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UNPATH NUMBER OF A COMPLETE MULTIPARTITE GRAPH

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In [1] J. Akiyama, G. Exoo and F. Harary suggested the problem to determine the unpath number of a complete multipartite graph. We shall solve this problem.

The unpath number $Y(G)$ of an undirected graph G is the maximum number of edge-disjoint connected graphs into which G can be decomposed and none of which is a path. (The authors of [1] call them non-paths.) This concept was introduced in [2].

A complete multipartite graph is a graph G with the property that there exists a partition $\mathcal{P}$ of the vertex set $V(G)$ of G such that two vertices of G are adjacent if and only if they belong to distinct classes of $\mathcal{P}$. We call $\mathcal{P}$ the defining partition of G. If the number of classes of $\mathcal{P}$ is n, this graph is also called complete n-partite. If $\mathcal{P}=\{M_1, \dots, M_n\}$ and $|M_i|=m_i$ for $i=1, \dots, n$, the graph G thus defined will be denoted by $K(m_1, \dots, m_n)$.

Theorem. *If G is a finite complete multipartite graph, then*

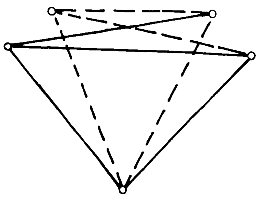
$$Y(G)=[\tfrac{1}{3}|E(G)|], \tag{1}$$

where $E(G)$ is the edge set of G.

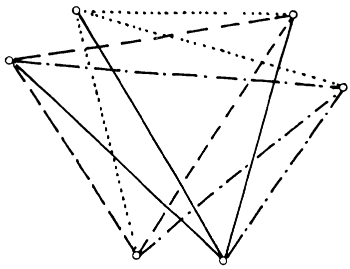
Proof. Any connected graph which is not a path has at least three edges. This implies that $Y(G)\leqq[\frac{1}{3}|E(G)|]$ for an arbitrary graph G. Hence it remains to construct the decomposition of G into $[\frac{1}{3}|E(G)|]$ edge-disjoint connected non-paths. In [1] the equality (1) was proved for complete bipartite graphs (i.e. n-partite graphs for $n=2$). We start by proving it for $K(1, 1, 1)$, $K(1, 1, 2)$, $K(1, 2, 2)$ and $K(2, 2, 2)$. For $K(1, 1, 1)$ and $K(1, 1, 2)$ this is trivial. For $G\cong K(1, 2, 2)$ we have $[\frac{1}{3}|E(G)|]=2$, for $G\cong K(2, 2, 2)$ we have $[\frac{1}{3}|E(G)|]=4$. The required decompositions are seen in Fig. 1.

Now consider a graph $G\cong K(m_1, \dots, m_n)$ such that $n\geqq3$ and each of the numbers $m_1, \dots, m_n$ is equal to 1 or 2. By induction according to n we shall prove that this graph can be decomposed into edge-disjoint connected non-paths whose number is equal to $[\frac{1}{3}|E(G)]$. Moreover, we prove that at most two of these nonpaths have more than three edges and each of those exceptional non-paths is isomorphic to some of the graphs in Fig. 2.

We have proved this for $n = 3$. Let $n_0 \geqq 4$ and suppose that the assertion holds for $n = n_0 - 1$. Let $n = n_0$; we have a graph $G \cong K(m_1, \ldots, m_n)$, where $n = n_0$ and each m_i is equal to 1 or to 2. Let G' be the graph obtained from G by deleting the set M_n; then $G' \cong K(m_1, \ldots, m_{n-1})$. Let $V(G')$, $E(G')$ be the vertex set and the edge



K(1,2,2)



K(2,2,2)

Fig 1

set of G' respectively. Let a (or b) be the rest in dividing $|V(G')|$ (or $|E(G')|$ respectively) by 3. Con ider the required decomposition of G'; it exists according to the induction hypothesis. First let $m_n = 1$; denote the element of M_n by x. The degree of x in G is equal to $|V(G')|$. If $a = 0$, this degree is divisible by 3. We choose a partition of the set of edges incident with x into three-element classes and form the corresponding stars with the centre x. The required decomposition of G consists of the decomposition of G' and of these stars. If $b = 0$, then $[\frac{1}{3}|E(G')|] = \frac{1}{3}|E(G')|$ and each non-path of the decomposition of G' has three edges. We construct the stars as in the preceding case; a of them will have four edges, the remaining ones will have three edges each. Thus there remains to be considered the case when both a and b are non-zero. If $b = 1$, then one of the non-paths of the decomposition of G' has four edges, the remaining ones have three edges each. If $a = 1$, we proceed analogously to the preceding case. If $a = 2$, we take the exceptional non-path H in G' and choose an edge e in it with the property that after deleting e from H a graph is obtained which consists of a non-path H' and eventually of an isolated vertex (as H must be isomorphic to one of the graphs in Fig. 2, such an edge e exists). We take all non-paths of the decomposition of G' except H, further we take H', the triangle induced by the end vertices of e and the vertex x and the three-edge stars with the centre x and with edges not belonging to this triangle; thus the required decomposition of G is finished. If $b = 2$, then the decomposition of G' contains either one non-path H with five edges, or two non-paths H_1', H_2 with four edges each; the remaining non-paths have three edges each. In the first case we choose two edges e_1, e_2 of H such that after deleting then from H a non-path H' and eventually an isolated vertex occurs. In the second case we choose analogously an edge e_1 in H_1 and an edge e_2 in H_2 and define analogously H_1' and H_2'. If e_1 and e_2 have a common end vertex z, we construct

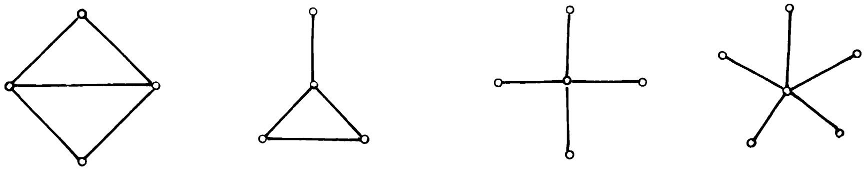


Fig. 2

a star with the centre z and with the edges e_1, e_2, xz. We take all non-paths of the decomposition of G' except H or except H_1, H_2; further we take H' or H_1', H_2', the mentioned star and the stars with the centre x constructed analogously to the preceding case (if $a=2$, then one of them has four edges and the remaining ones have three edges each; if $a=1$, then each of them has three edges). If e_1, e_2 have no common end vertex, then with both of them we do the same as with e in the case $a=2$, $b=1$. Now consider $m_n=2$; let $M_n=\{x, y\}$. If $a=0$ or $b=0$, we proceed analogously to the case $m_n=1$. Let $b=1$ and $a\neq 0$. We take an edge e of G' analogously to the case $m_n=1$, choose an end vertex z of e and form a star with the centre z and with the edges e, xz, yz. If $b=2$ and $a\neq 0$, we take analogously the edges e_1, e_2. If $a=2$, we form two triangles, one induced by the end vertices of e_1 and the vertex x, the other induced by the end vertices of e_2 and the vertex y. If $a=1$, we form only one of them. The rest of the procedure is analogous to that in the case $m_n=1$. Thus the assertion is proved for all graphs $K(m_1, \ldots, m_n)$, where each of the numbers $m_1, \ldots, m_n$ is equal to 1 or to 2.

Now consider a complete multipartite graph $G\cong K(m_1, \ldots, m_n)$ for arbitrary values of m_i. For each $i=1, \ldots, m$ we choose a subset M_i' of M_i whose cardinality is equal to the rest in dividing m_i by 3. If i, j are two distinct numbers from the numbers $1, \ldots, m$, by G_{ij} we denote the subgraph of G induced by the set $M_i\cup M_j$; this is a complete bipartite graph. Choose a partition of M_j-M_j' into three-element classes and construct all stars with a centre in M_i and with the set of terminal vertices equal to a class of this partition. Further choose a partition of M_i-M_i' into three-element classes and construct all stars with a centre in M_j' and with the set of terminal vertices equal to a class of this partition. If we do this in each G_{ij}, then either the required decomposition of G is done, or all edges of G not belonging to these stars induce a subgraph G' of G which is a complete multipartite graph in which the classes of the defining partition are exactly all sets M_i' which are non-empty. Each of these classes has at most two vertices, therefore we may decompose G' as described above and the required decomposition of G is finished.

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АНТИЦЕПНОЕ ЧИСЛО ПОЛНОГО МНОГОДОЛЬНОГО ГРАФА

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Резюме

Антицепное число $Y(G)$ графа G есть максимальное число реберно-непересекающихся связных графов, в которые граф G можно разложить, причем никакой из них не является цепью. Если G есть конечный полный многодольный граф, то $Y(G)=[\frac{1}{3}|E(G))]$, где $E(G)$ есть множество вершин графа G. Это является решением проблемы, которую задали Дж. Акияма, Дж. Эксу и Ф. Харари.