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COVERING GRAPHS AND SUBDIRECT DECOMPOSITIONS OF PARTIALLY ORDERED SETS

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Covering graphs of partially ordered sets (and, in particular, of lattices) were investigated in several papers; cf., e.g., [1], [2], [3], [6], [7], [8], [9], [13].

The notion of an almost discrete partially ordered set was introduced in [5]; cf. also Section 1 below.

Let $\mathscr{L}$ be an almost discrete partially ordered set and let $C(\mathscr{L})$ be the covering graph of $\mathscr{L}$. The relations between certain types of subdirect decompositions of $C(\mathscr{L})$ and subdirect decompositions of $\mathscr{L}$ will be studied in the present paper.

1. Preliminaries

The covering graph $C(\mathscr{L})$ of a partially ordered set $\mathscr{L}=(L;\leqq)$ is defined to be the undirected graph whose vertices are the elements of L and whose edges are those pairs (a, b) of elements of L for which either a covers b or b covers a.

A partially ordered set $\mathscr{L}$ is said to be almost discrete if, whenever $a, b \in L$ and $a<b$, then there are elements $a_0, a_1, a_2, \ldots, a_n \in L$ such that $a_0=a$, $a_n=b$ and a_i covers a_{i-1} for $i=1, 2, \ldots, n$.

All partially ordered sets dealt with in this paper are assumed to be almost discrete.

Weak direct product decompositions of lattices and partially ordered sets were studied in [4] and [5]. Weak direct product decompositions of graphs were investigated in [11]. In [2], the relations between two-factor direct decompositions of a partially ordered set $\mathscr{L}$ and the two-factor direct decompositions of $C(\mathscr{L})$ were dealt with.

The basic notions and denotations concerning direct and subdirect product decompositions of partially ordered sets and graphs will be recalled in Sections 2 and 3 below.

Let $\mathscr{L}_1$ be a partially ordered set with four elements a_1, a_2, b_1, b_2 such that a_i is covered by b_j $(i, j=1,2)$ and that there are no other covering relations in $\mathscr{L}_1$. (*Cf. Fig. 1a.) In* [5] *it was noted that* $C(\mathscr{L}_1)$ can be expressed as a nontrivial direct product while $\mathcal{L}_1$ is directly indecomposable. (Cf. also [2].) Hence there does not exist, in general, a one-to-one correspondence between the direct product decompositions of a partially ordered set and the direct product decompositions of its covering graph.

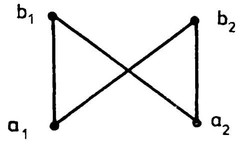

Fig. 1a

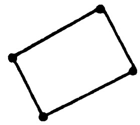
Fig. 1b

Each subdirect decomposition of $\mathcal{L}$ induces a subdirect decomposition of $C(\mathcal{L})$. Let us have a subdirect decomposition

$$\varphi\colon C(\mathcal{L}) \to (\text{sub})\ \Pi_{i\in I}\mathcal{G}_i \tag{1}$$

of the covering graph $C(\mathcal{L})$. Then the condition that

(α) φ induces a subdirect decomposition of $\mathcal{L}$

need not be valid in general. As a counter-example the partially ordered set $\mathcal{L}_1$ can be used again.

A subset K of L is called saturated, if, whenever $a, b \in K$ and a covers b in the partially ordered set $\mathcal{K} = (K; \leqq)$ (with the inherited partial order), then a covers b in $\mathcal{L}$.

It will be shown below that the following condition is necessary for (α) to be valid:

(β) If K is a saturated subset of L such that $\mathcal{K}$ is isomorphic to $\mathcal{L}_1$, then there is $i \in I$ such that card $\varphi_j(K) = 1$ for each $j \in I \setminus \{i\}$.

(Here, φ_j denotes the natural map of L onto the set of all vertices of $\mathcal{G}_j$ corresponding to the subdirect decomposition $\varphi\colon C(\mathcal{L}) \to (\text{sub})\ \Pi_{i\in I}\mathcal{G}_i$.)

For a certain type of subdirect decompositions φ of $C(\mathcal{L})$ there will be found necessary and sufficient conditions for (α) to be valid. These subdirect decompositions will be said to be of type (γ) (for $C(\mathcal{L})$ connected this type includes direct and weak direct decompositions.)

Let x_1, x_2, x_3, x_4 be distinct elements of a partially ordered set $\mathcal{P}$ such that (x_1, x_2), (x_2, x_3), (x_3, x_4) and (x_4, x_1) are edges of $C(\mathcal{P})$. Then $Q = (x_1, x_2, x_3, x_4)$ is said to be an elementary quadruple in $\mathcal{P}$. The following two lemmas are easy to verify (cf. also [8], Lemma 1.1 and 1.2).

1.1. Lemma. *Let Q be an elementary quadruple in $\mathcal{P}$. Then $(Q; \leqq)$ is isomorphic either to $\mathcal{L}_1$ or to the partially ordered set in Fig. 1.b.*

1.2. Lemma. *Let $Q=(x_1, x_2, x_3, x_4)$ be an elementary quadruple in $\mathcal{P}$ such that $(Q;\leqq)$ is not isomorphic to $\mathcal{L}_1$. Then we have either* (i) *x_1 is covered by x_2 and x_4 is covered by x_3, or* (ii) *x_1 covers x_2 and x_4 covers x_3.*

If x, y are vertices of a graph, then their distance $d(x, y)$ is defined in the usual way (cf. e.g, [2]); if for any two vertices x, y we have $d(x, y)<\infty$, then the graph is called connected.

2. Subdirect decompositions of graphs

We begin by recalling the notion of the direct product of graphs. Then we introduce some definitions concerning subdirect products of graphs which will be applied in the sequel.

Let $\mathcal{G}_i=(V_i, H_i)$ $(i \in I)$ be graphs; V_i or H_i is the set of all vertices or the set of all edges of $\mathcal{G}_i$, respectively. Let V be the cartesian product of the sets V_i; the elements of V will be denoted as $a=(a_i)_{i\in I}$ with $a_i \in V_i$ for each $i \in I$. Further let $\mathcal{G}=(V, H)$ be the graph such that H is the set of those pairs (a, b) of elements of V which fulfil the following condition: there exists $j \in I$ such that $(a_j, b_j)\in H_j$ and $a_i=b_i$ for each $i \in I\setminus\{j\}$. Then $\mathcal{G}$ is said to be the direct product of the graphs $\mathcal{G}_i$ and we write $\mathcal{G}=\Pi_{i\in I}\mathcal{G}_i$; the graphs $\mathcal{G}_i$ are called direct factors of $\mathcal{G}$.

If $a \in V$, $a=(a_i)_{i\in I}$, then we denote also $a_i=a(\mathcal{G}_i)$ (the component of a in $\mathcal{G}_i$). For $X\subseteq V$ we put $X(\mathcal{G}_i)=\{x(\mathcal{G}_i): x\in X\}$.

Let x_0 be a fixed element of V and let $y \in V$, $j\in I$. We denote by $y_j^-[x_0]$ that element $z \in V$, for which $z_j=y_j$ and $z_i=x_{0i}$ for each $i\in I\setminus\{j\}$. If no misunderstanding can arise, then we write y_j^- istead of $y_j^-[x_0]$.

Let $X\subseteq V$ such that $X(\mathcal{G}_i)=V_i$ is valid for each $i\in I$. Next let H_0 be the set of all pairs (x_1, x_2) of elements of X such that $(x_1, x_2)\in H$. Then the graph $\mathcal{G}_0=(X, H_0)$ is said to be a subdirect product of the graphs $\mathcal{G}_i$ and we denote this fact by writing

$$\mathcal{G}_0=(\text{sub})\ \Pi_{i\in I}\mathcal{G}_i. \tag{1'}$$

Our considerations would be trivial if card $V=1$. Hence let card $V>1$. In this case we can suppose that card $V_i>1$ for each $i\in I$. Let us consider the following conditions for $\mathcal{G}_0$:

(γ_1) There exists $x_0=(x_{0i})_{i\in I}\in X$ such that, whenever $j\in I$ and $y\in X$, then $y_j^-\in X$.

(γ_2) If $p, q, r, s \in X$ and $i\in I$ with $(p, q)\in H$, $(r, s)\in H$, $p_i=r_i$, $q_i=s_i$, $p_i\neq q_i$ and $r_i\neq s_i$, then there are distinct elements $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ $(n\geqq 1)$ in X such that $y_1=q$, $y_n=s$, $x_1=p$, $x_n=r$, $(x_t)_i=(x_{t+1})_i$, $(y_t)_i=(y_{t+1})_i$ for $t=1, 2, \dots, n-1$, $(x_t, y_t)\in H$ for $t=2, 3, \dots, n-1$ and $(x_t, x_{t+1})\in H$, $(y_t, y_{t+1})\in H$ for $t=1, 2, \dots, n-1$. (Cf. Fig. 2.)

The subdirect product decomposition (1) of the graph $\mathcal{G}_0$ is said to be of type (γ) if the conditions (γ_1) and (γ_2) are satisfied.

A condition analogous to the condition (γ_1) was introduced by F. Šik [12] for subdirect products of lattice ordered groups.

By a graph isomorphism φ of a graph $\mathcal{G}_1 = (V_1, H_1)$ onto a graph $\mathcal{G}_2 = (V_2, H_2)$ we mean a bijection φ: $V_1 \to V_2$ such that $(a, b) \in H_1$ iff $(\varphi(a), \varphi(b)) \in H_2$.

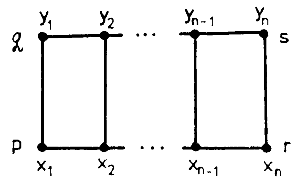

Fig. 2

Let $\mathcal{G}' = (V', H')$ be a graph and let φ be an isomorphism of $\mathcal{G}'$ onto $\mathcal{G}_0$, where $\mathcal{G}_0$ is as above. Then

$$\varphi: \mathcal{G}' \to \Pi_{i \in I} \mathcal{G}_i \tag{1''}$$

is said to be a subdirect product representation of the graph $\mathcal{G}'$. If $\mathcal{G}_0$ is of type (γ), then φ is said to be of type (γ).

$\mathcal{G}_0$ is called a weak direct product of graphs $\mathcal{G}_i$ if the following condition is satified:

(γ_3) there is $x_0 \in X$ such that, for each $x \in V$, x belongs to X if and only if the set $\{i \in I: x_{0i} \neq x_i\}$ is finite.

2.1. Lemma. *If $\mathcal{G}_0$ is connected, then $(\gamma_3) \Rightarrow (\gamma_2)$.*

Proof. Suppose that $\mathcal{G}_0$ is connected and that (1') is a weak direct decomposition of $\mathcal{G}_0$. Let p, q, r and s be elements of X which fulfil the assumptions of (γ_2). In the case of $p = r$ we have $q = s$ and hence (γ_2) holds. Let $p \neq r$; then $q \neq s$. There are elements $a_1, a_2, \ldots, a_m \in X$ with $a_1 = p$, $a_m = r$ such that $(a_k, a_{k+1}) \in H$ for $k = 1, 2, \ldots, m-1$. Let $k \in \{2, 3, \ldots, m-1\}$. Since (γ_3) is valid, there exists $b_k \in X$ such that $(b_k)_i = p_i$ and $(b_k)_j = (a_k)_j$ for each $j \in I \setminus \{i\}$. Thus for each $k \in \{1, 2, \ldots, m-1\}$ we have either $b_k = b_{k+1}$ or $(b_k, b_{k+1}) \in H$. Hence there are elements $c_1, c_2, \ldots, c_n$ in X $(n \leqq m)$ such that $c_1 = p$, $c_n = r$, $(c_t)_i = p_i$ for $t = 1, 2, \ldots, n$ and $(c_t, c_{t+1}) \in H$ for $k = 1, 2, \ldots, n-1$. Again, because (γ) holds, there are elements $d_1, d_2, \ldots, d_n$ in X such that for each $t \in \{1, 2, \ldots, n\}$ we have $(d_t)_i = q_i$ and $(d_t)_j = (c_t)_j$ for each $j \in I \setminus \{i\}$. Now it suffices to put $x_t = c_t$ and $y_t = d_t$ for $t = 1, 2, \ldots, n$.

Clearly $(\gamma_3) \Rightarrow (\gamma_1)$. Hence if $\mathcal{G}_0$ is connected, then each weak direct product of $\mathcal{G}_0$ is of type (γ). Similarly, if $\mathcal{G}_0$ is connected, then each direct product of $\mathcal{G}_0$ is of type (γ). Weak direct products of graphs were investigated by Miller [11].

If (1') is a subdirect product decomposition of type (γ), then it need not be a weak direct product decomposition of $\mathcal{G}_0$ (cf. examples below).

3. Subdirect products of partially ordered sets

Let $\mathscr{L}=(L; \leqq)$ be a partially ordered set. If $a, b \in L$ and a is covered by b (i.e., if the interval $[a, b]$ of $\mathscr{L}$ is prime), then we write a $a \prec b$ or $b \succ a$. Each nonempty subset of L is partially ordered by the inherited partial order. $\mathscr{L}$ is said to be connected if for each pair $x, y \in L$ there are elements $x_1, x_2, \ldots, x_n \in L$ $(i \geqq 1)$ such that $x_1=a$, $x_n=b$ and for each $i \in \{1, 2, \ldots, n-1\}$ either x_i covers x_{i+1} or x_i is covered by x_{i+1}. Hence the partially ordered set $\mathscr{L}$ is connected iff the graph $C(\mathscr{L})$ is connected.

The direct product of partially ordered sets is defined in the usual way (cf. e.g., [6]).

Let I be a nonempty set of indices and for each $i \in I$ let $\mathscr{L}_i=(L_i; \leqq)$ be a partially ordered set. Let $\mathscr{L}=(L; \leqq)$ be the direct product of the system $\{\mathscr{L}_i\}_{i \in I}$; $\mathscr{L}$ is denoted by $\Pi_{i \in I}\mathscr{L}_i$. For elements and subsets of L we use denotations analogous to those in Section 2.

Let $L_0 \subseteq L$, $\mathscr{L}_0=(L_0; \leqq)$. Assume that $L_0(\mathscr{L}_i)=L_i$ is valid for each $i \in I$. Then $\mathscr{L}_0$ is said to be a subdirect product of the partially ordered sets $\mathscr{L}_i$ and we write

$$\mathscr{L}_0=(\text{sub})\ \Pi_{i \in I}\mathscr{L}_i. \tag{2}$$

The following lemma is easy to verify.

3.1. Lemma. *Let $\mathscr{L}$ and $\mathscr{L}_0$ be as above (i.e., $\mathscr{L}_0$ fulfils (2)). Assume that L_0 is a saturated subset of L. Let $a=(a_i)_{i \in I}$ and $b=(b_i)_{i \in I}$ be elements of L_0. Then a is covered by b in $\mathscr{L}_0$ iff there is $j \in I$ such that $a_j \prec b_j$ and $a_i=b_i$ bor each $i \in I \setminus \{j\}$.*

As a corollary wė obtain:

3.2. Lemma. *Let $\mathscr{L}$ and $\mathscr{L}_0$ be as in 3.1. Then*

$$C(\mathscr{L}_0)=(\text{sub})\ \Pi_{i \in I}C(\mathscr{L}_i). \tag{2'}$$

The subdirect product decomposition (2′) of the graph $C(\mathscr{L}_0)$ is said to be induced by the subdirect product decomposition (2).

More generally, let $\mathscr{L}'=(L', \leqq)$ be a partially ordered set and let

$$\varphi\colon \mathscr{L}' \to \Pi_{i \in I}\mathscr{L}_i=\mathscr{L} \tag{3}$$

be an isomorphism of $\mathscr{L}'$ into $\mathscr{L}$ such that

$$\varphi(\mathscr{L}')=(\text{sub})\ \Pi_{i \in I}(\mathscr{L}_i); \tag{3.1}$$

then φ is said to be a subdirect product representation of $\mathscr{L}'$. The subdirect product representation φ of $\mathscr{L}'$ is said to induce a subdirect product representation of the graph $C(\mathscr{L}')$ if

$$\varphi\colon C(\mathscr{L}') \to \Pi_{i \in I}C(\mathscr{L}_i) \tag{3'}$$

is a subdirect product representation of the graph $C(\mathscr{L}')$.

From 3.2 we infer:

3.3. Lemma. *Let φ be as in* (3). *Assume that φ is a subdirect product representation of $\mathcal{L}'$ such that $\varphi(L')$ is a saturated subset of L. Then φ induces a subdirect product representation of the graph $C(\mathcal{L}')$.*

Let us remark that if $\varphi(L')$ fails to be saturated in L, then the assertion of 3.3 need not hold.

Conversely, let us start by having a subdirect product decomposition of the graph $C(\mathcal{L}')$:

$$\psi\colon C(\mathcal{L}')\to\Pi_{i\in I}\mathcal{G}_i=\mathcal{G},\ \text{where}\ \mathcal{G}_i=(V_i,H_i). \tag{4}$$

Then ψ is said to induce a subdirect product representation of $\mathcal{L}'$ if there are partially ordered sets $\mathcal{L}_i=(V_i;\leqq_i)$ such that

(i) $$C(\mathcal{L}_i)=G_i \text{ for each } i\in I, \text{ and} \tag{4'}$$

(ii) $$\psi\colon\mathcal{L}'\to\Pi_{i\in I}\mathcal{L}_i=\mathcal{L} \tag{4'}$$

is a subdirect product representation of $\mathcal{L}'$.

In the following lemma we use the denotations introduced above.

3.4. Lemma. *Let* (4) *be valid. Assume that ψ induces a subdirect product representation* (4') *of $\mathcal{L}'$. Then $\psi(\mathcal{L}')$ is a saturated subset of $L=\Pi_{i\in I}\mathcal{L}_i$.*

Proof. Let $x, y\in L'$, $x<y$. Hence (x, y) is an edge in $C(\mathcal{L}')$ and in view of (4), $(\psi(x),\psi(y))$ is an edge in $\mathcal{G}$. Thus there is $j\in I$ such that $(\psi(x)(j),\psi(y)(j))$ is an edge in $\mathcal{G}_j$ and for each $i\in I\setminus\{j\}$ we have $\psi(x)(i)=\psi(y)(i)$.

From (4') (ii) and from the relation $x<y$ we infer that $\psi(x)<\psi(y)$ holds in $\mathcal{L}$. Thus $\psi(x)(k)\leqq\psi(y)(k)$ for each $k\in I$. Hence $\psi(x)(j)<\psi(y)(j)$. In view of (4'i), $\psi(x)(j)<\psi(y)(j)$ is valid in $\mathcal{L}_j$. We obtain that $\psi(x)<\psi(y)$ is fulfilled in $\mathcal{L}$. Therefore $\psi(L')$ is saturated in L.

Again, let φ be as in (3) and suppose that φ is a subdirect product representation of $\mathcal{L}'$. Let $j\in I$ be fixed. We denote by L_j^* the set of all elements $a^*=(a_i)_{i\in I\setminus\{j\}}$ belonging to the direct product $\Pi_{i\in I\setminus\{j\}}L_i$ and having the property that there exists $x\in L'$ such that $(\varphi(x))_i=a_i$ for each $i\in I\setminus\{j\}$. Under the above denotations put $\varphi_{(j)}(x)=((\varphi(x))_j, a^*)$. Then

$$\varphi_{(j)}\colon\mathcal{L}'\to\mathcal{L}_j\times\mathcal{L}_j^* \tag{5}$$

is a subdirect product representation of $\mathcal{L}'$. If $\varphi(L')$ is a saturated subset of L, then for each $j\in I$, $\varphi^{(j)}(L')$ is a saturated subset of $L_j\times L_j^*$.

Similarly, let us have a subdirect product representation of the graph $C(\mathcal{L}')$ (cf. (4)). Then for each $j\in I$ we define $\mathcal{G}_j^*=(V_j^*, H_j^*)$ analogously to the case of $\mathcal{L}_k^*$ and we obtain that

$$\psi_{(j)}\colon C(\mathcal{L}')\to\mathcal{G}_j\times\mathcal{G}_j^* \tag{6}$$

is a subdirect product representation of the graph $C(\mathscr{L}')$. The representation ψ is of type (γ) iff all $\psi_{(j)}(j \in I)$ are of type (γ).

4. The conditions (β) and (β_1) — necessity

The considerations performed at the end of Section 3 suggest to investigate first two-factor subdirect decompositions.

4.1. Lemma. *Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be partially ordered sets, $\mathscr{L} = (\text{sub})\,(\mathscr{P}_1 \times \mathscr{P}_2)$. Let Q be a saturated subset of L such that $\mathscr{Q} = (Q\,;\,\leqq)$ is isomorphic to $\mathscr{L}_1$. Then either $Q(\mathscr{P}_1)$ or $Q(\mathscr{P}_2)$ is a one-element set.*

Proof. We may suppose that $Q = \{a, b, c, d\}$, where $a < c$, $a < d$, $b < c$, $b < d$, a is incomparable with b and c is incomparable with d. We denote $a(\mathscr{P}_1) = a_1$, $a(\mathscr{P}_2) = a_2$, and similarly for other elements of Q. Assume that $Q(\mathscr{P}_2)$ fails to be a one-element set. We have to verify that card $Q(\mathscr{P}_1) = 1$.

There exist elements $x, y \in Q$ such that $x(\mathscr{P}_2) \neq y(\mathscr{P}_2)$ and (x, y) is an edge in $C(\mathscr{L})$. Without loss of generality we may suppose that $x = a$ and $y = c$. Hence $a_2 \neq c_2$. Since $a < c$, we infer that

$$c_1 = a_1 \text{ and } a_2 < c_2. \tag{4.1}$$

Suppose that $d_1 \neq a_1$. Then, because of $a < d$, we must have

$$a_1 < d_1 \text{ and } a_2 = d_2. \tag{4.2}$$

Next, from $b < c$ it follows that there are two possibilities:

either

$$b_1 = a_1 \text{ and } b_2 < c_2, \tag{4.2α}$$

or

$$b_1 < a_1 \text{ and } b_2 = c_2. \tag{4.2β}$$

In the case (4.2α) we would have $(a_1, b_2) = b < d = (d_1, a_2)$, and in view of (4.2) (because of $a_1 \neq d_1$) the relation $b_2 = a_2$ would be valid, implying $b = (a_1, a_2) = a$, which is a contradiction.

In the case (4.2β), $(b_1, c_2) = b < c = (a_1, c_2)$, thus $b_1 < a_1$. At the same time, $(b_1, c_2) = b < d = (d_1, a_2)$. If $b_1 = d_1$, then $c_2 < a_2$, contradicting (4.1). Hence $b_1 < d_1$, and so $c_2 = a_2$, which is impossible in view of (4.1).

Therefore $d_1 = a_1$ must be valid. Thus $a_2 < d_2$.

Now suppose that $b_1 \neq a_1$. Then from $b < c$ and $b < d$ we infer that we must have $c_2 = b_2 = d_2$, thus $c = d$, which is a contradiction. Therefore $Q(\mathscr{P}_1)$ is a one-element set.

4.2. Corollary. *Let* (3) *and* (3.1) *be valid such that $\varphi(L')$ is a saturated subset* *of L. Let Q be a saturated subset of L' such that $\mathcal{Q}=(Q;\leqq)$ is isomorphic to $\mathcal{L}_1$. Then there exists $j\in I$ such that* card $(\varphi(Q))(\mathcal{L}_i)=1$ *for each $i\in I\setminus\{j\}$.*

Proof. The set $\varphi(Q)$ is a saturated subset of L. There exists $j\in I$ such that card $(\varphi(Q))(\mathcal{L}_j)\neq 1$. Now it suffices to apply Lemma 4.1 for the subdirect decomposition (5).

4.3. Proposition. *Let $\mathcal{L}'$ be a partially ordered set and let ψ be a subdirect product representation of the graph $C(\mathcal{L}')$ described in (4). Suppose that ψ induces a subdirect product representation of $\mathcal{L}'$. Then the condition (β) is fulfilled.*

Proof. Let K be as in (β). Let $\mathcal{L}_i$ $(i\in I)$ be as in (4′). According to 3.4, $\psi(L')$ is a saturated subset of $\Pi_{i\in I}L_i$. Since K is a saturated subset of L', the set $\psi(K)$ is saturated in $\Pi_{i\in I}L_i$. Thus in view of 4.2, the condition (β) holds.

Let us consider the following condition for the subdirect product decomposition (1′):

(β_1) The condition (γ_1) holds and whenever u and v are distinct elements of X such that the relation $u_i^-\leqq v_i^-$ is valid in $\mathcal{L}$ for each $i\in I$, then there are $j\in I$ and $z\in X$ which have the following property: the relations $z_j^->u_j^-$ and $z_j^-\leqq v_j^-$ hold in $\mathcal{L}$.

The subdirect product representation (1″) is said to fulfil (β_1) if $\varphi(G')=G_0$ fulfils the condition (β_1).

In view of the denotations introduced above it suffices to consider the case when ψ is an identity on $\mathcal{L}'$; this assumption (which simplifies the notations) will be applied in 4.4 and also in 5.1, 5.2, 5.3, 5.5. below.

4.4. Lemma. *Let the assumption of 4.3 be fulfilled. Moreover, assume that in $\psi(C(\mathcal{L}'))$ the condition (γ_1) holds. Then the condition (β_1) is valid.*

Proof. Because ψ induces a subdirect product representation of $\mathcal{L}'$, for each $x,y\in L'$ and for each $i\in I$ we have

$$x_i\leqq y_i \Leftrightarrow x_i^-\leqq y_i^-.$$

Let u, $v\in L'$ such that $u\neq v$ and $u_i^-\leqq v_i^-$ is valid for each $i\in I$. Hence $u_i\leqq v_i$ holds for each $i\in I$, thus $u<v$. There exists $z\in L'$ such that $u<z\leqq v$. Then there is $j\in I$ such that

$$u_j<z_j \text{ and } u_i=z_i \text{ for each } i\in I\setminus\{j\};$$

moreover, $z_i\leqq v_i$ for each $i\in I$. Therefore

$$u_j^-<z_j^- \text{ and } z_i^-\leqq v_i^- \text{ for each } i\in I.$$

Hence (β_1) holds.

5. The conditions (β) and (β_1) — sufficiency

Let $\mathscr{L}'$ be a partially ordered set. In this section it will be shown that if (4) is a subdirect product representation of $C(\mathscr{L}')$ such that

(i) the subdirect representation ψ is of type (γ),

(ii) the conditions (β) and (β_1) are fulfilled,

then ψ induces a subdirect product representation of the partially ordered set $\mathscr{L}'$.

5.1. Lemma. *Assume that the conditions* (β) *and* (γ) *are fulfilled. Let $p, q, r, s \in L'$ and $i \in I$. Suppose that $(p, q) \in H$, $(r, s) \in H$, $p_i = r_i$, $q_i = s_i$, $p_i \neq q_i$, $r_i \neq s_i$ and $p < q$. Then $r < s$.*

Proof. Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be as in (γ_2) (with X replaced by L'). If $n = 1$, then $p = r$, $q = s$. Let $n > 1$. Thus $Q = (x_1, x_2, y_2, y_1)$ is an elementary quadruple in $\mathscr{L}'$ and Q is a saturated subset of L'.

We have to verify that $\mathscr{Q} = (Q; \leqq)$ fails to be isomorphic to $\mathscr{L}_1$. By way of contradiction, suppose that $\mathscr{Q}$ is isomorphic to $\mathscr{L}_1$. Since the condition (β) is fulfilled and because of card $Q(\mathscr{G}_i) > 1$, we have card $Q(\mathscr{G}_j) = 1$ for each $j \in J \setminus \{i\}$. Therefore the natural map of Q into V_i is an injection, which is a contradiction (since $(x_1)_i = (x_2)_i$). Hence in view of 1.1 and 1.2, the relation $x_2 < y_2$ is valid. Now it suffices to apply the induction on n.

5.1.1. Corollary. *Assume that* (β) *and* (γ) *are fultilled. Let $p, q \in L'$, $i \in I$. Suppose that $(p, q) \in H$ and $p_i^- \neq q_i^-$. Then the relations $p < q$ and $p_i^- < q_i^-$ are equivalent.*

Let $i \in I$ and let x, y be distinct elements of V_i. We put $x < y$ if there are distinct elements $x_0, x_1, \ldots, x_n$ in V_i such that (i) $x_0 = x$, $x_n = y$, and (ii) for each $j \in \{0, 1, \ldots, n-1\}$ there exist elements $u, v \in L'$ such that $u < v$ and $u_i^- = x_j$, $v_i^- = x_{j+1}$.

In view of 5.1 and 5.1.1 we obtain:

5.1.2. Corollary. *Assume that* (β) *and* (γ) *are fulfilled. Let $a, b \in L'$ and $i \in I$. Then the relations $a_i \leqq b_i$ and $a_i^- \leqq b_i^-$ are equivalent.*

5.2. Lemma. *Assume that* (β) *and* (γ) *are fulfilled. Let $a, b \in L'$ and $i \in I$. If $a \leqq b$, then $a_i \leqq b_i$.*

Proof. The case $a = b$ is obvious; let $a < b$. There are elements $c_0, c_1, \ldots, c_n \in L'$ such that $c_0 = a$, $c_n = b$ and $c_i < c_{i+1}$ is valid for $i = 1, 2, \ldots, n-1$. We proceed by induction on n.

Let $n = 1$. Then $a < b$, hence (a, b) is an edge in $C(\mathscr{L}')$. Thus we have either (i) $a_i = b_i$, or (ii) $a_i \neq b_i$ and $a_j = b_j$ for each $j \in I \setminus \{i\}$. Let (ii) be valid. Then from 5.1 we obtain $a_i^- < b_i^-$, whence $a_i < b_i$.

Next suppose that $n > 1$. Then $(c_0)_i \leqq (c_1)_i$; moreover, from the induction assumption we infer that $(c_1)_i \leqq (c_n)_i$, completing the proof.

5.3. Lemma. *Assume that* (β), (β_1) *and* (γ) *are fulfilled. If $a, b \in L'$ and if $a_i \leqq b_i$ is valid for each $i \in I$, then $a \leqq b$.*

Proof. By way of contradiction, assume that the assertion of the lemma does not hold in general. Then there are distinct elements u and v in L' such that $u \not< v$ and $u_i \leqq v_i$ for each $i \in I$. In view of 5.1.2 we have $u_i^- \leqq v_i^-$ for each $i \in I$. Hence the assumptions of (β_1) are satisfied; let j and z be as in (β_1).

Since the partially ordered set $\mathscr{L}'$ is almost discrete we can also assume that the following minimality condition is fulfilled:

if $c, d \in L'$ such that $c_i \leqq d_i$ is valid for each $i \in I \setminus \{j\}$ and either $u_j^- < c_j^- < d_j^- \leqq v_j^-$ or $u_j^- \leqq c_j^- < d_j^- < v_j^-$, then $c < d$.

Then 5.1 yields that $u < z$ is valid. Moreover, from the above minimality condition we obtain (by taking $c = z$, $d = v$) that $z \leqq v$ holds. Therefore $a < b$.

From 5.2, 5.3, 4.3 and 4.4 we obtain immediately:

5.4. Theorem. *Let $\mathscr{L}'$ be a partially ordered set. Let ψ be a subdirect product representation of the graph $C(\mathscr{L}')$ described in* (4). *Suppose that ψ is of type* (γ). *Then the following conditions are equivalent:*

(i) *The conditions* (β) *and* (β_1) *hold.*

(ii) *ψ induces a subdirect product representation of $\mathscr{L}'$.*

5.5. Lemma. *Assume that $\mathscr{L}'$ is connected. Further suppose that* (β) *and* (γ_3) *are fulfilled. Then* (β_1) *holds.*

Proof. The method applied here is similar to that used in the proof of 2.1. Let u and v be elements of X fulfilling the assumptions of (β_1). Since $u \neq v$, there exists $j \in I$ such that $u_j^- < v_j^-$. According to the definition of u_j^- and v_j^- we have $(u_j^-)_i = (v_j^-)_i$ for each $i \in I \setminus \{j\}$. Hence there are elements $a_1, a_2, \ldots, a_m$ in X such that $a_1 = u_j^-$, $a_m = v_j^-$ and $a_k < a_{k+1}$ for $k = 1, 2, \ldots, m-1$. In view of (γ_3) for each $k \in \{1, 2, \ldots, m\}$ there exists $b_k \in X$ such that $(b_k)_j = (a_k)_j$ and $(b_k)_i = u_j^-)_i$ for each $i \in I \setminus \{j\}$. Hence $b_1 = u_j$, $b_m = v_j^-$ and for each $k \in \{1, 2, \ldots, m-1\}$ we have either $b_k = b_{k+1}$ or $(b_k, b_{k+1}) \in H$. Moreover, if $(b_k, b_{k+1}) \in H$, then in view of 5.1 and 2.1 we infer that $b_k < b_{k+1}$ is valid (we take now j instead of i).

Hence, after changing the indices if needed, we obtain elements $c_1, c_2, \ldots, c_n$ ($n \leqq m$) in X such that $c_1 = u_j^-$, $c_n = v_j^-$, $c_k < c_{k+1}$ for $k = 1, 2, \ldots, n-1$, $(c_k)_i = (u_j^-)_i$ for each $i \in I \setminus \{j\}$ and each $k \in \{1, 2, \ldots, n\}$. According to ($\gamma_3$) there exists $z \in X$ such that $z_i = (c_2)_i$ for each $i \in I \setminus \{j\}$ and $z_j = (c_2)_j$. Then $z_j^- = c_2$ and hence z has the desired properties; therefore (β_1) is valid.

From 5.4, 5.5 and 4.3 we obtain:

5.6. Corollary. *Let $\mathscr{L}'$ be a connected partially ordered set. Let ψ (described in* (4)) *be a weak direct product (or direct product) representation of $C(\mathscr{L}')$. Then ψ induces a weak direct product (or direct product) representation of $\mathscr{L}'$ iff the condition* (β) *holds.*

(If ψ is a direct product representation of $\mathscr{L}'$ where $\mathscr{L}'$ is connected and if card $G_i > 1$ for each $i \in I$, then I must be finite; cf. [5].)

Now let $\mathscr{L}'$ be a semilattice. Then no saturated subset of $\mathscr{L}'$ with the inherited partial order is isomorphic to $\mathscr{L}_1$. Thus 5.6 yields:

5.7. Corollary. *Let $\mathscr{L}'$ be a semilattice. Let ψ be a weak direct product (or direct product) representation of $C(\mathscr{L}')$. Then ψ induces a weak direct product (or direct product) representation of $\mathscr{L}'$.*

Corollary 5.7 generalizes a result of [2] (cf. [2], Thm. 4) concerning two-factor direct decompositions of $C(\mathscr{L}')$, where $\mathscr{L}'$ is a lattice; in the proof of Thm. 4 in [2] a theorem of Kolibiar [10] on direct products of quasiordered sets was applied.

6. Examples

Let $\mathscr{L}'$ be a partially ordered set.

6.1. Let $\mathscr{G}_1$, $\mathscr{G}_2$ be graphs and let ψ be an isomorphism of $C(\mathscr{L}')$ onto $\mathscr{G}_1 \times \mathscr{G}_2$.

If $\mathscr{L}'$ is not connected, then (β) need not imply that ψ induces a direct product representation of $\mathscr{L}'$. In fact, let $L' = \{a, b, c, d\}$; L' is partially ordered such that $a \prec b$, $d \prec c$, and no other covering relations are defined on L'. Let $\mathscr{G}_1 = (V_1, H_1)$ and $\mathscr{G}_2 = (V_2, H_2)$ be graphs such that $V_1 = \{u, v\}$, $H_1 = \{(u, v)\}$ and $V_2 = \{x, y\}$, $H_2 = \emptyset$. Consider the mapping ψ: $L' \to V_1 \times V_2$ defined by $\psi(a) = (u, x)$, $\psi(b) = (v, x)$, $\psi(c) = (u, y)$, $\psi(d) = (v, y)$. Then the condition (β) holds, ψ is an isomorphism of $C(\mathscr{L}')$ onto $\mathscr{G}_1 \times \mathscr{G}_2$ and ψ does not induce a direct product representation of $\mathscr{L}'$.

6.2. Let ψ be a subdirect product representation of $C(\mathscr{L}')$ of type (γ). Then ψ need not be a direct representation (a weak direct representation) of $C(\mathscr{L}')$. Example: let N be the set of all positive integers with the natural linear order; put $N_1 = N_2 = N$, $P = N_1 \times N_2$. Let $m \in N$, $m > 1$ and let Q be the set of all elements $q = (x, y) \in P$ such that some of the following conditions is fulfilled: (i) $x = 1$; (ii) $y = 1$; (iii) $x + y \leqq m$. The set Q is partially ordered by the inherited partial order. Let ψ be the identity on Q. Then ψ: $C(Q) \to C(N_1) \times C(N_2)$ is a subdirect product representation of $C(Q)$; ψ is of type (γ) and ψ is not a direct (weak direct) product representation of $C(Q)$.

6.3. Let ψ be a subdirect product representation of $C(\mathscr{L}')$ fulfilling (γ_1) and (γ_2). Then ψ need not fulfil (β_1). Example: Let P be as in 6.2. Let Q_1 be the set of all elements $q = (x, y)$ of P such that one of the following conditions holds: (i) $x = 1$; (ii) $y = 1$; (iii) $x = y = 2$; (iv) $x = y = 3$. We define a partial order $\leqq_1$ on Q_1 as follows: for distinct elements (x_1, y_1), (x_2, y_2) we put $(x_1, y_1) <_1 (x_2, y_2)$ if either a) $x_1 = x_2 = 1$ and $y_1 < y_2$ or b) $y_1 = y_2 = 1$ and $x_1 < x_2$. Let ψ be the identity on Q_1. Then ψ: $C(Q_1) \to C(N_1) \times C(N_1)$ fulfils the conditions (γ_1) and (γ_2), but it does not fulfil the condition (β_1).

6.4. Let ψ be a subdirect product representation of $C(\mathscr{L}')$ fulfilling (γ_1) and (β_1). Then ψ need not fulfil (γ_2). Example: Let $\mathscr{L}'$ be as in 6.1 with the distinction that we put $c \prec d$ instead of $d \prec c$. Then (γ_1) and (β_1) hold, but (γ_2) does not hold.

Let us also remark that if ψ fulfils (γ), then $\mathscr{L}'$ need not be connected.

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ПОКРЫВАЮЩИЕ ГРАФЫ И ПОЛУПРЯМЫЕ РАЗЛОЖЕНИЯ ЧАСТИЧНО УПОРЯДОЧЕННЫХ МНОЖЕСТВ

Ján Jakubík

Резюме

В статье исследуются условия для того, чтобы полупрямое разложение покрывающего графа $C(\mathscr{L})$ почти дискретного частично упорядоченного множества $\mathscr{L}$ индуцировало полупрямое разложение $\mathscr{L}$.