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ON THE CONNECTIVITY FOR DARBOUX FUNCTIONS

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Let A be a set of connected points of $f \in D^*$. Then every bilateral accumulation point of A belongs to A. If A is a set such that any bilateral accumulation point of A belongs to A, then there is a function $f \in D^*$ such that A is the set of points at which f is connected.

In [1] Bruckner and Ceder described what it means for a real function to be Darboux at a point and in [2] Garret, Nelms and Kellum introduced the idea of a connected function at a point. In [3] Rosen showed that the set of points at which f is Darboux and the set of points at which f is connected are G_δ-sets. In [4] Snoha showed that if A is a G_δ-set then there exists a discontinuous function f such that A is the set of points at which f is connected. It is clear that if f is a Darboux function then not every G_δ-set is the set of points at which f is connected. Snoha posed the following problem: Find the characterization of a set $A \subset R$ for which there exists a Darboux function f such that f is discontinuous at every point of R and A is the set of points at which f is connected.

The present paper gives an answer as regards functions of class D^*, where D^* denotes the class of all functions $f: R \to R$ for which $f(J) = R$ for every non degenerate interval $J \subset R$. It is clear that every function $f \in D^*$ is the Darboux function.

Let $C_+(f, x)[C_-(f, x)]$ denote the right-hand side the left-hand side cluster set of a function f at a point x. For any subset M of the plane R^2, $\pi(M)$ denotes the X-projection of M. l_x denotes the vertical line containing a point $(x, 0)$. No distinction is made between a function and its graph.

Definition 1. *We say that a function f is connected from the right [the left] at a point x (we write $x \in \text{Cted}_+(f)$ $[x \in \text{Cted}_-(f)]$ if and only if*

1. $f(x) \in C_+(f, x)$ $[f(x) \in C_-(f, x)]$,
2. *if* $a, b \in C_+(f, x)$ $[a, b \in C_-(f, x)]$ *and M is a continuum such that*
   C1. $l_x \cap M \subset \{x\} \times (a, b)$,
   C2. *$\pi(M)$ is a non degenerate set with the left [the right] endpoint x, then $M \cap f \neq \emptyset$.*

The function f is connected at a point x (we write $x \in \mathrm{Cted}\,(f)$) if f is connected from both the right and the left at x.

Definition 2. *If in the definition 1 each M is a horizontal interval, then we obtain the definitions of the Darboux function from the right-hand side [the left-hand side] at a point x and the Darboux function at a point x.*

Theorem. *If $f \in D^*$ and a set A is a set of all points at which f is connected, then every point of a bilateral accumulation of A belongs to A. If A is a set such that every point of a bilateral accumulation of A belongs to A, then there exists a function $f \in D^*$ such that* $\mathrm{Cted}(f) = A$.

Proof. Let $f \in D^*$ and $A = \mathrm{Cted}(f)$. Let x be a bilateral accumulation point of A. It is clear that $C_+(f, x) = R$. Let $a, b \in C_+(f, x)$ and M be a continuum satisfying the conditions C1 and C2 of the definition 1. Then there exists $x' \in A$ such that $x' > x$, $l_{x'} \cap M \subset \{x'\} \times (a, b)$ and $\pi(M) \cap \{u\colon u \geqslant x'\}$ is a non-degenerate set with the left end-point x' and there exists a continuum M' satisfying the conditions C1 and C2 of the definitions 1 for the point x' and $M' \subset M \cap \{(u, y)\colon u \geqslant x'\}$. Then $M' \cap f = \emptyset$, which implies $M \cap f = \emptyset$. We have shown that $x \in \mathrm{Cted}_+(f)$. In a similar way we prove that $x \in \mathrm{Cted}_-(f)$. Hence $x \in A$.

Let A be a set such that every bilateral accumulation point of A belongs to A. A one-point set cannot be a component of the complement of A since A contains all its bilateral accumulation points. Therefore the complement of A is a union of a finite or an infinite countable system of intervals. Every component of $R \setminus A$ can have a form: (a, b), $\langle a, b)$, $(a, b\rangle$, $(-\infty, a)$, $(-\infty, a\rangle$, $(b, +\infty)$, $\langle b, +\infty)$, $(-\infty, +\infty)$, $\langle a, b\rangle$.

We define a function $\psi\colon R \setminus A \to R$ as follows: If I is a component of $R \setminus A$ with end-points a, b and $a, b \in R$, then

$$\psi(x) = \mathrm{ctg}\,\frac{\pi}{b-a}(x-a) \quad \text{if } x \in \left(a, \frac{a+b}{2}\right\rangle \text{ and } a \in A,$$

$$\psi(x) = \mathrm{ctg}\,\frac{\pi}{b-a}(x-a) \quad \text{if } x \in \left(\frac{a+b}{2}, b\right) \text{ and } b \in A,$$

$$\psi(x) = x - \frac{a+b}{2} \quad \text{if } x \in \left\langle a, \frac{a+b}{2}\right\rangle \text{ and } a \notin A,$$

$$\psi(x) = x - \frac{a+b}{2} \quad \text{if } x \in \left(\frac{a+b}{2}, b\right\rangle \text{ and } b \notin A.$$

If I is a component of $R \setminus A$ of the form $(a, +\infty)$ or $(-\infty, b)$, $a, b \in R$, then $\psi|_I$ is a strictly monotonic continuous function on I and $|\lim_{x \to a^+} \psi(x)| = +\infty$ or $|\lim_{x \to b^-} \psi(x)| = +\infty$, respecively. If I is a component of $R \setminus A$ of the form $\langle a, +\infty)$

or $(-\infty, b\rangle$ $a, b \in R$, then $\psi|_I$ is a strictly monotonic continuous function on I. If $A=\emptyset$, then $\psi(x)=x$.

Let $\mathscr{A}$ be a family of the cardinality c of dense pairwise disjoint sets in R. Let $\mathfrak{M}$ be the family of all continuax in R^2 such that their X-projections are nondegenerate sets. Since these families are of the same cardinality, there exists a one-to-one function $T: \mathfrak{M} \to \mathscr{A}$. We define a function $\varphi: R \to R$ as follows:

$$\varphi(x)=\min\{y: (x, y)\in M\} \text{ if } x \in T(M), \{y: (x, y) \in M\} \neq \emptyset$$

and $\min\{y: (x, y) \in M\} \neq \psi(x)$, and $\varphi(x)=0$ otherwise. We will show that $\varphi \in D^*$. Let $z \in R$ and J be a non-degenerate interval. Since ψ is strictly monotonic on each component of its domain, there exists a non-degenerate closed interval $I \subset J$ such that $\psi \cap \{(u, z): u \in I\}=\emptyset$. Since $I \times \{z\} \in \mathfrak{M}$, $\varphi(I) \cap \{z\} \neq \emptyset$.

We shall prove that $\mathrm{Cted}(\varphi)=A$. Let $x \in A$ and M be a continuum satisfying the conditions C1 and C2 of definition 1. There exists a continuum $M' \subset M$ such that $M' \cap \psi=\emptyset$. From the definition of φ it follows that $M' \cap \varphi \neq \emptyset$. Let $x \notin A$. Then there exists a one-side connected neighbourhood $U \subset R \setminus A$ of the point x.

Let $M=\{(u, y): u \in U, y=\psi(u)\}$. Then there exists a continuum $M_1 \subset M$ satisfying the conditions C1 and C2 of definition 1, but $M_1 \cap \varphi=\emptyset$. This ends the proof of the theorem.

If we replace "$f \in D^*$" in the Theorem by the expression "f is a Darboux function dense between graphs of two continuous functions g and h such that $g(x)<h(x)$ for all $x \in R$", then the Theorem is also true.

The condition "g and h are continuous" is essential because:

Example. Let $g, h, \psi: R \to R$ be defined as follows: $g(x)=\left|\sin\frac{1}{x}\right|$ if $x \neq 0$ and $g(0)=0$, $h(x)=-1$ and $\psi(x)=x$. Then, according to the construction used in the second part of the proof, there exists a function φ contained in the set $W=\{(x, y): f(x)<y<g(x),\ y \neq \psi(x)\}$ and satisfying the condition: every continuum contained in W has a non-empty intersection with φ. Points $\frac{1}{n\pi}$ and $\frac{-1}{n\pi}$ belong to $\mathrm{Cted}(\varphi)$, however, $0 \notin \mathrm{Cted}(\varphi)$.

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О СВЯЗНОСТИ ДЛЯ ФУНКЦИЙ ДАРБУ

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Резюме

Пусть Cted f будет множеством всех точек связности некоторой действительной функции f. В этой работе дана следующая характеризация Cted f для плотной в R^2 действительной функции Дарбу f: каждая двухсторонняя граничная точка множества Cted f принадлежит к Cted f.