Beloslav Riečan

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Mathematica Slovaca, Vol. 33 (1983), No. 2, 153--163

Persistent URL: http://dml.cz/dmlcz/136326

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ON MEASURES AND INTEGRALS WITH VALUES IN ORDERED GROUPS

BELOSLAV RIEČAN

In the paper presented we shall formulate and prove two main results: the measure extension theorem and the Daniell integral extension theorem, both with values in partially ordered groups.

These results are analogies of corresponding results holding in linear ordered spaces. Probably the first result of this kind was published in [4] (see also [2]) and it was concerned with the extension of linear continuous operators in regular K-spaces. A more general result was published in [6]. A special case of this result is the measure extension theorem as well as the Kantorovič theorem; both with values in regular K-spaces.

Of course, regular K-spaces present a quite special kind of linear ordered spaces. The measure extension problem for values in linear ordered spaces was definitively solved by J. D. M. Wright in [12]. He found a sufficient and necessary condition (σ-distributivity of X) for every measure with values in X could be extended from a ring to the generated σ-ring. Other proofs of the Wright theorem were published in [3] and [8].

In the paper we present some improvements of the preceding results. Firstly we suppose that the rang space G is a group instead of previous assumption that G is a linear space. Of course, if G is moreover a linear space, the corresponding mentioned results are special cases of our theorems. Secondly G need not be a lattice; we assume only that G is a partially ordered group. Thirdly we study from a unique point of view the measure as well as the integral. This permits to obtain simultaneously the measure extension theorem as well as the Daniell integral extension theorem. (This method was first used in [1] and [5]; see also [6], [10] and [11].) Fourthly we admit weaker assumptions on the domain of studied maps. Thus we obtain the measure extension theorem for measures on Boolean algebras. On the other hand we obtain the theory of the Daniell integral for maps defined on a subgroup of a lattice ordered group.

Our constructions and proof are very similar to that of Fremlin ([3]), but more general and simpler (see [8]).

Recall that measures and integrals with values in ordered groups were studied in [7], [9], [10] and [11].

1. Range space

Our range space will consist of an ordered commutative group G, i. e. such a commutative group $(G, +)$, which is a partially ordered set $(G, \leqq)$ and such that $a \leqq b$ implies $a+c \leqq b+c$ for every $c \in G$. G is called monotonously σ-complete if every increasing bounded sequence has the least upper bound. If p is a positive integer and $x \in G$, then we define $px = x + x + \ldots + x$ (p times), i. e. $1x = x$ and $px = (p-1)x + x$ for $p > 1$.

Usually we shall assume that G is an l-group, i. e. such an ordered group which becomes a lattice. The lattice operations will be denoted by $a \vee b$, $a \wedge b$ and similarly $\bigvee_{i=1}^{\infty} a_i$, $\bigwedge_{i=1}^{\infty} a_i$. In any l-group we can define $a^+ = a \vee O$ (where O is the neutral element), $a^- = (-a) \vee O$. It is well known that $a = a^+ - a$ for every $a \in G$. We shall write $a_n \nearrow a$, if $a_n \leqq a_{n+1}$ $(n = 1, 2, \ldots)$ and $a = \bigvee_{i=1}^{\infty} a_i$, the symbol $a_n \searrow a$ has an analogous meaning.

The group G will be sometimes assumed to have some further properties, namely the following two ones.

An ordered group G is called σ-distributive if $a_{ij} \searrow O$ $(j \to \infty,\ i = 1, 2, \ldots)$ implies

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = O.$$

An ordered group G satisfies the condition (P) if for any bounded sequence $(a_{n,i,j})_{n,i,j}$ of elements of G such that $a_{n,i,j} \searrow O$ $(j \to \infty,\ n, i = 1, 2, \ldots)$ and any $a > O$ there is a bounded sequence $(a_{i,j})_{i,j}$ of elements of G such that $a_{i,j} \searrow O$ $(j \to \infty)$ and for every $\varphi \in N^N$

$$a \wedge \Big(\sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n-1)} \Big) \leqq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Proposition 1. *Every monotonously σ-complete l-group satisfies the condition (P).*

Proof. See [13].

2. Assumptions

We shall extend mappings whose domains and ranges are partially ordered sets of special types. First we list the assumptions concerning the structure X containing the domain of a given mapping.

The structure $(X, \vee, \wedge, +, -)$ satisfies the following properties:

(1) $(X, \vee, \wedge)$ is a boundedly σ-complete lattice, i. e. every bounded sequence has the least upper bound and the greatest lower bound.

(2) If $a \leqq b$ and $c \leqq d$, then $a + c \leqq b + d$, $c - b \leqq d - a$.

(3) If $a_n \nearrow a$, $b_n \nearrow b$, then $a_n + b_n \nearrow a + b$, $a_n \wedge b_n \nearrow a \wedge b$.

(4) If $a_n \searrow a$, $b_n \searrow b$, then $a_n \vee b_n \searrow a \vee b$.

(b) If $b_n \nearrow b$, $c_n \searrow c$, then $b_n - c_n \nearrow b - c$, $c_n - b_n \searrow c - b$.

Let X satisfy the preceding conditions, $A \subset X$ be a subset closed under $\vee$, $\wedge$, $+$, $-$ and such that to every $x \in X$ there is an $a \in A$ with $x \leqq a$. Let G be an ordered, commutative, monotonously σ-complete group and let $J_0 \colon A \to G$ satisfy the following conditions:

(i) If $a, b \in A$, $a \leqq b$, then $J_0(a) \leqq J_0(b)$ and $J_0(b) = J_0(a) + J_0(b - a)$.

(ii) $J_0(a) \leqq J_0(b) + J_0(a - b)$ for every $a, b \in A$.

(iii) $J_0(a + b) \leqq J_0(a) + J_0(b)$ for every $a, b \in A$.

(iv) If $a_n \nearrow a$, $b_n \searrow b$, $a \geqq b$, $a_n, b_n \in A$ $(n = 1, 2, \ldots)$ and $(J_0(a_n))_{n=1}^{\infty}$, $(J_0(b_n))_{n=1}^{\infty}$ are bounded, then

$$\bigwedge_{n=1}^{\infty} J_0(b_n) \leqq \bigvee_{n=1}^{\infty} J_0(a_n).$$

3. Construction

Lemma 1. *If $a_n, b_n \in A$ $(n = 1, 2, \ldots)$, $a_n \nearrow a$, $b_n \nearrow b$, $a \leqq b$ and $(J_0(a_n))_{n=1}^{\infty}$, $(J_0(b_n))_{n=1}^{\infty}$ are bounded, then $\bigvee_{n=1}^{\infty} J_0(a_n) \leqq \bigvee_{n=1}^{\infty} J_0(b_n)$.*

Proof. By the assumption (3) $a_n \wedge b_m \nearrow a_n \wedge b = a_n (m \to \infty)$, hence by (i) and (iv)

$$J_0(a_n) = \bigvee_{m=1}^{\infty} J_0(a_n \wedge b_m) \leqq \bigvee_{m=1}^{\infty} J_0(b_m).$$

Definition 1. *By A^+ we denote the set of all $b \in X$ for which there exists a sequence $(a_n)_{n=1}^{\infty}$ of elements of A such that $a_n \nearrow b$ and $(J_0(a_n))_{n=1}^{\infty}$ is bounded. Further we define $J^+ \colon A^+ \to G$ by the formula*

$$J^+(b) = \bigvee_{n=1}^{\infty} J_0(a_n),$$

where $a_n \in A$, $a_n \nearrow b$. The symbols A^-, J^- will have an analogous meaning.

Proposition 2. *If $a, b \in A^+$ or $a, b \in A^-$ resp. and $a \leqq b$, then $J^+(a) \leqq J^+(b)$ or $J^-(a) \leqq J^-(b)$ resp.*

Proof. It follows from Lemma 1 or the dual assertion resp.

Proposition 3. *If $b \in A^+$, $c \in A^-$, $b \geqq c$, then $J^+(b) \geqq J^-(c)$.*

Proof. It follows from (iv) and the definition of J^+ and J^-.

Proposition 4. *For every $b \in A^+$, $c \in A^-$ we have*

$$J^-(c) \leqq J^-(c - b) + J^+(b),$$

$$J^+(b) \leqq J^+(b-c) + J^-(c).$$

Proof. The assertion follows from (ii) and (5).

Proposition 5. *Let $b_n \in A^+$, $c_n \in A^-$ $(n=1, 2, \ldots)$, $b_n \nearrow b$, $c_n \searrow c$, $(J^+(b_n))_{n=1}^{\infty}$, $(J^-(c_n))_{n=1}^{\infty}$ be bounded. Then $b \in A^+$, $c \in A^-$ and*

$$J^+(b) = \bigvee_{n=1}^{\infty} J^+(b_n), \quad J^-(c) = \bigwedge_{n=1}^{\infty} J^-(c_n).$$

Proof. Since $b_n \in A^+$, there are $d_m^n \in A$ such that $d_m^n \nearrow b_n$ $(m \to \infty)$. Put $d_n = \bigvee_{m=1}^{n} d_m^n$. Then $d_n \leqq b_n$, $d_n \in A$, $J_0(d_n) = J^+(d_n) \leqq J^+(b_n)$ $(n=1, 2, \ldots)$ and $d_n \nearrow b$. Therefore $b \in A^+$ and

$$J^+(b) = \bigvee_{n=1}^{\infty} J_0(d_n) \leqq \bigvee_{n=1}^{\infty} J^+(b_n) \leqq J^+(b).$$

The second assertion can be proved similarly.

Proposition 6. *If $b \in A^+$, $c \in A^-$, $b \geqq c$, then $J^+(b) = J^+(b-c) + J^-(c)$.*

Proof. Let $c_n \in A$, $c_n \searrow c$. Then $b \wedge c_n \in A^+$, hence there are $d_{k,n} \in A$, $d_{k,n} \nearrow b \wedge c_n (k \to \infty)$. Since $d_{k,n} \leqq b$, $d_{k,n} \leqq c_n$, we have by (2) $b - d_{k,n} \geqq b - c_n$, hence by (iv), (i) and Proposition 2

$$J^+(b) = J_0(d_{k,n}) + J^+(b - d_{k,n}) \geqq J_0(d_{k,n}) + J^+(b - c_n).$$

Therefore by Proposition 4 $(b \wedge c_n \geqq c,\ b \wedge c_n \in A^+,\ c \in A^-)$

$$J^+(b) \geqq \bigvee_{k=1}^{\infty} J_0(d_{k,n}) + J^+(b - c_n) = J^+(b \wedge c_n) +$$
$$+ J^+(b - c_n) \geqq J^-(c) + J^+(b - c_n).$$

Proposition 5 implies

$$J^+(b) \geqq J^-(c) + \bigvee_{n=1}^{\infty} J^+(b - c_n) = J^-(c) + J^+(b-c).$$

The opposite inequality follows trom Proposition 4.

Proposition 7. *For every a, $b \in A^+$ we have*

$$J^+(a+b) \leqq J^+(a) + J^+(b).$$

Proof. It follows from (iii) and (3)

Definition 2. *We shall write $x \in L$ if there is $w \in G$ and $a_{i,j}$, $b_{i,j} \in G$, $a_{i,j} \searrow 0$, $b_{i,j} \searrow 0$ $(j \to \infty,\ i=1, 2, \ldots)$ such that for every $\varphi \in N^N$ there are $x_1^\varphi \in A^-$, $x_2^\varphi \in A^+$ such that $x_1^\varphi \leqq x \leqq x_2^\varphi$ and*

$$J^+(x_2^\varphi) - \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \leqq w \leqq J^-(x_1^\varphi) + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}.$$

Proposition 8. *If $x \in L$, then*

$$\bigwedge\{J^+(x_2);\ x_2 \geqq x,\ x_2 \in A^+\} = \bigvee\{J^-(x_1);\ x_1 \leqq x,\ x_1 \in A^-\}.$$

Proof. If $x_1 \leqq x$, $x_1 \in A^-$, then $x_1 \leqq x_2^\varphi$, hence by Proposition 3 $J^-(x_1) \leqq J^+(x_2^\varphi)$. It follows that

$$J^-(x_1) - w \leqq J^+(x_2^\varphi) - w \leqq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

By the σ-distributivity of G one obtains

$$J^-(x_1) - w \leqq 0 = \bigwedge_{\varphi} \bigvee_{i} a_{i,\varphi(i)},$$

hence

$$J^-(x_1) \leqq w$$

and therefore the element w is an upper bound of the set $\{J^-(x_1);\ x_1 \leqq x,\ x_1 \in A^-\}$. Let $z \in G$ be any upper bound of this set. We show that $z \geqq w$. Indeed, $z \geqq J^-(x_1^\varphi)$, hence

$$w - z \leqq w - J^-(x_1^\varphi) \leqq \bigvee_{i} b_{i,\varphi(i)}$$

for every $\varphi \in N^N$ and therefore

$$w - z \leqq 0 = \bigwedge_{\varphi} \bigvee_{i} b_{i,\varphi(i)}$$

by the σ-distributivity of G. We have just proved the equality

$$w = \bigvee\{J^-(x_1),\ x_1 \leqq x,\ x_1 \in A^-\}.$$

The equality

$$w = \bigwedge\{J^+(x_2);\ x_2 \geqq x,\ x_2 \in A^+\}$$

can be proved similarly.

Definition 3. *For $x \in L$ we define*

$$J(x) = \bigwedge\{J^+(x_2);\ x_2 \geqq x,\ x_2 \in A^+\} = \\ = \bigvee\{J^-(x_1);\ x_1 \leqq x,\ x_1 \in A^-\}.$$

4. General results

Theorem 1. *Let G be a monotonously σ-complete, σ-distributive, commutative ordered group satisfying the condition (P). Let $x, y \in L$, $x \leqq y$. Then $y - x \in L$ and $J(y-x) = J(y) - J(x)$.*

Proof. Follows immediately from the condition (P) and the definition of L.

Theorem 2. *Let G be a monotously σ-complete, σ-distributive, commutative ordered group satisfying the condition (P). Let $x_n \in L$ $(n=1, 2, \ldots)$, $x_n \nearrow x$. Then $x \in L$ and $J(x) = \bigvee_{n=1}^{\infty} J(x_n)$.*

Proof. By the assumption there is $a \in A$ such that $a \geqq x$. By Theorem 1, $x_1, x_2 - x_1, x_3 - x_2, \ldots \in L$, hence there are $a_{n,i,j} \in G$, $a_{n,i,j} \searrow 0$ $(j \to \infty)$ such that for every $\varphi \in N^N$ there are $y_n \in A^+$ with $y_1 \geqq x_1$, $y_n \geqq x_n - x_{n-1}$, $y_n \leqq a$, $(n=2, 3, \ldots)$ and

$$J(x_1) > J^+(y_1) - \bigvee_i a_{1,i,\varphi(i)}$$

$$J(x_n - x_{n-1}) > J^+(y_n) - \bigvee_i a_{n,i,\varphi(i+n-1)}, \qquad (n=2, 3, \ldots).$$

By the condition (P) there are $a_{i,j} \in G$ such that $a_{i,j} \searrow 0 (j \to \infty)$ and

$$(J_0(a) - J(x_1)) \wedge \Big(\sum_n \bigvee_i a_{n,i,\varphi(i+n-1)}\Big) \leqq \bigvee_i a_{i,\varphi(i)}$$

for every $\varphi \in N^N$. Hence by Theorem 1 and Proposition 7 we have

$$J(x_n) = J(x_1) + \sum_{i=1}^{n} (J(x_i) - J(x_{i-1})) > \sum_{i=1}^{n} J^+(y_i) - \sum_{k=1}^{n} \bigvee_i a_{k,i,\varphi(i+k-1)} \geqq$$

$$\geqq J^+\Big(\sum_{i=1}^{n} y_i\Big) - \sum_{k=1}^{n} \bigvee_i a_{k,i,\varphi(i+k-1)}.$$

If we put $u_n = a \wedge \sum_{i=1}^{n} y_i$, then

$$J^+(u_n) - J(x_n) \leqq J_0(a) - J(x_1),$$

hence

$$J^+(u_n) - J(x_n) \leqq (J_0(a) - J(x_1)) \wedge \Big(\sum_n \bigvee_i a_{n,i,\varphi(i+n-1)}\Big) \leqq$$

$$\leqq \bigvee_i a_{i,\varphi(i)}.$$

Put $u = \bigvee_{n=1}^{\infty} u_n$. Then $u \in A^+$, $u \geqq x$,

$$J^+(u) - \bigvee_i a_{i,\varphi(i)} \leqq \bigvee_n J(x_n)$$

by Proposition 5.

Since $x_n \in L$, there are $b_{n,i,j} \searrow 0 (j \to \infty)$ such that for every $\varphi \in N^N$ there are $v_n \in A^-$ with $v_n \leqq x_n$ and

$$J^-(v_n) + \bigvee_i b_{n,i,\varphi(n+i-1)} > J(x_n).$$

Put $b_{0,i,j}=\bigvee_n J(x_n)-J(x_j)$. Finally fix $j\geqq\varphi(1)$ (hence $b_{0,i,j}\leqq b_{o,i,\varphi(1)}$) and put $v=v_j$. By the condition (P) there is $b_{i,j}\searrow 0(j\to\infty)$ such that

$$(J_0(a)-J^-(v))\wedge\Big(\sum_n\bigvee_i b_{n,i,\varphi(i+n-1)}\Big)<\bigvee_i b_{i,\varphi(i)}$$

for every $\varphi\in N^N$. Since

$$\bigvee_{n=1}^{\infty} J(x_n)=J(x_j)+b_{0,i,j}$$

$$\leqq J^-(v)+\bigvee_i b_{k,i,\varphi(j+i-1)}+\bigvee_i b_{0,i,\varphi(i)},$$

and

$$\bigvee_{n=1}^{\infty} J(x_n)-J^-(v)\leqq J_0(a)-J^-(v),$$

we have

$$\bigvee_n J(x_n)\leqq J^-(v)+\bigvee_i b_{i,\varphi(i)}.$$

Hence we have constructed two sequences $(a_{i,j})_{i,j}$, $(b_{i,j})_{i,j}$ of elements of G such that $a_{i,j}\searrow 0$, $b_{i,j}\searrow 0$ $(j\to\infty)$ and such that to every $\varphi\in N^N$ there are $u\in A^+$, $v\in A^-$ with $v\leqq x\leqq u$ and

$$J^+(u)-\bigvee_i a_{i,\varphi(i)}<w=\bigvee_{n=1}^{\infty} J(x_n)<J^-(v)+\bigvee_i b_{i,\varphi(i)}.$$

Therefore $x\in L$ and

$$J(x)=w=\bigvee_{n=1}^{\infty} J(x_n).$$

Corollary. *Let G satisfy the assumptions of Theorem 1. Let $x_n\in L$, $x_n\searrow x$, $(J(x_n))_{n=1}^{\infty}$ be bounded. Then $x_1-x\in L$ and*

$$\bigwedge_n J(x_n)=J(x_1)-J(x_1-x).$$

Proof. By (5) $x_1-x_n\nearrow x_1-x$. By Theorem 1 $J(x_1-x_n)=J(x_1)-J(x_n)$, hence $(J(x_1-x_n))_{n=1}^{\infty}$ is bounded and by Theorem 2 $x_1-x\in L$ and

$$J(x_1-x)=\bigvee_n J(x_1-x_n)=\bigvee_n (J(x_1)-J(x_n))=J(x_1)-\bigwedge_n J(x_n).$$

5. Measure theory

Definition 4. *By a measure with values in an ordered group G we mean a mapping $\mu\colon \mathscr{R}\to G$ defined on a ring $\mathscr{R}$ of subsets of a set and satisfying the following conditions:*

1. $\mu(\emptyset)=0$, $\mu(A)\geqq 0$ *for every* $A\in\mathcal{R}$.

2. μ *is* σ*-additive, i. e.*

$$\mu(A)=\sum_{n=1}^{\infty}\mu(A_n)=\bigvee_{n=1}^{\infty}\sum_{i=1}^{n}\mu(A_i)$$

whenever $A_n,\ A\in\mathcal{R}$ $(n=1,2,\ldots)$, $A=\bigcup_{n=1}^{\infty}A_n$ *and* $A_n\cap A_m=\emptyset$ $(n\neq m)$.

Evidently, a mapping $\mu\colon \mathcal{R}\to G$ is a measure iff $\mu(\emptyset)=0$, $\mu\geqq 0$, μ is additive and continuous from below.

Theorem 3. *Let G be a monotonously σ-complete, σ-distributive, commutative ordered group satisfying the condition (P). Let $\mu\colon \mathcal{R}\to G$ be a measure defined on an algebra $\mathcal{R}$ of subsets of a set. Then there is exactly one measure $\bar{\mu}\colon \sigma(\mathcal{R})\to G$ extending μ and defined on the σ-algebra $\sigma(\mathcal{R})$ generated by $\mathcal{R}$.*

Proof. Let X be the family of all subsets of a given set. Let X be ordered by the inclusion. For every $E,\ F\in X$ put $E+F=E\cup F$, $E-F=E\setminus F$ (the set-theoretical union or difference resp.). Evidently X satisfies all assumptions listed in Section 2. Further put $A=\mathcal{R}$ and $J_0=\mu$. Our general results (from Section 4) can be applied now to the triple X, A, J_0. Hence we can use first A^+, J^+, A^-, J^- and then L and J. By Theorem 2 L is a monotone family containing $\mathcal{R}$, hence $L\supset\sigma(\mathcal{R})$. Put $\bar{\mu}=J\,|\,\sigma(\mathcal{R})$. By Theorem 2, $\bar{\mu}$ is continuous from below. We shall prove that $\bar{\mu}$ is additive.

Let $E,\ F\in\mathcal{R}$, $E\cap F=\emptyset$. Then there are $a_{i,j}\searrow 0$, $b_{i,j}\searrow 0 (j\to\infty)$ such that for every $\varphi\in N^N$ there are $C\subset E$, $D\subset F$ such that $C\in A^-$, $D\in A$ and

$$J(E)-\bigvee_i a_{i,\varphi(i)}<J^-(C),\ J(F)-\bigvee_i b_{i,\varphi(i)}<J^-(D).$$

Evidently $C\cap D\subset E\cap F=\emptyset$, hence C, D are disjoint and therefore $J^-(C)+J^-(D)=J^-(C\cap D)+J^-(C\cup D)=J^-(C\cup D)$. Choose (by the property (P)) such $c_{i,j}\searrow 0 (j\to\infty)$ that for every $\varphi\in N^N$

$$\bigvee_i a_{i,\varphi(i)}+\bigvee_i b_{i,\varphi(i)}\leqq\bigvee_i c_{i,\varphi(i)}.$$

We have

$$\bar{\mu}(E)+\bar{\mu}(F)-\bar{\mu}(E\cup F)\leqq J^-(C\cup D)+\bigvee_i c_{i,\varphi(i)}-\bar{\mu}(E\cup F)\leqq$$

$$\leqq\bigvee_i c_{i,\varphi(i)},$$

hence by the σ-distributivity of G, $\bar{\mu}(E)+\bar{\mu}(F)-\bar{\mu}(E\cup F)\leqq 0$.

The opposite inequality can be obtained similarly using Proposition 7.

If $\nu\colon \sigma(\mathcal{R})\to G$ is another measure extending μ, then $\mathcal{K}=\{E\in\sigma(\mathcal{R})\colon \nu(E)=\bar{\mu}(E)\}$ is a monotone family containing $\mathcal{R}$, hence $\mathcal{K}\supset\sigma(\mathcal{R})$ and therefore ν coincides with μ on $\sigma(\mathcal{R})$.

6. The Daniell integral

Now we present a theorem on the extension of continuous linear operators as a consequence of our general theory. Our main result will be concerned with lattice ordered groups. Of course, the first proposition will be formulated more generally.

Proposition 9. *Let G satisfy the assumptions of Theorem 1 and X, A, J_0 satisfy the assumptions of Section 1. Moreover let $+$ be continuous (i. e. also $x_n \searrow x$, $y_n \searrow y$ implies $x_n + y_n \searrow x + y$) and J_0 be additive (i. e. $J_0(x+y) = J_0(x) + J_0(y)$ for all $x, y \in A$). Then the set L is closed under the operation $+$ and J is additive, too.*

Proof. Evidently A^+, A^- are closed under $+$ and J^+, J^- are additive. Let $x, y \in L$. Then there are $a_{i,j} \searrow 0$, $b_{i,j} \searrow 0$, $c_{i,j} \searrow 0$, $d_{i,j} \searrow 0 (j \to \infty)$ such that for every $\varphi \in N^N$ there are $x_1, y_1 \in A^-$, $x_2, y_2 \in A^+$ such that $x_1 \leqq x \leqq x_2$, $y_1 \leqq y \leqq y_2$ and

$$J^+(x_2) - \bigvee a_{i,\varphi(i)} \leqq J(x) \leqq J^-(x_1) + \bigvee b_{i,\varphi(i)},$$

$$J^+(y_2) - \bigvee c_{i,\varphi(i)} \leqq J(y) \leqq J^-(y_1) + \bigvee d_{i,\varphi(i)}.$$

If we choose (property (P))$e_{i,j} \searrow 0$, $f_{i,j} \searrow 0 (j \to \infty)$ such that $\bigvee a_{i,\varphi(i)} + \bigvee c_{i,\varphi(i)} \leqq \bigvee e_{i,\varphi(i)}$ and $\bigvee b_{i,\varphi(i)} + \bigvee d_{i,\varphi(i)} \leqq \bigvee f_{i,\varphi(i)}$ for every $\varphi \in N^N$, then

$$J^+(x_2 + y_2) - \bigvee e_{i,\varphi(i)} \leqq J(x) + J(y) \leqq J^-(x_1 + y_1) + \bigvee f_{i,\varphi(i)},$$

where $x_1 + y_1 \leqq x + y \leqq x_2 + y_2$ and $x_1 + y_1 \in A^-$, $x_2 + y_2 \in A^+$. Hence $x + y \in L$ and $J(x+y) = J(x) + J(y)$.

Theorem 4. *Let G be a commutative, ordered, σ-distributive and σ-complete group satisfying the condition (P). Let X be a σ-complete l-group, A its l-subgroup majorizing X (i. e. to every $x \in X$ there is $a \in A$ such that $x \leqq a$). Let $J_0 : A \to G$ be an additive, positive and order continuous mapping (i. e. $x_n \nearrow x \Rightarrow J_0(x_n) \nearrow J_0(x)$). Then there is a subgroup L of the group X and an additive, positive and order continuous mapping J: $L \to G$ extending J_0 and satisfying the following additional condition:*

If $x_n \nearrow x$, $x_n \in L$ $(n = 1, 2, \ldots)$ and $(J(x_n))_{n=1}^{\infty}$ is bounded, then $x \in L$.

If moreover G is complete, then L is a lattice group.

Proof. In Proposition 9 we proved that J is additive and L is closed under the operation $+$. Now it suffices to prove that $x \in L$ implies $-x \in L$. Let $a_{i,j} \searrow 0$, $b_{i,j} \searrow 0$ $(j \to \infty)$ be corresponding sequences, $x_1 \leqq x \leqq x_2$ be corresponding elements, $x_1 \in A^-$, $x_2 \in A^+$ and

$$J^+(x_2) - \bigvee a_{i,\varphi(i)} \leqq J(x) \leqq J^-(x_1) + \bigvee b_{i,\varphi(i)}.$$

Evidently $-x_2 \leqq -x \leqq -x_1$, $-x_2 \in A^-$, $-x_1 \in A^+$ and $J^+(-x_1) = -J^-(x_1)$, $J^-(-x_2) = -J^+(x_2)$. Therefore

$$J^+(-x_1) - \bigvee b_{i,\varphi(i)} \leqq -J(x) \leqq J^-(-x_2) + \bigvee a_{i,\varphi(i)}.$$

It follows that $-x \in L$ and moreover $J(-x) = -J(x)$. (Of course, the equality $J(-x) = -J(x)$ is a consequence of additivity of J.)

If G is complete, we can define for every $x \in X$

$$J^*(x) = \bigwedge \{J^+(b) : b \in A^+, b \geqq x\},$$

$$J_*(x) = \bigvee \{J^-(c) : c \in A^-, c \leqq x\}.$$

Evidently $J^*(x) \geqq J_*(x)$ for every $x \in X$ and $x \in L$ iff $J^*(x) = J_*(x)$. Now let $x, y \in L$, i. e. $J^*(x) = J_*(x)$, $J^*(y) = J_*(y)$. Since J_0 is additive and $x + y = x \vee y + x \wedge y$, we have

$$J^*(x) + J^*(y) \geqq J^*(x \vee y) + J^*(x \wedge y),$$

$$J_*(x) + J_*(y) \leqq J_*(x \vee y) + J_*(x \wedge y).$$

Therefore

$$J^*(x \vee y) \leqq J^*(x) + J^*(y) - J^*(x \wedge y) \leqq$$

$$\leqq J_*(x) + J_*(y) - J_*(x \wedge y) \leqq J_*(x \vee y),$$

hence $x \vee y \in L$. On the other hand $x \wedge y = -((-x) \vee (-y))$.

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Received January 27, 1981

Katedra teórie pravdepodobnosti
a matematickej štatistiky
Matematicko-fyzikálnej fakulty UK
Mlynská dolina
842 15 Bratislava

О МЕРАХ И ИНТЕГРАЛАХ СО ЗНАЧЕНИЯМИ В УПОРЯДОЧЕННЫХ ГРУППАХ

Beloslav Riečan

Резюме

В статье формулированы и доказаны для результата для отображений со значениями в частично упорядоченных группах: теорема о продолжении меры и теорема о продолжении интеграла Даниеля. Приведены некоторые улучшения известных результатов. Во первых, допустимым множеством значений является группа, а не только линейное пространство. Во вторых, это множество не должно быть структурой. В третьих, обе теории рассматриваются с единой точки зрения. Таким образом теорема о продолжении меры и теорема о продолжении интеграла Даниеля являются следствиями одной общей теоремы о продолжении отображений определенных на подструктуре данной структур некоторого типа. И, наконец, в четвертых, приводятся более слабые условия наложенные на область определения продолжаемого отображения. Таким образом получается теорема о продолжении меры на Булевых алгебрах, и с другой стороны, теория интеграла Даниела на структурно упорядоченных группах.