the Theorem.

Remark. It is an opened question to gain further information about the set \( X \subset U_{ad} \) of solutions of Problem P. We have only verified that \( X \neq \emptyset \). The core of the problem is that we have been solving the control problem governed by the variational inequality and hence the minimized functional \( J \) with respect to \( e \in U_{ad} \) is not convex.

2. The Example

We shall investigate the optimal control problem for the thickness function of a thin plate with an obstacle.

Let \( \Omega \subset R^2 \) be the middle plane of a plate. We assume that \( \Omega \) has the Lipschitz boundary \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \). We suppose that a part \( \Gamma_1 \) of the boundary of the plate is clamped, a part \( \Gamma_2 \) is simply supported and a part \( \Gamma_3 \) is free. An obstacle for the deflection of the plate can be described by the function \( \bar{q}: \Omega \to R \) satisfying the inequality \( \bar{q}(x, y) \leq 0 \) on \( \Gamma_1 \cup \Gamma_2 \).

We denote

(29) \( V = \left\{ v \in H^2(\Omega) \mid v = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_1 \right\} \),

where \( H^2(\Omega) \) is a Sobolev space of all functions from \( L_2(\Omega) \) which have the
distributive derivatives up to the 2-nd order from $L_2(\Omega)$. The boundary conditions are satisfied in the sense of traces ([4]).

Functions expressing deflections of the plate belong to the set

$$(30)\quad K = \{ v \in V, v(x, y) \geq \varphi(x, y) \text{ a.g. in } \Omega \}$$

The thickness functions $e: \Omega \to R$ play the role of controls. We assume the set $U_{ad}$ of admissible controls in the form

$$(31)\quad U_{ad} = \{ e \in H^2(\Omega) \parallel e \parallel_{H^2(\Omega)} \leq M, e(x, y) \geq m > 0 \text{ on } \Omega \}$$

Due to the imbedding theorems in the Sobolev space $H^2(\Omega)$, $K$ is a convex closed subset of $V$ and $U_{ad}$ a convex closed and bounded subset of the space $U = H^2(\Omega)$.

The operators $A(e): K \to V^*$, $e \in U_{ad}$, of the problem are of the form

$$[A(e)u, v] = \frac{E}{12(1-\mu^2)} \int_{\Omega} e^3(x, y) \left[ \left( \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} \right) \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right] dx \, dy,$$

$$+ 2(1-\mu) \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} + \left( \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 u}{\partial x^2} \right) \frac{\partial^2 v}{\partial y^2} \right] dx \, dy,$$

$u \in K, \quad v \in V, \quad e \in U_{ad}, \quad \mu \in (0, 1)$

The operators $A(e)$ satisfy the assumptions (1)—(5). If the outer force has the form of linear bounded functional $f \in V^*$, then a deflection of the plate $u(e) \in K$ is a solution of a variational inequality

$$(33)\quad [A(e), v - u(e)] \geq [f, v - u(e)] \quad \text{for every } v \in K$$

For simplicity we do not consider the operator $B: U_{ad} \to V^*$. A cost functional can be of the form

$$(34)\quad J(e) = \int_{\Omega} (Tu(e) - z_d)^2 \, dx \, dy, \quad e \in U_{ad},$$

where $I: V \to L_2(\Omega)$ is the identity operator, $z_d \in L_2(\Omega)$, or

$$(35)\quad \hat{J}(e) = \int_{\Gamma_3} (Tu(e) - z_d)^2 \, ds, \quad e \in U_{ad},$$

where $T: V \to L_2(\Gamma_3)$ is the operator of traces, $z_d \in L_2(\Gamma_3)$. The optimality conditions for the functionals (34) and (35) mean the minimizing of the distance between the deflection of the plate $u(e)$ and the prscribed function $z_d$ on $\Omega$, or $\Gamma_3$. Due to the Theorem 1. there exists the optimal thickness function $e_0: \Omega \to R$ which minimizes the functional $J$ or $\hat{J}$ on the set of admissible functions $U_{ad}$.

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ОДНА ЗАДАЧА ОПТИМАЛЬНОГО УПРАВЛЕНИЯ
ДЛЯ ЭЛЛИПТИЧЕСКОГО ВАРИАЦИОННОГО НЕРАВЕНСТВА

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Резюме

В работе рассматривается задача оптимального управления для эллиптического вариационного неравенства с управлениями в операторе и в правой части. Доказывается существование оптимального управления. Показывается пример оптимизации толщины тонкой пластины с препятствием.