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ON AN ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE DIFFERENTIAL EQUATION OF THE FOURTH ORDER

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In the paper presented an asymptotic behaviour of solutions of the linear differential equation of the fourth order of the form

$$y^{(iv)}+p(t)y''+q(t)y'-(-1)^m r(t)y=0, \quad m=1,2 \tag{E}$$

is investigatesd. Five Theorems and five corresponding Corollaries and two examples are shown.

Throughout the paper the functions $p(t)$, $r(t)$ and $q(t)$ will be supposed continuous and continuously differentiable to the order which stands in the Theorems and $r(t)>0$ on the interval $[a, \infty)$.

Asymptotic and oscillatory properties of the differential equation

$$y^{(iv)}+a(t)y'+b(t)y=0 \tag{a}$$

were studied in papers [5, 6, 8, 9] and elsewhere. The form (a) is the so-called second canonical form of the linear differential equation of the fourth order (see [4]).

The aim of the present paper is to show asymptotic formulae of the first canonical form

$$y^{(iv)}+p(t)y''+q(t)y'+r(t)y=0 \tag{b}$$

of the linear differential equation of the fourth order. Equation (E) is a special case of equation (b).

In paper [4] it is proved that if the differential equation

$$z''+\frac{1}{10}p(t)z=0 \tag{c}$$

has a solution $z(t)\neq 0$, then the differential equation (b) can be transformed into the form (a). Since such functions $p(t)$ will be considered that will not be known whether the equation (c) has a nonzero solution, the asymptotic behaviour of solutions of the equation (b) will be studied. Some results can be found in [3] under the condition $|p(t)-a|\to 0$ as $t\to\infty$, where a is a positive constant.

The paper gives new results on the asymptotic behaviour of solutions of equation (b). Some of them (Theorem 1 and 4) generalize the results in [8].

The equation (E) is equivalent to the system of linear differential equations of the first order

$$z'(t)=A(t)z(t) \tag{S}$$

where

$$A(t)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (-1)^m r(t), & -q(t), & -p(t), & 0 \end{bmatrix}$$

and $z(t)=(y(t),\ y'(t),\ y''(t),\ y'''(t))^T$.

Let $T(t)$ be a diagonal and nonsingular matrix. If we change $z(t)$ by setting $z(t)=T^{-1}(t)w(t)$ and subsutitute in (S), we obtain

$$w'(t)=[T(t)A(t)T^{-1}(t)+T'(t)T^{-1}(t)]w(t)\,. \tag{1}$$

The form of system (1) depends on the matrix $T(t)$. For the following purpose we choose the matrix $T(t)$ in the form

$$T(t)=\text{dia}\,[r^{3/4}(t),\ r^{1/2}(t),\ r^{1/4}(t),\ 1]\,.$$

Then the system (1) has the form

$$w'(t)=[A_0r^{1/4}(t)+A_1q(t)r^{-1/2}(t)+A_2p(t)r^{-1/4}(t)+A_3r^{-1}(t)r'(t)]w(t)\,, \tag{2}$$

where $A_3=\text{dia}\left[\frac{3}{4},\frac{1}{2},\frac{1}{4},0\right]$ and $A_0=(a_{ij})$, $A_1=(b_{ij})$, $A_2=(c_{ij})$ are the matrixes of the fourth degree such that $a_{12}=a_{23}=a_{34}=1$, $a_{41}=(-1)^m$ and all the others $a_{ij}=0$; $b_{ij}=0$ for $i\neq 4$, $j\neq 2$, $b_{42}=-1$; $c_{ij}=0$ for $i\neq 4$, $j\neq 3$ and $c_{43}=-1$.

Let $\int_a^\infty r^{1/4}(t)\,dt=\infty$; then the function $s=\omega(t)=\int_a^t r^{1/4}(\tau)\,d\tau$ has the derivative $\omega'(t)=r^{1/4}(t)>0$, and so $\omega(t)$ has an inverse function $t=\alpha(s)$ defined on $[0,\infty)$.

Putting $t=\alpha(s)$ we get

$$x'(s)=[A_0+A_1f(s)+A_2g(s)+A_3h(s)]x(s)\,, \tag{3}$$

where

$$x(s)=w(\alpha(s)),\ f(s)=q(\alpha(s))r^{-3/4}(\alpha(s))\,,$$
$$g(s)=p(\alpha(s))r^{-1/2}(\alpha(s))$$

and

$$h(s)=r'(\alpha(s))r^{-5/4}(\alpha(s)) .$$

The system (3) is a special case of the linear system

$$x'=(A_0+V(s)+R(s))x . \tag{4}$$

There is proved a following theorem for the system (4) in [1], p. 92:

Theorem I. ([1], p. 92) *Let A_0 be a constant matrix with characteristic roots μ_j, $j=1, 2, \ldots, n$, all of which are distinct. Let the matrix V be differentiable and satisfy*

$$\int_0^\infty |V'(s)|\, ds<\infty$$

and let $V(s)\to 0$ as $s\to\infty$. Let the matrix R be integrable and let

$$\int_0^\infty |R(s)|\, ds<\infty .$$

Let the roots of $\det(A_0+V(s)-\lambda E)=0$ *be denoted by $\lambda_j(s)$, $j=1, 2, \ldots, n$. Clearly, by reordering the μ_j if necessary, $\lim_{s\to\infty}\lambda_j(s)=\mu_j$. For a given k, let*

$$D_{kj}(s)=\mathrm{Re}\,(\lambda_k(s)-\lambda_j(s)) .$$

Suppose all j, $1\leqq j\leqq n$, fall into one of two classes I_1 and I_2, where

$$j\in I_1 \text{ if } \int_0^s D_{kj}(\sigma)\, d\sigma\to\infty \text{ as } s\to\infty$$

and

$$\int_{s_1}^{s_2} D_{kj}(\sigma)\, d\sigma>-K \quad (s_2\geqq s_1\geqq 0) ,$$

$$j\in I_2 \text{ if } \int_{s_1}^{s_2} D_{kj}(\sigma)\, d\sigma<K \quad (s_2\geqq s_1\geqq 0) ,$$

where k is fixed and where K is a constant. Let p_k be a characteristic vector of A_0 associated with μ_k, so that

$$A_0p_k=\mu_kp_k .$$

Then there is a solution $\varphi_k(s)$ of (4) and a s_0, $0\leqq s_0<\infty$ such that

$$\lim_{s\to\infty}\varphi_k(s)\exp\left[-\int_{s_0}^s \lambda_k(\sigma)\, d\sigma\right]=p_k .$$

If the hypothesis is satisfied for all k, $1 \leqq k \leqq n$, *then* $\varphi_k(s)$, $k = 1, 2, \ldots, n$ *form a fundamental system of solutions of (4).*

The following theorem will also be needed.

Theorem II. (Hinton [2]) *Let* $r(t) > 0$ *on an interval* $[a, \infty)$ *and* $r''(t)/r^{1+1/n}(t)$ be in $L[a, \infty)$, *where the symbol* $L[a, \infty)$ *will refer to the set of all complex-valued functions which are Lebesgue integrable on the interval* $[a, \infty)$, for $n = 1, 2, \ldots$ Then

(i) $r^{1/n}(t)$ *is not in* $L[a, \infty)$

(ii) $[r'(t)/r^{1+1/n}(t)]'$ *is in* $L[a, \infty)$

(iii) $[r'(t)/r^{1+1/2n}(t)]^2$ *is in* $L[a, \infty)$.

Applying Theorem I to the system (3) we obtain the following theorems.

Theorem 1. (i) *Let* $r''(t)/r^{5/4}(t)$, $q'(t)/r^{3/4}(t)$, $q^2(t)/r^{5/4}(t)$, $p'(t)/r^{1/2}(t)$ *and* $p^2(t)/r^{3/4}(t)$ *be in* $L[a, \infty)$.

Then there is a fundamental system of solutions $z_k(t)$, $k = 1, 2, 3, 4$ *of the system* (S) *and* $t_0 \geqq a$ *such that*

$$\lim_{t\to\infty} T(t) z_k(t) r^{-3/8}(t) \exp\left[-\int_{t_0}^{t}\left(\mu_k r^{1/4}(\tau) - \frac{\bar{\mu}_k}{4}\frac{p(\tau)}{r^{1/4}(\tau)} - (-1)^m \frac{\mu_k^2}{4}\frac{q(\tau)}{r^{1/2}(\tau)}\right) d\tau\right] = p_k \,. \tag{5}$$

(ii) *If in addition we suppose that* $r'(t)/r(t)$ *is in* $L[a, \infty)$, *then there is a fundamental system of solutions* $z_k(t)$, $k = 1, 2, 3, 4$ *of the system* (S) *and* $t_0 \geqq a$ *such that*

$$\lim_{t\to\infty} T(t) z_k(t) \exp\left[-\int_{t_0}^{t} \left(\mu_k r^{1/4}(\tau) - \frac{\bar{\mu}_k}{4}\frac{p(\tau)}{r^{1/4}(\tau)} - (-1)^m \frac{\mu_k^2}{4}\frac{q(\tau)}{r^{1/2}(\tau)}\right) d\tau\right] = p_k \,, \tag{6}$$

where μ_k *are the roots of the characteristic equation* $\mu^4 - (-1)^m = 0$ *of the matrix* A_0 *and* $p_k = (1, \mu_k, \mu_k^2, \mu_k^3)^T$ are the characteristic vectors of the matrix A_0.

Proof. We show that all hypotheses of Theorem I for the system (3) are satisfied.

The characteristic equation of the matrix A_0 is

$$\mu^4 - (-1)^m = 0, \quad m = 1, 2 \,. \tag{7}$$

The roots of (7) are $\mu_{1,2} = \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$, $\mu_{3,4} = -\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$ for $m = 1$ and $\mu_{1,2} = \pm 1$, $\mu_{3,4} = \pm i$ for $m = 2$. So the characteristic roots of the matrix A_0 are distinct. The vectors $p_k = (1, \mu_k, \mu_k^2, \mu_k^3)^T$, $k = 1, 2, 3, 4$ are characteristic vectors of A_0 corresponding to μ_k.

(i) Denote

$$V(s)=A_1f(s)+A_2g(s)+A_3h(s) \text{ and } R(s)=0$$

in the system (3).

In order to be $\int_0^\infty |V'(s)|\,ds<\infty$ it is sufficient to prove that

$$\int_0^\infty |f'(s)|\,ds<\infty,\quad \int_0^\infty |g'(s)|\,ds<\infty \quad\text{and}\quad \int_0^\infty |h'(s)|\,ds<\infty\,.$$

If we put $\alpha(s)=t$, then from the definition of the functions $f(s)$, $g(s)$ and $h(s)$ there follows

$$\int_0^\infty |f'(s)|\,ds=\int_0^\infty |[q(\alpha(s))r^{-3/4}(\alpha(s))]'|\,ds=$$

$$=\int_0^\infty \left|\left[q'(\alpha(s))r^{-3/4}(\alpha(s))-\frac{3}{4}\,q(\alpha(s))r^{-7/4}(\alpha(s))r'(\alpha(s))\right]\alpha'(s)\right|\,ds\leqq$$

$$\leqq\int_a^\infty |q'(t)r^{-3/4}(t)|\,dt+\frac{3}{4}\int_a^\infty |q(t)r^{-7/4}(t)r'(t)|\,dt\,.$$

The first integral is in $L[a,\infty)$ by hypothesis. By apply the Cauchy inequality to the second integral we get

$$\int_a^\infty |q(t)r'(t)r^{-7/3}(t)|\,dt=\int_a^\infty |q(t)r^{-5/8}(t)|\cdot|r'(t)r^{-9/8}(t)|\,dt\leqq$$

$$\leqq\left[\int_a^\infty q^2(t)r^{-5/4}(t)\,dt\right]^{1/2}\cdot\left[\int_a^\infty (r'(t)r^{-9/8}(t))^2\,dt\right]^{1/2},$$

since $q^2(t)r^{-5/4}(t)$ is in $L[a,\infty)$ by hypothesis and $r'(t)r^{-9/8}(t)$ is in $L[a,\infty)$ by Theorem II of point (iii). Therefore $\int_0^\infty |f'(s)|\,ds<\infty$.

Similarly (by hypothesis and by Theorem II) we get

$$\int_0^\infty |g'(s)|\,ds=\int_0^\infty |[p(\alpha(s))r^{-1/2}(\alpha(s))]'|\,ds\leqq$$

$$\leqq\int_a^\infty |p'(t)r^{-1/2}(t)|\,dt+\frac{1}{2}\int_a^\infty |p(t)r'(t)r^{-3/2}(t)|\,dt\leqq$$

$$\leqq\int_a^\infty |p'(t)r^{-1/2}(t)|\,dt+\frac{1}{2}\left[\int_a^\infty p^2(t)r^{-3/4}(t)\,dt\right]^{1/2}\cdot$$

$$\cdot\left[\int_a^\infty (r'(t)r^{-9/8}(t))^2\,dt\right]^{1/2}<\infty$$

and

$$\int_0^\infty |h'(s)|\, ds = \int_0^\infty \left| \left[\frac{r'(\alpha(s))}{r^{5/4}(\alpha(s))} \right]' \right| ds = \int_a^\infty \left| \left[\frac{r'(t)}{r^{5/4}(t)} \right]' \right| dt < \infty .$$

Consequently $\int_0^\infty |V'(s)|\, ds < \infty$.

Similarly we get

$$\int_0^\infty f^2(s)\, ds = \int_a^\infty q^2(t) r^{-5/4}(t)\, dt < \infty ,$$

$$\int_0^\infty g^2(s)\, ds = \int_a^\infty p^2(t) r^{-3/4}(t)\, dt < \infty ,$$

$$\int_0^\infty h^2(s)\, ds = \int_a^\infty [r'(t) r^{-9/8}(t)]^2\, dt < \infty .$$

Since $f'(s)$ and $f^2(s)$ are in $L[a, \infty)$ then $f(s) \to 0$ as $s \to \infty$. By the same way we get $g(s) \to 0$ and $h(s) \to 0$ as $s \to \infty$. Therefore $V(s) \to 0$ as $s \to \infty$.

Evidently $\int_0^\infty |R(s)|\, ds < \infty$ because $R(s) = 0$.

The characteristic equation of the matrix $A_0 + V(s)$ is

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 , \tag{8}$$

where

$$a_1 = -\frac{3}{2} h , \qquad a_2 = \frac{11}{16} h^2 + g ,$$

$$a_3 = -\frac{3}{32} h^3 - \frac{5}{4} gh + f , \qquad a_4 = \frac{3}{8} gh^2 - \frac{3}{4} fh - (-1)^m .$$

Since $f(s) \to 0$, $g(s) \to 0$, $h(s) \to 0$ as $s \to \infty$ we get that $a_1 \to 0$, $a_2 \to 0$, $a_3 \to 0$, $a_4 \to -(-1)^m$ and $P(\lambda(s)) \to \mu^4 - (-1)^m$ as $s \to \infty$. Hence the roots $\lambda_k(s)$ of (8) converge to the roots of (7). Thut we may write for $s \in [0, \infty)$

$$\lambda(s) = \mu + \delta(s) ,$$

where $\delta(s) \to 0$ as $s \to \infty$. In order to find whether the hypotheses of Theorem I are satisfied we show that the function $\delta(s)$ may be written as a sum

$$\delta(s) = \beta(s) + \gamma(s) , \tag{10}$$

where $\beta(s) = c_1 f(s) + c_2 g(s) + c_3 h(s)$ for some numbers c_1, c_2, c_3 and $\gamma(s) \to 0$ as

$s\to\infty$ and $\gamma(s)$ is in $L[0, \infty)$, Then $\beta(s)\to 0$ as $s\to\infty$ (because $f(s)\to 0$, $g(s)\to 0$, $h(s)\to 0$ as $s\to\infty$) and $\gamma(s)\to 0$ as $s\to\infty$ (this follows from (10)).

Substituting $\lambda(s)=\mu+\beta(s)+\gamma(s)$ into (8) we get

$$P(\mu+\beta(s)+\gamma(s))=\gamma^4+[4(\mu+\beta)+a_1]\gamma^3+$$
$$+[6(\mu+\beta)^2+3(\mu+\beta)a_1+a_2]\gamma^2+$$
$$+[4(\mu+\beta)^3+3(\mu+\beta)^2a_1+2(\mu+\beta)a_2+a_3]\gamma+P(\mu+\beta)=0\,, \qquad (12)$$

where

$$P(\mu+\beta)=\beta^4+(4\mu+a_1)\beta^3+(6\mu^2+3\mu a_1+a_2)\beta^2+$$
$$+(4\mu^3+3\mu^2a_1+2\mu a_2+a_3)\beta+P(\mu)\,.$$

The equation (12) may be written as

$$\gamma(s)H(s)=-P(\mu+\beta(s))\,, \qquad (14)$$

where

$$H(s)=\gamma^3+[4(\mu+\beta)+a_1]\gamma^2+[6(\mu+\beta)^2+3(\mu+\beta)a_1+a_2]\gamma+$$
$$+4(\mu+\beta)^3+3(\mu+\beta)^2a_1+2(\mu+\beta)a_2+a_3\,.$$

Since $a_1(s)\to 0$, $a_2(s)\to 0$, $a_3(s)\to 0$, $\beta(s)\to 0$, $\gamma(s)\to 0$ as $s\to\infty$ then $H(s)\to 4\mu^3$ as $s\to\infty$. If μ_k, $k=1, 2, 3, 4$ are the roots of the equation (7), then $H_k(s)\to 4\mu_k^3$ as $s\to\infty$.

Thus for every $\varepsilon>0$ there is a number $s_0\in[0, \infty)$ such that

$$|4\mu_k^3-H_k(s)|<\varepsilon \quad \text{for} \quad s\in[s_0, \infty)\,.$$

From this it follows that

$$|H_k(s)>4|\mu_k^3|-\varepsilon=4-\varepsilon \qquad (16)$$

because $|\mu_k^3|=1$. If we put $\varepsilon=1$, then from (14) and (16) we get

$$3|\gamma_k(s)|<|P(\mu_k+\beta_k(s))|, \quad k=1, 2, 3, 4 \ \text{for} \ s\in[s_0, \infty)\,. \qquad (17)$$

Put $\beta_k(s)$ in $P(\mu_k+\beta_k(s))$ such that

$$4\mu_k^3\beta_k(s)-\frac{3}{2}\,h(s)\mu_k^3+g(s)\mu_k^2+f(s)\mu_k=0\,,$$

i.e.

$$\beta_k(s)=\frac{3}{8}\,h(s)-\frac{\bar{\mu}_k}{4}\,g(s)-(-1)^m\,\frac{\mu_k^2}{4}\,f(s)\,,$$

then $P(\mu_k+\beta_k(s))$ is in $L[0, \infty)$; (because each term of $P(\mu_k+\beta_k(s))$ consist of functions f^2 or g^2 or h^2 or fg or hg or hf, which are in $L[0, \infty)$) and consenquently from (17) it follows that $\gamma_k(s)$ is in $L[0, \infty)$.

The roots $\lambda_k(s)$ of the equation $P(\lambda) \doteq 0$ may be written as

$$\lambda_k(s) = \mu_k + \frac{3}{8} h(s) - \frac{\bar{\mu}_k}{4} g(s) - (-1)^m \frac{\mu_k^2}{4} f(s) + \gamma_k(s) ,$$

where μ_k are the roots of the equation $\mu^4 - (-1)^m = 0$, $m = 1, 2$.

Then $D_{kj} = \mathrm{Re}\,(\lambda_k(s) - \lambda_j(s))$ for all $k, j = 1, 2, 3, 4$ may have the following forms

a) $D_{kj} = G(s)$
b) $D_{kj} = c + F(s) + G(s)$
c) $D_{kj} = -c + F(s) + G(s)$

where $c > 0$ is a constant, $F(s)$, $G(s)$ are functions such that $F(s) \to 0$, $G(s) \to 0$ as $s \to \infty$ and $G(s)$ is in $L[0, \infty)$.

In the case of a) $j \in I_2$, because of $G(s)$ being a continuous function on $[0, \infty)$ and

$$\int_0^\infty D_{kj}(s)\, ds = \int_0^\infty G(s)\, ds < \infty$$

it follows that there exists a number $K > 0$ such that

$$\int_{s_1}^{s_2} D_{kj}(s)\, ds < K \quad \text{for all } \ 0 \leqq s_1 \leqq s_2 .$$

In the case of b) $j \in I_1$, since $F(s) \to 0$ as $s \to \infty$, the exists a number $s' \in [0, \infty)$ such that for every number $s > s'$ there is

$$c + F(s) + G(s) \geqq \frac{c}{2} + G(s)$$

Then

$$\int_0^\infty D_{kj}(s)\, ds = \int_0^\infty (c + F(s) + G(s))\, ds = \infty$$

since

$$\int_0^\infty \left(\frac{c}{2} + G(s)\right) ds = \infty$$

and

$$\int_{s_1}^{s_2} D_{kj}(s)\, ds > -K \quad \text{for all } \ s_2 \geqq s_1 \geqq 0 \ \text{ and some } \ K > 0 .$$

In the case of c) $j \in I_2$, because from $F(s) \to 0$ as $s \to \infty$ it follows that there exists a number $s'' \in [0, \infty)$ such that

$$-c + F(s) + G(s) < -\frac{c}{2} + G(s) \text{ on the interval } [s'', \infty)$$

and

$$\int_0^\infty D_{kj}(s)\, ds = \int_0^\infty (-c + F(s) + G(s))\, ds < \int_0^\infty \left(-\frac{c}{2} + G(s)\right) ds = -\infty$$

and also

$$\int_{s_1}^{s_2} D_{kj}(s)\, ds < K \text{ for every } s_2 \geqq s_1 \geqq 0$$

and some $K > 0$.

Thus all assumptions of Theorem I are satisfied, so that there are four linearly independent solutions $x_k(s)$, $k = 1, 2, 3, 4$ of (3) and a number s_0, $0 \leqq s_0 < \infty$ such that

$$x_k(s) \exp\left[-\int_{s0}^{s} \lambda_k(\sigma)\, d\sigma\right] \to p_k \text{ as } s \to \infty, \text{ i.e.}$$

$$x_k(s) \exp\left[-\int_{s_0}^{s} \left(\mu_k + \frac{3}{8}\frac{r'(\alpha(\sigma))}{r^{5/4}(\alpha(\sigma))} - \frac{\bar{\mu}_k}{4}\frac{p(\alpha(\sigma))}{r^{1/2}(\alpha(\sigma))} - \right.\right.$$

$$\left.\left. -(-1)^m \frac{\mu_k^2}{4}\frac{q(\alpha(\sigma))}{r^{3/4}(\alpha(\sigma))} + \gamma_k(\sigma)\right) d\sigma\right] \to p_k \text{ as } s \to \infty .$$

Denoting $\exp\left[\int_{s0}^{\infty} \gamma_k(s)\, ds\right] = b_k$, the formula (18) may be written as

$$w_k(t) r^{-3/8}(t) \exp\left[-\int_{t_0}^{t} \left(\mu_k r^{1/4}(\tau) - \frac{\bar{\mu}_k}{4}\frac{p(\tau)}{r^{1/4}(\tau)} - (-1)^m \frac{\mu_k^2}{4}\frac{q(\tau)}{r^{1/2}(\tau)}\right) d\tau\right] \to$$

$$\to p_k b_k r^{-3/8}(t_0) \text{ as } s \to \infty .$$

Since $w(t) = T(t) z(t)$ and the system (3) is a linear one, there are solutions z_k, $k = 1, 2, 3, 4$ of the system (S) with properties (5). Hence part (i) is proved completely.

(ii) To prove the second part of Theorem 1 we denote

$$V_{12}(s) = A_1 f(s) + A_2 g(s) \text{ and } R_3(s) = A_3 h(s)$$

in the system (3).

The integral $\int_0^\infty |R_3(s)|\,ds$ is in $L[0,\infty)$, because

$$\int_0^\infty |h(s)|\,ds = \int_a^\infty |r'(t)r^{-1}(t)|\,dt < \infty$$

by hypothesis.

The matrix $V_{12}(s)$ is the special case of the matrix $V(s)$, therefore

$$\int_0^\infty |V'_{12}(s)|\,ds < \infty \quad \text{and} \quad V_{12}(s) \to 0 \quad \text{as} \quad s \to \infty\,.$$

The characteristic roots of the matrix $A_0 + V_{12}(s)$ have the form

$$\lambda_k(s) = \mu_k - \frac{\bar{\mu}_k}{4}\, g(s) - (-1)^m \frac{\mu_k^2}{4} f(s) + \gamma_k(s)\,,$$

$k = 1, 2, 3, 4$.

Thus all assumptions of Theorem I are satisfied. Then there are four linearly independent solutions $x_k(s)$ of (3) and a number s_0, $0 \leqq s_0 < \infty$ such that

$$x_k(s) \exp\left[-\int_{s_0}^{s} \left(\mu_k - \frac{\bar{\mu}_k}{4} \frac{p(\alpha(\sigma))}{r^{1/2}(\alpha(\sigma))} - \right.\right.$$

$$\left.\left. -(-1)^m \frac{\mu_k^2}{4} \frac{q(\alpha(\sigma))}{r^{3/4}(\alpha(\sigma))} + \gamma_k(\sigma) \right) d\sigma \right] \to p_k \quad \text{as} \quad s \to \infty\,.$$

By a similar procedure as the assertion (5) we get the assertion (6).

Putting in the system (3)

(i) $V_{13}(s) = A_1 f(s) + A_3 h(s)$ and $R_2(s) = A_2 g(s)$
(ii) $V_1(s) = A_1 f(s)$ and $R_{23}(s) = A_2 g(s) + A_3 h(s)$

we obtain

Theorem 2. (i) *Let* $r''(t)r^{-5/4}(t)$, $q'(t)r^{-3/4}(t)$, $q^2(t)r^{-5/4}(t)$ *and* $p(t)r^{-1/4}(t)$ *be in* $L[a,\infty)$.

Then there is a fundamental system of solutions $z_k(t)$, $k = 1, 2, 3, 4$ *of the system* (S) *and a number* t_0, $t_0 \geqq a$ *such that*

$$\lim_{t\to\infty} T(t) z_k(t) r^{-3/8}(t) \exp\left[-\int_{t_0}^{t} \left(\mu_k r^{1/4}(\tau) - (-1)^m \frac{\mu_k^2}{4} \frac{q(\tau)}{r^{1/2}(\tau)} \right) d\tau \right] = p_k\,.$$

(ii) *If in addition we suppose that* $r'(t)r^{-1}(t)$ *is in* $L[a,\infty)$, *then there is a fundamental system of solutions* $z_k(t)$, $k = 1, 2, 3, 4$ *of the system* (S) *and* $t_0 \geqq a$ *such that*

$$\lim_{t\to\infty} T(t)z_k(t) \exp\left[-\int_{t_0}^{t} \left(\mu_k r^{1/4}(\tau) - (-1)^m \frac{\mu_k^2}{4} \frac{q(\tau)}{r^{1/2}(\tau)}\right) d\tau = p_k ,\right.$$

where μ_k and p_k are the same as in Theorem 1.

Denoting in the system (3)

(i) $V_{23}(s) = A_2 g(s) + A_3 h(s)$ and $R_1(s) = A_1 f(s)$
((ii) $V_2(s) = A_2 g(s)$ and $R_{13}(s) = A_1 f(s) + A_3 h(s)$

we get

Theorem 3. (i) *Let $r''(t)r^{-5/4}(t)$, $p^2(t)r^{-3/4}(t)$, $p'(t)r^{-1/2}t)$ and $q(t)r^{-1/2}(t)$ be in $L[a, \infty)$.*

Then there are four linearly independent solutions $z_k(t)$, $k = 1, 2, 3, 4$ of the system (S) and a number $t_0 \geqq a$ such that

$$\lim_{t\to\infty} T(t)z_k(t)r^{-3/8}(t) \exp\left[-\int_{t_0}^{t} \left(\mu_k r^{1/4}(\tau) - \frac{\bar{\mu}_k}{4} \frac{p(\tau)}{r^{1/4}(\tau)}\right) d\tau\right] = p_k .$$

(ii) *If in addition we suppose that $r'(t)r^{-1}(t)$ is in $L[a, \infty)$, there are four linearly independent solutions $z_k(t)$, $k = 1, 2, 3, 4$ of the system (S) and $t_0 \geqq a$ such that*

$$\lim_{ts\to\infty} T(t)z_k(t) \exp\left[-\int_{t_0}^{t} \left(\mu_k r^{1/4}(\tau) - \frac{\bar{\mu}_k}{4} \frac{p(\tau)}{r^{1/4}(\tau)}\right) d\tau\right] = p_k ,$$

where μ_k and p_k are the same as in Theorem 1.

If in system (3) we denote

$V_3(s) = A_3 h(s)$ and $R_{12}(s) = A_1 f(s) + A_2 g(s)$ we get

Theorem 4. *Let $r''(t)r^{-5/4}(t)$, $q(t)r^{-1/2}(t)$, $p(t)r^{-1/4}(t)$ be in $L[a, \infty)$.*

Then there is a fundamental system of solutions $z_k(t)$, $k = 1, 2, 3, 4$ of the system (S) and $t_0 \geqq a$ such that

$$\lim_{t\to\infty} T(t)z_k(t)r^{-3/8}(t) \exp\left[-\int_{t_0}^{t} \mu_k r^{1/4}(\tau)\, d\tau\right] = p_k ,$$

where μ_k and p_k are the same as in Theorem 1.

If in the system (3) we put

$V_0(s) = 0$ and $R_{123}(s) = A_1 f(s) + A_2 g(s) + A_3 h(s)$ we obtain

Theorem 5. *Let $q(t)r^{-1/2}(t)$, $p(t)r^{-1/4}(t)$ and $r'(t)r^{-1}(t)$ is in $L[a, \infty)$ and $\int_a^\infty r^{1/4}(t)\, dt = \infty$.*

Then there is a fundamental system of solutions $z_k(t)$, $k = 1, 2, 3, 4$ of the system (S) and a number $t_0 \geqq a$ such that

$$\lim_{t\to\infty} T(t)z_k(t)\exp\left[-\int_{t_0}^{t} \mu_k r^{1/4}(\tau)\,d\tau\right] = p_k ,$$

where μ_k and p_k are the same as in Theorem 1.

Theorms 2, 3, 4 and 5 may be proved in the same way as Theorem 1.

Theorems 1—5 result in Corollaries 1—5 respectively. The hypotheses of Corollaries are the same as in the Theorems.

Corollary 1. (i) *There is a fundamental system of solutions $y_k(t)$, $k=1, 2, 3, 4$ of the differential equation* (E) *and a number t_0, $t_0 \geqq a$ such that*

$$y_k^{(j)} = \mu_k^j r^{(2j-3)/8}(t)\exp\left[\int_{t_0}^{t}\left(\mu_k r^{1/4}(\tau) - \frac{\bar{\mu}_k}{4}\frac{p(\tau)}{r^{1/4}(\tau)} - \right.\right.$$

$$\left.\left. -(-1)^m \frac{\mu_k^2}{4}\frac{q(\tau)}{r^{1/2}(\tau)}\right)d\tau\right](1+o(1)),\ j=0, 1, 2, 3 . \tag{19}$$

(ii) *There is a fundamental system of solutions $y_k(t)$, $k=1, 2, 3, 4$ of equation* (E) *and a number $t_0 \geqq a$ such that*

$$y_k^{(j)} = \mu_k^j r^{(j-3)/4}(t)\exp\left[\int_{t_0}^{t}\left(\mu_k r^{1/4}(\tau) - \frac{\bar{\mu}_k}{4}\frac{p(\tau)}{r^{1/4}(\tau)} - \right.\right.$$

$$\left.\left. -(-1)^m \frac{\mu_k^2}{4}\frac{q(\tau)}{r^{1/2}(\tau)}\right)d\tau\right](1+o(1)),\ j=0, 1, 2, 3 .$$

Corollary 2. (i) *There is a fundamental system of solutions $y_k(t)$, $k=1, 2, 3, 4$ of the equation* (E) *and $t_0 \geqq a$ such that*

$$y_k^{(j)} = \mu_k^j r^{(2j-3)/8}(t)\exp\left[\int_{t_0}^{t}\left(\mu_k r^{1/4}(\tau) - \right.\right.$$

$$\left.\left. -(-1)^m \frac{\mu_k^2}{4}\frac{q(\tau)}{r^{1/2}(\tau)}\right)d\tau\right](1+o(1)),\ j=0, 1, 2, 3 .$$

(ii) *There is a fundamental system of solutions $y_k(t)$, $k=1, 2, 3, 4$ of equation* (E) *and $t_0 \geqq a$ such that*

$$y_k^{(j)} = \mu_k^j r^{(j-3)/4}(t)\exp\left[\int_{t_0}^{t}\left(\mu_k r^{1/4}(\tau) - \right.\right.$$

$$\left.\left. -(-1)^m \frac{\mu_k^2}{4}\frac{q(\tau)}{r^{1/2}(\tau)}\right)d\tau\right](1+o(1)),\ j=0, 1, 2, 3 .$$

Corollary 3. (i) *There is a fundamental system of solutions $y_k(t)$, $k=1, 2, 3, 4$ of* (E) *and $t_0 \geqq a$ such that*

$$y_k^{(j)}=\mu_k^j r^{(2j-3)/8}(t)\exp\left[\int_{t_0}^{t}\left(\mu_k r^{1/4}(\tau)-\right.\right.$$

$$\left.\left.-\frac{\bar{\mu}_k}{4}\frac{p(\tau)}{r^{1/4}(\tau)}\right)\mathrm{d}\tau\right](1+o(1)),\ j=0,1,2,3\,.$$

(ii) *There is a fundamental system of solutions* $y_k(t)$, $k=1, 2, 3, 4$ *of* (E) *and* $t_0\geqq a$ *such that*

$$y_k^{(j)}=\mu_k^j r^{(j-3)/4}(t)\exp\left[\int_{t_0}^{t}\left(\mu_k r^{1/4}(\tau)-\right.\right.$$

$$\left.\left.-\frac{\bar{\mu}_k}{4}\frac{p(\tau)}{r^{1/4}(\tau)}\right)\mathrm{d}\tau\right](1+o(1)),\ j=0,1,2\,.$$

Corollary 4. *There is a fundamental system of solutions* $y_k(t)$, $k=1, 2, 3, 4$ *of* (E) *and* $t_0\geqq a$ *such that*

$$y_k^{(j)}=\mu_k^j r^{(2j-3)/8}(t)\exp\left[\int_{t_0}^{t}\mu_k r^{1/4}(\tau)\,\mathrm{d}\tau\right](1+o(1))\,,\ j=0,1,2,3\,.$$

Corollary 5. *There is a fundamental system of solutions* $y_k(t)$, $k=1, 2, 3, 4$ *of* (E) *and* $t_0\geqq a$ *such that*

$$y_k^{(j)}=\mu_k^j r^{(j-3)/4}(t)\exp\left[\int_{t_0}^{t}\mu_k r^{1/4}(\tau)\,\mathrm{d}\tau\right](1+o(1)),\ j=0,1,2,3\,.$$

Proof of Corollary 1. (i) Since the system (S) is equivalent to the equation (E) for the fundamental system of solutions $z_k(t)$, $k=1,2,3,4$ of (S) it follows that

$$z_k=(y_k,\, y_k',\, y_k'',\, y_k''')^T\,,$$

where the functions $y_k(t)$, $k=1, 2, 3, 4$ are four linearly independent solutions of the equation (E). From the formula (5) we get

$$\lim_{t\to\infty}(r^{3/8}(t)y_k(t),\, r^{1/8}(t)y_k'(t),\, r^{-1/8}(t)y_y''(t),\, r^{-3/8}(t)y_k'''(t))^T\cdot$$

$$\cdot\exp\left[-\int_{t_0}^{t}\left(\mu_k r(\tau)-\frac{\bar{\mu}_k}{4}\frac{p(\tau)}{r^{1/4}(\tau)}-(-1)^m\frac{\mu_k^2}{4}\frac{q(\tau)}{r^{1/2}(\tau)}\right)\mathrm{d}\tau\right]=$$

$$=(1,\, \mu_k,\, \mu_k^2,\, \mu_k^3)^T\,,$$

or

$$\lim_{t\to\infty} r^{(3-2j)/8}(t)y_k^{(j)}\exp\left[-\int_{t_0}^{t}\left(\mu_k r^{1/4}(\tau)-\frac{\bar{\mu}_k}{4}\frac{p(\tau)}{r^{1/4}(\tau)}-\right.\right.$$

$$\left.\left.-(-1)^m\frac{\mu_k^2}{4}\frac{q(\tau)}{r^{1/2}(\tau)}\right)\mathrm{d}\tau\right]=\mu_k^j,\ \text{where } j=0,1,2,3$$

and so the formula (19) holds.

The proof of the second part of Corollary 1 is analogous to the first.

In the same way we prove Corollaries 2, 3, 4, 5.

Remark. Theorems 1 and 4 generalize the result in [8] in which $p(t) \equiv 0$ is supposed.

Example 1. Let $q(t)/t^2$ and $p(t)/t$ be in $L[a, \infty)$, $a>0$. Then the differential equations

a) $$y^{(iv)} + p(t)y'' + q(t)y' + 64t^4 y = 0 ,$$

b) $$y^{(iv)} + p(t)y'' + q(t)y' - 16t^4 y = 0$$

satisfy the assumptions of Corollary 4 and therefore their solu tions are

a) $y(t) = t^{-3/2}[e^{t^2}(c_1 \cos t^2 + c_2 \sin t^2) + e^{-t^2}(c_3 \cos t^2 + c_4 \sin t^2)](1 + o(1))$,

b) $y(t) = t^{-3/2}(c_1 e^{t^2} + c_2 e^{-t^2} + c_3 \cos t^2 + c_4 \sin t^2)(1 + o(1))$,

where c_1, c_2, c_3, c_4 are arbitrary numers.

Example 2. Let $p(t)$ be in $L[a, \infty)$, $a>0$.

a) Then the differential equation

$$y^{(iv)} + p(t)y'' + \frac{\alpha}{t} y' - \beta^4 y = 0 ,$$

where α and $\beta>0$ are constants satisfies the hypotheses of Corollary 2 and so its solutions have the form

$$y(t) = [t^{-\alpha/4\beta^2}(c_1 e^{\beta t} + c_2 e^{-\beta t}) + t^{\alpha/4\beta^2}(c_3 \cos \beta t + c_4 \sin \beta t)](1 + o(1)) ,$$

where c_1, c_2, c_3, c_4 are arbitrary numbers.

b) The function

$$y(t) = [e^{\beta t/\sqrt{2}}(c_1 \cos(\beta t/\sqrt{2} - (\alpha/4\beta^2) \ln t) +$$

$$+ c_2 \sin(\beta t/\sqrt{2} - (\alpha/4\beta^2) \ln t)) + e^{-\beta t/\sqrt{2}}(c_3 \cos(\beta t/\sqrt{2} +$$

$$+ (\alpha/4\beta^2) \ln t) + c_4 \sin(\beta t/\sqrt{2} + (\alpha/4\beta^2) \ln t))](1 + o(1)) ,$$

where c_1, c_2, c_3, c_4 are arbitrary numers is the solution of the differential equation

$$y^{(iv)} + p(t)y'' + \frac{\alpha}{t} y' + \beta^4 y = 0 ,$$

because this equation also satisfies the assumptions of Corollary 2.

From these examples we see that the coefficients do not satisfy the assumptions of theorems in [3], [4] and therefore this paper gives new results on the asymptotic behaviour of the differential equation of the fourth order.

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ОБ АСИМПТОТИЧЕСКОМ ПОВЕДЕНИИ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ЧЕТВЕРТОГО ПОРЯДКА

Jozef Miklo

Резюме

В работа рассматриваются асимптотические поведения решений уравнения (E) при $t \to \infty$, если несобственные интегралы от некоторых дробей функций p, q и r являются конечными.