GRAPHS WITH PRESCRIBED
NEIGHBOURHOOD GRAPHS

BOHDAN ZELINKA

Let $G$ be an undirected graph without loops and multiple edges, let $u$ be a vertex of $G$. By $N_G(u)$ we denote the subgraph of $G$ induced by the set of all vertices which are adjacent to $u$ in $G$; we call it the neighbourhood graph of $u$ in $G$.

At the Symposium on Graph Theory in Smolenice in 1963 [1] A. A. Zykov has proposed the problem (by himself and B. A. Trachtenbrot) to characterize graphs $H$ with the property that there exists a graph $G$ such that $N_G(u) \cong H$ for each vertex $u$ of $G$.

We shall study the graphs $G$ with the property that for each vertex $u$ of $G$ the graph $N_G(u)$ is isomorphic to the complement of a path or of a circuit. Evidently a graph has the required property if and only if each of its connected components has this property and therefore we shall consider only connected graphs with such a property.

The symbol $P_n$ usually denotes the (simple) path of the length $n$, i.e. with $n + 1$ vertices. Its complement will be denoted by $\bar{P}_n$. Similarly $C_n$ denotes the circuit of the length $n$ and $\bar{C}_n$ denotes its complement.

Theorem 1. Let $G$ be a graph with the property that $N_G(u) \cong \bar{P}_n$ for each vertex $u$ of $G$, where $n \geq 4$. Then $G \cong \bar{C}_{n+4}$.

Proof. Let $u$ be a vertex of $G$. We have $N_G(u) \cong \bar{P}_n$. Let $v_0, v_1, \ldots, v_n$ be the vertices of $N_G(u)$ such that $\{v_{i-1}, v_i\}$ for $i = 1, \ldots, n$ are the unique non-adjacent pairs of its vertices. Consider $N_G(v_0)$. This graph contains the vertices $v_2, \ldots, v_n$, $u$ and its subgraph induced by these vertices is the complement of the graph consisting of a path of the length $n - 2$ and an isolated vertex. As $N_G(v_0) \cong \bar{P}_n$, there exists a vertex $w$ in it which is non-adjacent to $u$ and one of the vertices $v_2, v_n$ and adjacent to all other vertices $v_i$. Suppose that $w$ is non-adjacent to $v_2$. Then $v_2$ is adjacent to none of the vertices $v_1, v_3, w$. If $n = 4$, then the graph $N_G(v_2)$ contains a triangle with the vertices $u, v_0, v_4$. We have $N_G(v_2) \cong \bar{P}_4$; hence in the complement of $N_G(v_2)$ one of the vertices $u, v_0, v_4$ is the centre and the others are terminal vertices of the path of the length 4. Therefore there exists a vertex $z$ of $N_G(v_2)$ which is adjacent to exactly one of the vertices $u, v_0, v_4$. This vertex $z$ must be evidently distinct from the vertices $u, v_0, v_1, v_2, v_3, v_4, w$. But $N_G(u)$ contains the vertices $v_0, v_1, v_2, v_3, v_4$, the graph $N_G(v_0)$ contains $u, v_2, v_3, v_4, w$ and $N_G(v_4)$
contains $u, v_0, v_1, v_2, w$. As $z$ is adjacent to one of the vertices $u, v_0, v_2$, one of the graphs $N_G(u), N_G(v_0), N_G(v_2)$ contains at least six vertices and is not isomorphic to $P_4$, which is a contradiction. Now suppose $n \geq 5$. All vertices $v_1, v_2, v_3, w$ are contained in $N_G(v_3)$. In the complement of $N_G(v_3)$ the vertex $v_2$ has the degree at least 3 and thus this graph is not a path, which is a contradiction. Hence $w$ is non-adjacent to $v_n$ and $N_G(v_0)$ is the complement of the path with the vertices $v_2, \ldots, v_n, w, u$ (in this order). Analogously taking $N_G(v_n)$ instead of $N_G(v_0)$, we prove that there exists a vertex $x$ non-adjacent to $v_0$ and $u$ and adjacent to $v_1, \ldots, v_n$. Using the graph $N_G(v_2)$, we prove that also $x$ is adjacent to $w$. Therefore the set \{v_0, \ldots, v_n, w, u, x\} induces a subgraph $F$ of $G$ such that $F \equiv \tilde{C}_{n+4}$. We have $N_F(y) \equiv \tilde{P}_n$ for each vertex $y$ of $F$. As also $N_G(y) \equiv \tilde{P}_n$ for each vertex $y$ of $F$, we have $N_G(y) = N_F(y)$ and, as $G$ is connected, $G = F$ and $G \equiv \tilde{C}_{n+4}$. □

Evidently also the converse assertion is true and even without the assumption that $n \geq 4$. For each positive integer $n$ the graph $G \equiv \tilde{C}_{n+4}$ has the property that $N_G(u) \equiv \tilde{P}_n$ for each vertex $u$ of $G$. But for $n = 1$ not only the circuit $C_3 \equiv \tilde{C}_4$, but an arbitrary circuit of the length at least 4 has this property. For $n = 2$ the line graph of the graph obtained from an arbitrary regular graph of degree 3 by inserting one vertex onto each edge has the required property. For $n = 3$ we have $\tilde{P}_3 \equiv P_3$; in [2] it was proved that for any odd integer $n \geq 7$ there exists a graph $G$ with $n$ vertices such that $N_G(u) \equiv P_3$ for each $u$. This graph is constructed from a circuit $C_n$ by joining any two vertices having the distance $\frac{1}{2}(n - 1)$ in $C_n$ by an edge. (For $n = 7$ such a graph is isomorphic to $\tilde{C}_7$.)

**Theorem 2.** A graph $G$ with the property that $N_G(u) \equiv \tilde{C}_n$ for each vertex $u$ of $G$ exists if and only if $3 \leq n \leq 6$.

**Proof.** The graph $\tilde{C}_3$ consists of three isolated vertices. Every regular graph of degree 3 without triangles has the property that $N_G(u) \equiv \tilde{C}_3$ for each $u$. The graph $\tilde{C}_4$ has two connected components which are both isomorphic to the complete graph $K_2$. If $G$ is the line graph of a regular graph of degree 3 without triangles, then $N_G(u) \equiv \tilde{C}_4$ for each vertex $u$ of $G$. The graph $\tilde{C}_5 \equiv C_5$; the graph of the regular icosahedron has the required property. The graph with the required property for $n = 6$ is the complement of the Petersen graph.

Now let $n \geq 7$. Suppose that there exists a graph $G$ such that $N_G(u) \equiv \tilde{C}_n$ for each vertex $u$ of $G$. Let $u$ be a vertex of $G$. Let the vertices of $N_G(u)$ be $v_1, \ldots, v_n$ such that \{v_i, v_{i+1}\} for $i = 1, \ldots, n - 1$ and \{v_n, v_1\} are the unique pairs of non-adjacent vertices of $N_G(u)$. Consider $N_G(v_2)$. A subgraph of this graph is the graph consisting of the complement of the path with vertices $v_4, \ldots, v_n$ and of the vertex $u$ adjacent to all of these vertices. As $N_G(v_2) \equiv \tilde{C}_n$, there exists a vertex $w$ which is not adjacent to $v_n$ and $u$ and is adjacent to all the vertices $v_4, \ldots, v_{n-1}$. The graph $N_G(v_4)$ contains the vertices $v_{n-1}, v_n, v_1, w$. The subgraph of $N_G(v_4)$ induced by these vertices is the complement of the graph in which $v_n$ has the degree 3; hence the complement of $N_G(v_4)$ is not a circuit, which is a contradiction. □
Quite analogously to the part of Theorem 2 for \( n \geq 7 \) also the following two theorems can be proved.

**Theorem 3.** There exists no graph \( G \) with the property that \( N_G(u) \) for each vertex \( u \) of \( G \) is the complement of a one-way infinite path.

**Theorem 4.** There exists no graph \( G \) with the property that \( N_G(u) \) for each vertex \( u \) of \( G \) is the complement of a two-way infinite path.

**REFERENCES**


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Katedra tvárenia a plastov
Vysokej školy strojnej a textilnej
Hálkova 6
461 17 Liberec

ГРАФЫ С ПРЕДПИСАННЫМИ ГРАФАМИ ОКРЕСТНОСТЕЙ

Bohdan Zelinka

Резюме

Пусть \( G \) — неориентированный граф, пусть \( u \) — его вершина. Символом \( N_G(u) \) обозначаем подграф графа \( G \), порожденный множеством вершин смежных с \( u \). В статье изучаются графы \( G \) такие, что графы \( N_G(u) \) для всех вершин \( u \) графа \( G \) изоморфны дополнению цепи или контура.