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LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS AND WEIGHTED SOBOLEV SPACES: A MODIFIED APPROACH

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Dedicated to Academician Štefan Schwarz on the occasion of his 70th birthday

0. Introduction

0.1. The aim of this paper is to describe a method which makes it possible to solve elliptic boundary value problems in weighted Sobolev spaces. Let us illustrate our approach on the Dirichlet problem

$$-\Delta u + u = f \quad \text{on} \quad \Omega, \tag{0.1}$$

$$u|_{\partial\Omega} = g, \tag{0.2}$$

where Ω is a domain in $\mathbf{R}^N$ with boundary $\partial\Omega$.

0.2. A function $u \in W^{1,2}(\Omega)$ is called a *weak solution* of the problem (0.1), (0.2) if

$$u - \tilde{g} \in W_0^{1,2}(\Omega) \tag{0.3}$$

and if the identity

$$a(u, v) = \langle f, v \rangle \tag{0.4}$$

holds for every $v \in C_0^\infty(\Omega)$. Here $\tilde{g}$ is a function from $W^{1,2}(\Omega)$ such that $\tilde{g}|_{\partial\Omega} = g$, $a(u, v)$ is the bilinear form

$$a(u, v) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx + \int_\Omega uv \, dx \tag{0.5}$$

and f is a functional from the dual space $(W_0^{1,2}(\Omega))^*$ (the most usual case is $\langle f, v \rangle = \int_\Omega fv \, dx$ with $f \in L^2(\Omega)$).

Thus the concept of a weak solution in the Sobolev space $W^{1,2}(\Omega)$ is meaningful if the following conditions are satisfied:

$$g \in W^{1/2,2}(\partial\Omega), \quad f \in (W_0^{1,2}(\Omega))^*. \tag{0.6}$$

The existence of a (uniquely determined) weak solution can be proved by the Lax—Milgram Lemma, since the form $a(u, v)$ is *bounded:*

$$|a(u, v)| \leqq c_1 \|u\| \, \|v\| \quad \text{for} \quad \text{every} \quad u, v \in W_0^{1,2}(\Omega), \tag{0.7}$$

and $W_0^{1,2}(\Omega)$-*elliptic:*

$$a(u, u) \geqq c_2 \|u\|^2 \quad \text{for} \quad \text{every} \quad u \in W_0^{1,2}(\Omega). \tag{0.8}$$

0.3. We can ask whether the problem (0.1), (0.2) is solvable in a *weighted Sobolev space* $W^{1,2}(\Omega; h)$. Besides the natural effort to extend the theory of weak solutions to weighted spaces, the motivation of that question may be found in the fact that the given f, g need not satisfy the conditions (0.6); e.g., the function g in the boundary condition (0.2) can be so "misbehaving" that there is no function $\tilde{g} \in W^{1,2}(\Omega)$ such that $\tilde{g}|_{\partial\Omega} = g$, and the $W^{1,2}(\Omega)$-theory cannot be applied. Then we can try to seek a weight function h so that these difficulties might be avoided if we replace the space $W^{1,2}(\Omega)$ by the weighted Sobolev space $W^{1,2}(\Omega; h)$.

0.4. One way how to proceed is to introduce *formally* the weight h into the integral identity (0.4), i.e. in the bilinear form $a(u, v)$:

$$a(u, v) = \sum_{i=1}^{N} \int_\Omega \frac{\partial u}{\partial x_i} h^{1/2} \frac{\partial v}{\partial x_i} h^{-1/2} \, dx + \int_\Omega u h^{1/2} v h^{-1/2} \, dx.$$

Then this bilinear form can be considered on the cartesian product $W_0^{1,2}(\Omega; h) \times W_0^{1,2}(\Omega; h^{-1})$. The existence of the weak solution in $W^{1,2}(\Omega; h)$ — i.e. of a function $u \in W^{1,2}(\Omega; h)$ such that $u - \tilde{g} \in W_0^{1,2}(\Omega; h)$ with $\tilde{g} \in W^{1,2}(\Omega; h)$ and that the identity (0.4) holds for every $v \in C_0^\infty(\Omega)$ with $f \in (W_0^{1,2}(\Omega; h^{-1}))^*$ — can then be proved (only for certain weights h, of course!) by using a *generalized version* of the Lax—Milgram Lemma and starting with the "ordinary" (= non-weighted) boundedness (0.7) and ellipticity (0.8). This approach is described in detail in [3], Chapter 6, and [1], Section 13.

0.5. Our aim is to describe another method. We change a little the definition of the weak solution to the problem (0.1), (0.2) in order to facilitate the use of the "classical" Lax—Milgram Lemma, to simplify the calculations and to obtain generally a larger scale of admissible weights:

The function $u \in W^{1,2}(\Omega; h)$ will be called an *h-weak solution* of the Dirichlet problem (0.1), (0.2) if $u - \tilde{g} \in W_0^{1,2}(\Omega; h)$ (with $\tilde{g} \in W^{1,2}(\Omega; h)$ such that $\tilde{g}|_{\partial\Omega} = g$) and if the integral identity

$$\sum_{i=1}^{N} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial (vh)}{\partial x_i} \, dx + \int_\Omega uvh \, dx - \int_\Omega fvh \, dx \tag{0.9}$$

holds for every $v \in C_0^\infty(\Omega)$. The left hand side of (0.9) is a bilinear form $b(u, v)$ which is connected with the form $a(u, v)$ by the formula

$$b(u, v) = a(u, vh). \tag{0.10}$$

Hence we see that the difference between the weak solution from Section 0.2 and the h-weak solution is in the fact that in the latter case we work with test functions of the form vh with $v \in C_0^\infty(\Omega)$. Since the form $a(u, v)$ can be obtained from (0.1) by multiplying this equation by $v \in C_0^\infty(\Omega)$, integrating over Ω and using Green's formula, we easily see that the identity (0.9) corresponds to the equation

$$-h\Delta u + hu = hf \quad \text{on} \quad \Omega,$$

which is equivalent with (0.1) almost everywhere in Ω since the weight function h is supposed to be positive a.e. in Ω.

1. Notation and assumptions

1.1. The domain Ω. We shall suppose troughout this paper that Ω is a domain in $\mathbf{R}^N$ with locally lipschitzian boundary $\partial\Omega$. For a precise description see e.g. [3], [2].

1.2. The weighted Sobolev spaces. (i) A function $h = h(x)$ defined on Ω is called a *weight* if it is measurable and positive a.e. on Ω. We shall work with weights h which fulfil the condition

$$h \in L^1_{\text{loc}}(\Omega), \quad h^{-1} \in L^1_{\text{loc}}(\Omega). \tag{1.1}$$

Later we shall deal with special weights of the type

$$h(x) = d_M^\varepsilon(x) \quad \text{or} \quad h(x) = \exp(\varepsilon d_M(x)), \quad \varepsilon \in \mathbf{R}, \tag{1.2}$$

where $d_M(x) = \text{dist}(x, M)$ and $M \subset \partial\Omega$ is an m-dimensional manifold, $0 \leqq m \leqq N-1$. The weights (1.2) obviously fulfil the conditions (1.1).

(ii) Let h be a weight. We denote by $L^2(\Omega; h)$ the set of functions u measurable on Ω with a finite norm

$$\|u\|_h = \left(\int_\Omega |u(x)|^2 h(x)\, \mathrm{d}x\right)^{1/2}. \tag{1.3}$$

Let us denote

$$\mathrm{D}_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, N,$$

and let h_0, h_1 be weights. We denote by

$$W^{1,2}(\Omega; h_0, h_1)$$

the set of functions $u \in L^2(\Omega; h_0)$ such that $\mathrm{D}_i u \in L^2(\Omega; h_1)$, $i = 1, \ldots, N$, equipped with the norm

$$\|u\|_{1; h_0, h_1} = \left(\|u\|^2_{h_0} + \sum_{i=1}^N \|\mathrm{D}_i u\|^2_{h_1}\right)^{1/2}. \tag{1.4}$$

Further, we denote by

$$W_0^{1,2}(\Omega;\, h_0,\, h_1)$$

the closure of the set $C_0^\infty(\Omega)$ with respect to the norm (1.4).

[The first condition in (1.1) guarantees that $C_0^\infty(\Omega) \subset W^{1,2}(\Omega;\, h_0,\, h_1)$; the second implies that the spaces $W^{1,2}(\Omega;\, h_0,\, h_1)$ and $W_0^{1,2}(\Omega;\, h_0,\, h_1)$ are complete Hilbert spaces.]

If $h_0 = h_1 = h$, we shall write $W^{1,2}(\Omega;\, h)$ and $W_0^{1,2}(\Omega;\, h)$ instead of $W^{1,2}(\Omega;\, h,\, h)$ and $W_0^{1,2}(\Omega;\, h,\, h)$, respectively. The norm (1.4) will then be denoted by $\|u\|_{1;h}$.

For $h_0 = h_1 \equiv 1$ we obtain the *classical Sobolev spaces* $W^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$. The norm in these spaces will be denoted by $\|u\|_1$.

(iii) We shall say that a weight h *satisfies condition* $\mathbf{P}_1$ if there exists a weight h_0 and constants $\eta_1 = \eta_1(h, h_0) > 0$, $\eta_2 = \eta_2(h, h_0) \geqq 0$ so that

$$\|u\|_{h_0} \leqq \eta_1 \left(\sum_{i=1}^{N} \|D_i u\|_h^2\right)^{1/2} \quad \text{for every} \quad u \in W_0^{1,2}(\Omega;\, h) \tag{1.5}$$

and

$$|\nabla h(x)|^2 h^{-1}(x) \leqq \eta_2^2 h_0(x) \quad \text{for a.e.} \quad x \in \Omega. \tag{1.6}$$

We shall say that a weight h *satisfies condition* $\mathbf{P}_2$ if there exists a constant $\eta_3 = \eta_3(h) \geqq 0$ such that

$$|\nabla h(x)| \leqq \eta_3 h(x) \quad \text{for a.e.} \quad x \in \Omega. \tag{1.7}$$

1.3. Remarks. (i) The inequality (1.7) is a special case of the inequality (1.6) with $h_0 = h$, $\eta_2 = \eta_3$.

(ii) For $h = d_M^\varepsilon$ and $h_0 = d_M^{\varepsilon-2}$, $\varepsilon \in \mathbf{R}$, the condition (1.6) is satisfied with $\eta_2 = |\varepsilon|$. It follows from the imbedding theorems for weighted Sobolev spaces (see [1], [4]) that the estimate (1.5) holds with

$$\eta_1 = \frac{2c_1}{|\varepsilon - 1|} \quad \text{for} \quad \varepsilon \neq 1 \tag{1.8}$$

or

$$\eta_1 = \frac{2c_2}{|\varepsilon + N - m - 2|} \quad \text{for} \quad \varepsilon \neq m + 2 - N, \tag{1.9}$$

where $c_i = c_i(\Omega, M)$, $i = 1, 2$, are positive constants. Consequently the weight $h = d_M^\varepsilon$ fulfils condition $\mathbf{P}_1$.

(iii) The weight $h(x) = \exp(\varepsilon d_M(x))$ satisfies condition $\mathbf{P}_2$ with the constant $\eta_3 = |\varepsilon|$. Weights of this type are suitable for *unbounded* domains Ω.

1.4. Differential operators. For the sake of simplicity we shall deal with differential operators of the *second order*. The extension of all the results to operators of order $2k$, $k>1$, is straightforward (except only for some difficulties of technical character).

Let

$$(Lu)(x) \equiv \sum_{i,j=0}^{N} (-1)^i \, \mathrm{D}_i(a_{ij}(x) \, \mathrm{D}_j u(x)) \quad \text{with} \quad a_{ij} \in L^\infty(\Omega), \tag{1.10}$$

$i, j = 0, 1, \ldots, N$, and with $\mathrm{D}_0 u = u$. Let $a(u, v)$ be the corresponding bilinear form

$$a(u, v) = \sum_{i,j=0}^{N} \int_\Omega a_{ij} \, \mathrm{D}_j u \, \mathrm{D}_i v \, \mathrm{d}x. \tag{1.11}$$

We shall suppose that the operator L is *elliptic in* $W_0^{1,2}(\Omega)$, which means that there exists a constant $\lambda > 0$ such that

$$a(u, u) \geqq \lambda \|u\|_1^2 \quad \text{for every} \quad u \in W_0^{1,2}(\Omega). \tag{1.12}$$

1.5. Remark. A sufficient condition for (1.12) is the *algebraic condition*

$$\sum_{i,j=0}^{N} a_{ij}(x) \xi_i \xi_j \geqq \lambda |\xi|^2 \tag{1.13}$$

$$\text{for a.e.} \quad x \in \Omega \quad \text{and for every} \quad \xi \in \mathbf{R}^{N+1}.$$

The condition (1.13) can be weakened in various ways.

1.6. The bilinear form b. Let h be a weight on Ω satisfying the condition $\mathbf{P}_1$ or $\mathbf{P}_2$. We associate with the operator L from (1.10) a bilinear form $b(u, v)$ defined by the formula

$$b(u, v) = \sum_{i,j=0}^{N} \int_\Omega a_{ij}(x) \, \mathrm{D}_j u(x) \, \mathrm{D}_i[v(x) h(x)] \, \mathrm{d}x. \tag{1.14}$$

Obviously,

$$b(u, v) = a(u, vh)$$

with $a(u, v)$ from (1.11). We shall show that the form $b(u, v)$ is defined (and, moreover, bounded) on $W^{1,2}(\Omega; h) \times W^{1,2}(\Omega; h)$:

Let us denote

$$b_1(u, v) = \sum_{i,j=0}^{N} \int_\Omega a_{ij} \, \mathrm{D}_j u \, \mathrm{D}_i v \, h \, \mathrm{d}x,$$

$$b_2(u, v) = \sum_{i=1}^{N} \sum_{j=0}^{N} \int_\Omega a_{ij} \, \mathrm{D}_j u \, v \, \mathrm{D}_i h \, \mathrm{d}x.$$

For $u, v \in W^{1,2}(\Omega; h)$ we have by Hölder's inequality

$$|b_1(u, v)| \leqq \sum_{i,j=0}^{N} \int_\Omega |a_{ij}| \, |\mathrm{D}_j u| h^{1/2} |\mathrm{D}_i v| h^{1/2} \, \mathrm{d}x \leqq$$

$$\leqq \sum_{i,j=0}^{N} \|a_{ij}\|_\infty \|D_j u\|_h \|D_i v\|_h \leqq c_3 \|u\|_{1;h} \|v\|_{1;h}.$$

If h satisfies condition $\mathbf{P}_1$, we have similarly

$$|b_2(u, v)| \leqq \sum_{i=1}^{N} \sum_{j=0}^{N} \int_\Omega |a_{ij}|\,|D_j u| h^{1/2} |v|\,|D_i h| h^{-1/2}\,\mathrm{d}x \leqq$$

$$\leqq \sum_{i=1}^{N} \sum_{j=0}^{N} \|a_{ij}\|_\infty \|D_j u\|_h \left(\int_\Omega |v|^2 |D_i h|^2 h^{-1}\,\mathrm{d}x\right)^{1/2} \leqq$$

$$\leqq \mu \|u\|_{1;h} \eta_2 \|v\|_{h_0} \leqq \mu \eta_2 \eta_1 \|u\|_{1;h} \|v\|_{1;h};$$

if h satisfies condition $\mathbf{P}_2$, then simply

$$|b_2(u, v)| \leqq \sum_{i=1}^{N} \sum_{j=0}^{N} \int_\Omega |a_{ij}\, D_j u v|\,|D_i h|\,\mathrm{d}x \leqq \mu \eta_3 \|u\|_{1;h} \|v\|_{1;h}.$$

Consequently, we have

$$|b_2(u, v)| \leqq \mu \eta \|u\|_{1;h} \|v\|_{1;h} \tag{1.15}$$

with

$$\eta = \begin{cases} \eta_1 \eta_2 & \text{if } h \text{ satisfies } \mathbf{P}_1, \\ \eta_3 & \text{if } h \text{ satisfies } \mathbf{P}_2. \end{cases} \tag{1.16}$$

Since

$$b(u, v) = b_1(u, v) + b_2(u, v), \tag{1.17}$$

we have

$$|b(u, v)| \leqq (c_3 + \mu\eta) \|u\|_{1;h} \|v\|_{1;h}, \tag{1.18}$$

i.e. the form b is *bounded* on $W^{1,2}(\Omega; h) \times W^{1,2}(\Omega; h)$.

2. The Dirichlet boundary value problem

2.1. Definition. Let h be a weight function on Ω satisfying condition $\mathbf{P}_1$ or $\mathbf{P}_2$. Let L be the differential operator from (1.10). Let $f \in (W_0^{1,2}(\Omega; h))^*$ and $g \in W^{1,2}(\Omega; h)$.

We shall say that a function $u \in W^{1,2}(\Omega; h)$ is an *h-weak solution of the Dirichlet problem* (L, f, g), if

$$u - g \in W_0^{1,2}(\Omega; h),$$

$$b(u, v) = \langle f, v\rangle \quad \text{for every} \quad v \in C_0^\infty(\Omega). \tag{2.1}$$

2.2. Some estimates. Our aim is to prove the *existence* of an h-weak solution by means of the Lax—Milgram Lemma. Since in Section 1.6 we have proved *boundedness* of the form $b(u, v)$, we need the *$W_0^{1,2}(\Omega; h)$-ellipticity* of this form, i.e. an estimate

$$b(u, u) \geqq c\|u\|_{1;h}^2 \quad \text{for every} \quad u \in W_0^{1,2}(\Omega; h).$$

Let $u \in W_0^{1,2}(\Omega; h)$ and let us write

$$b(u, u) = a(uh^{1/2}, uh^{1/2}) + R(u, h). \tag{2.2}$$

The ellipticity of L — see (1.12) — implies

$$a(uh^{1/2}, uh^{1/2}) \geqq \lambda \|uh^{1/2}\|_1^2.$$

Analogously as in deriving the estimate (1.15) we obtain

$$\|uh^{1/2}\|_1^2 = \int_\Omega |u|^2 h \, \mathrm{d}x + \sum_{i=1}^N \int_\Omega |\mathrm{D}_i(uh^{1/2})|^2 \, \mathrm{d}x \geqq$$

$$\geqq \|u\|_h^2 + \sum_{i=1}^N \int_\Omega |\mathrm{D}_i u|^2 h \, \mathrm{d}x - \sum_{i=1}^N \int_\Omega |\mathrm{D}_i u| \, |u| \, |\mathrm{D}_i h| \, \mathrm{d}x -$$

$$-\frac{1}{4} \sum_{i=1}^N \int_\Omega |u|^2 |\mathrm{D}_i h|^2 h^{-1} \, \mathrm{d}x \geqq \|u\|_{1;h}^2 \left(1 - \eta - \frac{\eta^2}{4}\right)$$

and consequently, we have

$$a(uh^{1/2}, uh^{1/2}) \geqq \lambda \left(1 - \eta - \frac{\eta^2}{4}\right) \|u\|_{1;h}^2. \tag{2.3}$$

Further, by an analogous argument,

$$-R(u, h) = a(uh^{1/2}, uh^{1/2}) - b(u, u) =$$

$$= \sum_{i,j=0}^N \int_\Omega a_{ij} \, [\mathrm{D}_j(uh^{1/2}) \, \mathrm{D}_i(uh^{1/2}) - \mathrm{D}_j u \, \mathrm{D}_i(uh)] \, \mathrm{d}x =$$

$$= \frac{1}{2} \sum_{i=0}^N \sum_{j=1}^N \int_\Omega a_{ij} \, \mathrm{D}_i u \, \mathrm{D}_j h \, \mathrm{d}x - \frac{1}{2} \sum_{i=1}^N \sum_{j=0}^N \int_\Omega a_{ij} \, \mathrm{D}_i h \, \mathrm{D}_j u \, \mathrm{d}x +$$

$$+ \frac{1}{4} \sum_{i,j=1}^N \int_\Omega a_{ij} u^2 \, \mathrm{D}_i h \, \mathrm{D}_j h h^{-1} \, \mathrm{d}x \leqq \left(\eta + \frac{\eta^2}{4}\right) \mu \|u\|_{1;h}^2.$$

Hence, (2.2) in view of (2.3) yields the estimate

$$b(u, u) \geqq \left[\lambda\left(1 - \eta - \frac{\eta^2}{4}\right) - \mu \left(\eta + \frac{\eta^2}{4}\right)\right] \|u\|_{1;h}^2 \tag{2.4}$$

and the multiplicative constant in square brackets (= the *ellipticity constant* for the bilinear form b) is positive if only if

$$\eta < 2 \left(\sqrt{\frac{2 + \varkappa}{1 + \varkappa}} - 1\right), \tag{2.5}$$

where $\varkappa = \frac{\lambda}{\mu}$.

2.3. Remarks. (i) If we use the algebraic ellipticity condition (1.13), we obtain

$$b_1(u, u) \geqq \lambda \|u\|^2_{1;h}.$$

Together with the estimate (1.15) we derive from

$$b(u, u) \geqq b_1(u, u) - |b_2(u, u)|$$

the estimate

$$b(u, u) \geqq (\lambda - \mu\eta)\|u\|^2_{1;h}. \tag{2.6}$$

Consequently, for operators L which fulfil the (more restrictive) condition (1.13), we obtain that the bilinear form $b(u, v)$ is $W_0^{1,2}$ $\Omega; h)$-elliptic.

$$\eta < \varkappa = \frac{\lambda}{\mu}. \tag{2.7}$$

This estimate for η is better than the estimate (2.5); this fact shows that the stronger ellipticity condition (1.13) enables us to deal with a generally larger scale of weights.

(ii) From the previous considerations it is clear that the constant μ plays an important role. Let us note that all the estimates can be obtained for

$$\mu = \left(\sum_{i,j=0}^{N} \|a_{ij}\|^2_\infty \right)^{1/2},$$

but usually it is possible to choose μ in a better way, by using special properties of the particular operator L. See the following example.

2.4. Example. For the operator $Lu = -\Delta u + u$ from (0.1) we can take $\lambda = \mu = \varkappa = 1$. Obviously, the choice $\mu = 1$ is essentially better than the choice

$$\mu = \left(\sum_{i,j=0}^{N} \|a_{ij}\|^2_\infty \right)^{1/2} = \sqrt{N+1}.$$

2.5. Theorem. *Let L be the differential operator from (1.10), which is elliptic in the sense of (1.12). Then there exists $\eta_0 > 0$ such that for every weight h which satisfies condition $\mathbf{P}_1$ or $\mathbf{P}_2$ with $\eta < \eta_0$, η given by (1.16), and for every $f \in (W_0^{1,2}(\Omega; h))^*$ and $g \in W^{1,2}(\Omega; h)$ there exists one and only one h-weak solution of the Dirichlet problem (L, f, g). The h-weak solution $u \in W^{1,2}(\Omega; h)$ satisfies the estimate*

$$\|u\|_{1;h} \leqq c(\|f\|_* + \|g\|_{1;h}) \tag{2.8}$$

where $c > 0$ is independent of f and g.

Proof: Let us put $w = u - g$. Then the identity (2.1) can be rewritten in the form

$$b(w, v) = \langle f, v \rangle + b(g, v). \tag{2.9}$$

It follows from (1.18) that the right hand side is the value $\langle F, v\rangle$ of a continuous linear functional F on $W_0^{1,2}(\Omega; h)$. The same formula implies that $b(w, v)$ is bounded on $W_0^{1,2}(\Omega; h)\times W_0^{1,2}(\Omega; h)$. The estimate (2.4) shows that $b(u, v)$ is $W_0^{1,2}(\Omega; h)$-elliptic for every η satisfying (2.5) (for η_0 we take the right hand side in (2.5)). Now, it follows from the Lax—Milgram Lemma that there exists a uniquely determined w such that (2.9) holds for every $v \in W_0^{1,2}(\Omega; h)$ (and, a *fortiori*, for every $v \in C_0^\infty(\Omega)$) and that $\|w\|_{1;h} \leqq c_4\|F\|_* \leqq c_4(\|f\|_* + (c_3 + \mu\eta)\|g\|_{1;h})$. The function $u = w + g$ is the h-weak solution and satisfies (2.8).

2.6. A weak solution in $W^{1,2}(\Omega; h_0, h)$. Let us suppose that the weight h satisfies condition $\mathbf{P}_1$. Then it follows from the estimate (1.5) that

$$\|u\|^2_{1;h_0,h} \leqq (1+\eta_1^2)\|u\|^2_{1;h} \quad \text{for every} \quad u \in C_0^\infty(\Omega),$$

which means that the space $W_0^{1,2}(\Omega; h_0, h)$ is larger and that the imbedding

$$W_0^{1,2}(\Omega; h) \subset W_0^{1,2}(\Omega; h_0, h)$$

holds. Therefore, it is meaningful to consider (h_0, h)-*weak solutions of the Dirichlet problem* (L, f, g). The definition of such a solution literally follows Definition 2.1, replacing the spaces $W^{1,2}(\Omega; h)$ and $W_0^{1,2}(\Omega; h)$ by the space $W^{1,2}(\Omega; h_0, h)$ and $W_0^{1,2}(\Omega; h_0, h)$, respectively.

If we suppose in addition that the weight h is such that the expression

$$|||u|||_{1;h} = \left(\sum_{i=1}^N \|D_i u\|_h^2\right)^{1/2}$$

is a norm on $W_0^{1,2}(\Omega; h)$ which is *equivalent to the norm* $\|u\|_{1;h}$, then an existence and uniqueness theorem analogous to Theorem 2.5 holds. The formulation and proof is left to the reader; let us only point out that the analogue of the estimate (2.6) will have the form

$$b(u, u) \geqq \left[\frac{\lambda}{1+\eta_1^2} - \mu\eta_2\right] \|u\|^2_{1;h_0,h}.$$

3. Power-type weights

3.1. Now we shall apply the results of Section 2 to the case of power-type weights $d_M^\varepsilon(x)$ introduced in Section 1.2 (i). For $h = d_M^\varepsilon$ we have $h_0 = d_M^{\varepsilon-2}$, and Remark 1.3 (ii) implies that the constant η from (1.16) has the form

$$\eta = 2|\varepsilon| \min\left(\frac{c_1}{|\varepsilon-1|}, \frac{c_2}{|\varepsilon+N-m-2|}\right) \tag{3.1}$$

(see (1.8) and (1.9)).

We shall write shortly $W^{1,2}(\Omega;\varepsilon)$ instead of $W^{1,2}(\Omega;d_M^\varepsilon)$ and $W^{1,2}(\Omega;\varepsilon,\varepsilon-2)$ instead of $W^{1,2}(\Omega;d_M^\varepsilon,d_M^{\varepsilon-2})$.

From Theorem 2.5 and Remark 2.3 (i) we conclude

3.2. Proposition. *Let L be the differential operator from (1.10) which is elliptic in the sense of (1.13). Then there exist numbers s_1, s_2, $s_1<0<s_2$, with the following property: For every $\varepsilon\in(s_1,s_2)$ and for every $f\in(W_0^{1,2}(\Omega;\varepsilon))^*$ and $g\in W^{1,2}(\Omega;\varepsilon)$, there exists one and only one d_M^ε-weak solution $u\in W^{1,2}(\Omega;\varepsilon)$ of the Dirichlet problem (L,f,g), which satisfies the estimate (2.7).*

Proof: Let us denote $\tau=\dfrac{c_2-c_1(N-m-2)}{c_2+c_1}$. Then for η given by (3.1) we have

$$\eta=\begin{cases}\dfrac{2c_1|\varepsilon|}{|\varepsilon-1|} & \text{for}\quad \varepsilon\leqq\tau,\\[2ex] \dfrac{2c_2|\varepsilon|}{|\varepsilon+N-m-2|} & \text{for}\quad \varepsilon>\tau.\end{cases}$$

By Theorem 2.5 and Remark 2.3 (i), the parameter η must satisfy the inequality (2.7). This means that we can put $(s_1,s_2)=I_1\cup I_2$, where

$$I_1=(\tau,+\infty)\cap\left(\frac{-\varkappa(N-m-2)}{2c_2+\varkappa},\frac{\varkappa(N-m-2)}{2c_2-\varkappa}\right),$$

$$I_2=(-\infty,\tau\rangle\cap\left(-\frac{\varkappa}{2c_1-\varkappa},\frac{\varkappa}{2c_1+\varkappa}\right).$$

We have $I_2\subset(-1,\frac{1}{3}\rangle$ and it can be easily verified that $0\in(s_1,s_2)$.

5.3. Remark. An analogous proposition can be formulated and proved for weak solutions in $W^{1,2}(\Omega;\varepsilon,\varepsilon-2)$ — see Section 2.6.

3.4. Example. Let $\Omega=(0,1)^N$, $0\leqq m\leqq N-1$, $M=\{x\in\bar\Omega, x_i=0 \text{ for } i=m+1, m+2,\dots,N\}$. Then $M\subset\partial\Omega$, $\dim M=m$ and

$$d_M^\varepsilon(x)=\left(\sum_{i=m+1}^{N}x_i^2\right)^{\varepsilon/2}.$$

Further, let us consider the operator $Lu=-\Delta u+u$; then L satisfies the condition (1.13) with $\lambda=1$ and we can take $\mu=1$ (see Example 2.4). Hence the inequality (2.7) has the form $\eta<1$. Let us show for which values of ε this condition is fulfilled.

We extend the function $u\in C_0^\infty(\Omega)$ by zero for $x_i\geqq1$, $i=m+1,\dots,N$. Using the generalized cylindrical coordinates $(x_1,\dots,x_m,\vartheta_1,\dots,\vartheta_{N-m-1},r)=(x',\vartheta,r)$, we have $d_M(x)=r$ and

$$\|u\|_{\varepsilon-2}^2=\int_{(0,1)^m}dx'\int_{(0,\pi/2)^{N-m-1}}\int_0^\infty|u(x',\vartheta,r)|^2r^{\varepsilon-2}r^{N-m-1}\,dr.$$

Applying the Hardy inequality (see e.g. [1]) to the inner integral under the assumption $\varepsilon \neq m+2-N$ and passing again to the cartesian coordinates we obtain

$$\|u\|^2_{\varepsilon-2} \leqq \frac{4}{|\varepsilon+N-m-2|^2} \int_\Omega \left|\frac{\partial u}{\partial r}\right|^2 d_M^\varepsilon \,\mathrm{d}x \leqq \frac{4}{|\varepsilon+N-m-2|^2} \|u\|^2_{1;\varepsilon}.$$

That means $c_2=1$ (cf. (3.1)).

Analogously, for $\varepsilon \neq -1$ we have the estimate

$$\|u\|^2_{\varepsilon-2} \leqq \frac{4}{|\varepsilon-1|^2} c_1^2 \|u\|^2_{1;\varepsilon},$$

where

$$c_1 = \begin{cases} 2^{(2-\varepsilon)/4}(N-m)^{-1/2} & \text{for } \varepsilon \leqq 0, \\ 2^{1/2}(N-m)^{-1/2} & \text{for } 0<\varepsilon\leqq 2, \\ 2^{\varepsilon/4}(N-m)^{-1/2} & \text{for } \varepsilon>2 \end{cases}$$

(see [1], [4]).

A more detailed discussion gives the following values for s_1, s_2 from Proposition 3.2 as well as the values of t_1, t_2 which define the corresponding interval (t_1, t_2) for the case of the space $W_0^{1,2}(\Omega;\varepsilon,\varepsilon-2)$ — see Remark 3.3:

Table 1

$N-m$	s_1	s_2	t_1	t_2
1	–0,48	0,26	–0,13	0,09
2	–0,78	0,33	–0,30	0,15
3	–1,04	1	–0,39	0,30
4	–1,30	2	–0,48	0,63
5	–1,54	3	–0,56	0,78

3.5. Remark. The intervals (s_1, s_2), (t_1, t_2) defined by values of Table 1 give, naturally, only the *sufficient* conditions for the existence of the h-weak solution to the problem (L, f, g) in question.

4. Concluding remarks

4.1. A comparison with the approach mentioned in Section 0.4 shows that the above-described method generally gives a *larger class of admissible weights*. In fact, investigating the $W_0^{1,2}(\Omega;h)$-ellipticity, i.e. the inequality

$$b(u,u)=a(u,uh)\geqq \lambda \|u\|^2_{1;h},$$

we obtain restrictive conditions on the weight h. However, for the approach from Section 0.4, we have to prove in addition that the inequality

$$a(uh^{-1}, u) \geqq \tilde{\lambda}\|u\|^2_{1;h^{-1}}$$

holds, and this eventually generates further restrictions on h.

4.2. Other boundary value problems. The main tools for establishing the existence and uniqueness of an h-weak solution of the Dirichlet problem (L, f, g) were the boundedness and $W_0^{1,2}(\Omega; h)$-ellipticity of the bilinear form $b(u, v)$, i.e. the validity of estimates of the type

$$|b(u, v)| \leqq c_5\|u\|_V\|v\|_V, \tag{4.1}$$

$$b(u, u) \geqq c_6\|u\|_V^2 \tag{4.2}$$

for every $u, v \in V = W_0^{1,2}(\Omega; h)$.

For other boundary value problems, we have to derive analogous estimates, but now for functions $u, v \in V$, where V is a larger space,

$$W_0^{1,2}(\Omega; h) \subset V \subset W^{1,2}(\Omega; h)$$

(e.g., we have $V = W^{1,2}(\Omega; h)$ for the *Neumann problem*). Moreover, terms of the types

$$\int_\Gamma uvh \, dS, \quad \int_\Gamma gv \, dS, \quad \Gamma \subset \partial\Omega,$$

can appear in the bilinear form $b(u, v)$ and on the right hand side of the identity (2.1), respectively. This fact requires a more detailed knowledge of the properties of *traces* of functions from weighted spaces.

Therefore, let us give only two examples:

(i) The weak analogue of the (mixed) boundary value problem

$$Lu = f \text{ on } \Omega, \quad u = g_1 \text{ on } \Gamma \subset \partial\Omega, \quad \frac{\partial u}{\partial \nu} = g_2 \text{ on } \partial\Omega - \Gamma$$

makes sense and a weak solution exists in the space $V = V(\varepsilon)$ for the same values of ε as mentioned in Table 1, if we choose V as the closure of the subset of all functions $u \in C^\infty(\bar{\Omega})$ such that $\operatorname{supp} u \cap \bar{\Gamma} = \emptyset$ in the norm of $W^{1,2}(\Omega, \varepsilon)$ with $h = d_M^\varepsilon$ and $M \subset \Gamma$.

(ii) If we consider the weak solution of the Neumann problem for power-type weights, we have to check that (4.2) holds for $V = W^{1,2}(\Omega; \varepsilon)$. A comparison of the interval of those ε's for which the imbedding theorems for this space hold, with the intervals from Table 1 shows that

(a) in the case $m = N - 1$, these intervals are *disjoint* and so *the Neumann problem is not (weakly) solvable* (by our method!);

(b) in the case $m = N - 2$, the Neumann problem is weakly solvable in $W^{1,2}(\Omega, \varepsilon)$ for $\varepsilon \in \langle 0, s_2)$ and in $W^{1,2}(\Omega; \varepsilon, \varepsilon - 2)$ for $\varepsilon \in \langle 0, t_2)$;

(c) in the case $m \leqq N-3$, the Neumann problem is weakly solvable in $W^{1,2}(\Omega; \varepsilon)$ and $W^{1,2}(\Omega; \varepsilon, \varepsilon-2)$ for the same values of ε as the Dirichlet problem.

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ЛИНЕЙНЫЕ ЭЛЛИПТИЧЕСКИЕ КРАЕВЫЕ ЗАДАЧИ И ВЕСОВЫЕ ПРОСТРАНСТВА С. Л. СОБОЛЕВА: МОДИФИЦИРОВАННЫЙ ПОДХОД

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Резюме

В работе указано, как при помощи некоторого видоизменения понятия слабого решения краевой задачи можно решать эти задачи в весовых пространствах С.Л. Соболева. Этот метод позволяет расширить класс краевых задач, решаемых методами функционального анализа. Главными средствами являются лемма Лакса и Мильграма и свойства весовых пространств. Все подробно указано на примере задачи Дирихле для уравнения второго порядка.