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ON TWO PROBLEMS OF QUANTUM LOGICS

ANATOLIJ DVUREČENSKIJ

The solution to the Gudder conjecture on an equivalence of the independence and the strong independence of observables on a logic of a separable Hilbert space, and its generalization to a logic of a Hilbert space whose dimension is a non-real measurable cardinal are given.

Further we bring a positive answer to a problem of the author: supposing a system of observables has a joint distribution in a state is then the state a valuation on the minimal sublogic generated by all ranges of the observables?

1. Definitions and preliminaries

Let $L$ be a logic, that is, let $L$ be a $\sigma$-lattice with the first and last elements 0 and 1, respectively, with an orthocomplementation $\perp: a \mapsto a^\perp$, $a, a^\perp \in L$, which satisfies: (i) $(a^\perp)^\perp = a$ for all $a \in L$; (ii) if $a < b$, then $b^\perp < a^\perp$; (iii) $a \lor a^\perp = 1$ for all $a \in L$; (iv) the orthomodular law holds in $L$: if $a < b$, then $b = a \lor (b \land a^\perp)$.

Two elements $a, b \in L$ are (i) orthogonal, and we write $a \perp b$, if $a < b^\perp$; (ii) compatible, and we write $a \leftrightarrow b$, if there are three mutually orthogonal elements $a_1, b_1, c$ such that $a = a_1 \lor c, b = b_1 \lor c$. A logic is called separable if every system of mutually orthogonal nonzero elements is at most countable.

An observable is a map $x: B(R_i) \to L$ such that (i) $x(\emptyset) = 0$; (ii) if $E \cap F = \emptyset$, $E, F \in B(R_i)$, then $x(E) \perp y(F)$; (iii) $x\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigvee_{i=1}^{\infty} x(E_i), E_i \in B(R_i)$. The range of an observable $x$, $\mathcal{R}(x) = \{x(E): E \in B(R_i)\}$, is a Boolean sub-$\sigma$-algebra of $L$. The spectrum of an observable $x$, $\sigma(x)$, is the smallest closed subset $C$ of $R_i$ such that $x(C) = 1$. If $\sigma(x)$ is a compact, then $x$ is said to be bounded. An observable $x$ is purely atomic if (i) $\sigma(x) = \{\lambda_1, \lambda_2, \ldots\}$; (ii) $x(\{\lambda_i\})$ is an atom of $L$ for any $i$. Two observables $x$ and $y$ are compatible if $x(E) \leftrightarrow y(F)$ for all $E, F \in B(R_i)$. There has been developed a calculus for compatible observables (see [13] and [15, Theorem 6.17]). Therefore we may speak of a sum of compatible observables, of a scalar multiple of an observable, etc.
A state is a map \( m: L \rightarrow [0, 1] \) such that (i) \( m(1) = 1 \); (ii) \( m \left( \bigvee_{i=1}^{n} a_i \right) = \sum_{i=1}^{n} m(a_i) \) if \( a_i \perp a_j, \ i \neq j \).

If \( x \) is an observable and \( f \) a Borel function, then the mean value of \( f \circ x \) in a state \( m \) is given by the formula \( m(f \circ x) = \int_{-\infty}^{\infty} f(t) \, dm_x(t) \) (provided the integral exists), where \( m_x(E) = m(x(E)), \ E \in B(R_i) \).

We say that the observables \( x_1, \ldots, x_n \) have a joint distribution in a state \( m \) if there is a probability measure \( m_{x_1 \ldots x_n} \) on \( B(R_n) \) such that

\[
m_{x_1 \ldots x_n}(E_1 \times \ldots \times E_n) = m \left( \bigwedge_{i=1}^{n} x_i(E_i) \right),
\]

\( E_i \in B(R_i), \ i = 1, \ldots, n. \)

The necessary and sufficient conditions for the existence of the joint distribution of observables in a given state may be found in [2, 4, 11, 12].

2. Joint measurability of Boolean subalgebras

One of important problems of the quantum logic theory is a determination of a joint distribution for noncompatible observables, as indicated, for example, in [10, Problem VII]. The answer to this question has been obtained in the above quoted papers [3, 4, 11, 12].

Here we shall investigate a similar problem for subalgebras of \( L \). If \( x \) is an observable, then its range \( \mathcal{R}(x) \) is a Boolean sub-\( \sigma \)-algebra of \( L \) with a countable set of generators. Conversely, for any Boolean sub-\( \sigma \)-algebra \( \mathcal{A} \) of \( L \) with a countable set of generators there is an observable \( x \) such that \( \mathcal{R}(x) = \mathcal{A} \) [15, Theorem 1,6]. Hence a study of a joint measurability (see the definition below) of Boolean sub-\( \sigma \)-algebras of \( L \) with countably many generators in a state \( m \) may be transformed to a study of a joint distribution of observables in a state. The general case of Boolean sub-\( \sigma \)-algebras will be solved in this section.

For any \( a \in L \) we put \( a^0 = a^+ = a \). Let \( a \in L \). The nonempty subset \( M \subset L \) is partially compatible with respect to \( a \) (in abbreviation, p.c. \( a \)) of (i) \( M \leftrightarrow a \), that is, \( b \leftrightarrow a \) for any \( b \in M \); (ii) the set \( M \wedge a = \{ b \wedge a : b \in M \} \) is compatible in \( L \), that is, \( b_1 \wedge a \leftrightarrow b_2 \wedge a \) for any \( b_1, b_2 \in M \). The condition (ii) may be equivalently expressed in the following way: \( M \wedge a \) is the set of mutually compatible elements belonging to a logic \( L_{(0, a)} = \{ c \in L : c < a \} \) (here the orthocomplementation of \( c, c' \), is defined via \( c' = a \wedge c^+ )\). Let \( F = \{ a_1, \ldots, a_n \} \subset L \). According to [11] put

\[
\text{com } F = \text{com } (a_1, \ldots, a_n) = \bigvee_{d \in D^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n}, \tag{2.1}
\]

where \( D = \{ 0, 1 \}, \ d = (d_1, \ldots, d_n) \in D^n \).

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The element \( \text{com } F \) is called the commutator of the finite set \( F \). For an arbitrary \( \emptyset \neq M \subseteq L \) we define a commutator of \( M \), \( \text{com } M \), via

\[
\text{com } M = \bigwedge \{ \text{com } F : F \text{-finite subset of } M \}
\]  

(2.2)

(if it exists in \( L \)).

From [11, Theorem 3.10] it follows that if \( \text{com } M \) exists, then \( M \) is p. c. \( \text{com } M \). Especially, \( M \) is the set of mutually compatible elements iff \( \text{com } M = 1 \).

Let \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) be Boolean subalgebras of \( L \), and let \( \mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i \). For any \( n \)-tuple \( (a_1, \ldots, a_n) \in \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \) define

\[
a^0(a_1, \ldots, a_n) = \text{com } (a_1, \ldots, a_n).
\]

Put

\[
a^0(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \bigwedge \{ a^0(a_1, \ldots, a_n) : a_i \in \mathcal{A}_i, \ i = 1, \ldots, n \}
\]  

(2.3)

if it exists in \( L \).

We say that the Boolean subalgebras \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) are jointly measurable in a state \( m \) if

\[
m(a^0(a_1, \ldots, a_n)) = 1 \text{ for any } a_i \in \mathcal{A}_i, \ i = 1, \ldots, n.
\]

An arbitrary system of Boolean subalgebras of \( L \), \( \{ \mathcal{A}_s : a \in S \} \), is jointly measurable in a state \( m \) if every finite subsystem of \( \{ \mathcal{A}_s : s \in S \} \) is jointly measurable.

In order to state a necessary and sufficient condition for a system of Boolean subalgebras to be jointly measurable in a state \( m \) we have to prove the following lemma.

**Lemma 2.1.** Let \( A_i = \{ a_{i_1}^1, \ldots, a_{i_k}^1 \} \subset \mathcal{A}_1, \ldots, A_s = \{ a_{s_1}^1, \ldots, a_{s_k}^s \} \subset \mathcal{A}_s \) be systems of orthogonal elements, \( 1 \leq s \leq n \). For any \( i = 1, \ldots, s \), define

\[
a_{ij}^t = \begin{cases} 
    a_{ij}^t & \text{if } t = 0, \\
    \bigwedge_{j=1}^k a_{ij}^t & \text{if } t = 1.
\end{cases}
\]

Then for any \( b_{s+1} \in \mathcal{A}_{s+1}, \ldots, b_n \in \mathcal{A}_n \) we have

\[
\bigwedge_{i=1}^k \ldots \bigwedge_{i=1}^k \text{com } (a_{ij}^1, \ldots, a_{ij}^t, b_{s+1}, \ldots, b_n) = \\
= \bigvee_{d \in D^{s+1}} \bigvee_{d_1 \in (0, 1)} \bigwedge_{i=1}^k \ldots \bigwedge_{i=1}^k a_{ih}^1 \wedge \ldots \wedge a_{ih}^t \wedge b^d = \\
= \text{com } (A_1 \cup \ldots \cup A_s \cup B),
\]  

(2.4)

where \( b^d = b_{s+1}^{d_1} \wedge \ldots \wedge b_n^{d_s}, \ d = (d_{s+1}, \ldots, d_n) \in D^{s+1}, \ B = \{ b_{s+1}, \ldots, b_n \} \).
Proof. We shall prove (2.4) by induction over \( s \) and \( k \).

(i) Let \( s = 1 \). In this case (2.4) converts to the form
\[
\bigwedge_{j=1}^{k} \text{com} (a_{j}^{1}, b_{2}, \ldots, b_{n}) = \bigvee_{d \in D^{n-1}} \left\{ \bigwedge_{j=1}^{k} d_{j} \wedge b_{d} \bigvee_{m=1}^{k} a_{j}^{1} \wedge b_{d} \right\} = \text{com} (a_{1}^{1}, \ldots, a_{n}^{1}, b_{2}, \ldots, b_{n}).
\]

(2.5)
It is evident that (2.5) holds for \( k = 1 \). Suppose it holds for any \( k_{1} \leq k \). Then putting 
\[ \text{com} (a, b_{2}, \ldots, b_{n}) = \text{com} (a, B) \]
we have
\[
\bigwedge_{j=1}^{k+1} \text{com} (a_{j}, B) = \bigwedge_{j=1}^{k} \text{com} (a_{j}, B) \wedge \bigwedge_{j=2}^{k+1} \text{com} (a_{j}, B) = \bigvee_{d \in D^{n-1}} \left\{ \bigwedge_{j=1}^{k} a_{j}^{1} \wedge b_{d} \bigvee_{j=2}^{k+1} a_{j}^{1} \wedge b_{d} \right\}.
\]
The elements in \( M = \bigcup_{d \in D^{n-1}} \left\{ a_{1} \wedge b_{d}^{1}, \ldots, a_{k+1} \wedge b_{d}^{1}, \bigwedge_{j=1}^{k} a_{j}^{1} \wedge b_{d}^{1}, \bigwedge_{j=2}^{k+1} a_{j}^{1} \wedge b_{d}^{1} \right\} \) form
the Foulis-Holland set. That is, from any triple of elements from \( M \) one of them is compatible with two others. We note that in \( M \) the only noncompatible pairs are 
\[
\bigwedge_{j=1}^{k} a_{j}^{1} \wedge b_{d}^{1}, \bigwedge_{j=2}^{k+1} a_{j}^{1} \wedge b_{d}^{1}, (d \in D^{n-1}).
\]
Therefore the minimal sublattice of \( L \) generated by \( M \) is distributive (see [6]) and this proves (2.5).

(ii) The general case of (2.4) may be proved by analogical reasonings. This rather technical but essentially simple step is left to the reader.

Q.E.D.

**Corollary 2.1.1.** Let \( K_{i} = \{ a_{i}^{1}, \ldots, a_{k_{i}}^{1} \} \subset \mathcal{A}_{i}, \ i = 1, \ldots, k, \) be finite decompositions of 1, that is, for any \( i = 1, \ldots, n \) we have \( a_{u}^{1} \perp a_{v}^{1} \) for \( 1 \leq u, v \leq k_{i} \), and 
\[
\bigwedge_{u=1}^{k_{i}} a_{u}^{1} = 1.
\]
Then
\[
\text{com} (K_{i} \cup \ldots \cup K_{n}) = \bigwedge_{\mathbf{a} \in K_{i}} \text{com} (a_{1}, \ldots, a_{n})
\]
(2.6)

The element \( a = \bigwedge \{ a_{t}^{1}: t \in T \} \) is said to be countably obtainable from \( \{ a_{t}^{1}: t \in T \} \) if there is a countable subset \( T_{1} \subset T \) such that \( a = \bigwedge \{ a_{t}^{1}: t \in T_{1} \} \).

**Theorem 2.2.** The commutator \( \text{com} (\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}) = \text{com} \left( \bigcup_{i=1}^{n} \mathcal{A}_{i} \right) \) exists iff
\( a^{0}(\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}) \) exists, and in that case \( \text{com} (\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}) = a^{0}(\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}) \). Moreover, if \( \text{com} (\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}) \) is countably obtainable from \( \{ \text{com}: \text{R-finite subset of } \mathcal{A} \} \), so is \( a^{0}(\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}) \) from \( \{ a^{0}(a_{1}, \ldots, a_{n}): a_{i} \in \mathcal{A}_{i}, i = 1, \ldots, n \} \).

Proof. Let \( a = \text{com} (\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}) \) exist in \( L \). Then, for any \( a_{i} \in \mathcal{A}_{i}, i = 1, \ldots, n, \)
we have $a < a^0(a_1, ..., a_n)$. Let now $c < a^0(a_1, ..., a_n)$ for any $a_i \in A_i$. Each finite subset $\emptyset \neq F \subset A$ generates a finite decomposition of 1, $K_i$, in each $A_i$ in the following way. If in $F$ there are no elements from some $A_i$, then we put $K_i = \{ 0, 1 \}$, otherwise we put $K_i = \{ a_1^{i_1} \wedge ... \wedge a_k^{i_k} : a_j \in F \cap A_i \}$. Then

$$com \ F = com (K_1 \cup ... \cup K_n).$$

From Corollary 2.1.1 we have

$$com \ F = \bigwedge_{a \in K_i} com (a_1, ..., a_n).$$

Hence $a^0(\mathcal{A}_1, ..., \mathcal{A}_n)$ exists and equals $a$. Conversely, let $a^0 = a^0(\mathcal{A}_1, ..., \mathcal{A}_n)$ exist. Then (2.7) implies $com \ F > a^0$ for any $F \subset \mathcal{A}$, and therefore $a = a^0$.

The last assertion is evident. Q.E.D.

Remark 1. Let $\{ A_s : s \in S \}$ be a system of Boolean subalgebras of a complete logic $L$. Put $A = \bigcup_{s \in S} A_s$. Then

$$com \ A = \bigwedge_{(s_{i_1}, ..., s_{i_n})} com (A_{s_{i_1}}, ..., A_{s_{i_n}}).$$

Theorem 2.3. Let $\mathcal{A}_1, ..., \mathcal{A}_n$ be Boolean subalgebras of $L$.

(i) If $a^0(\mathcal{A}_1, ..., \mathcal{A}_n)$ is countably obtainable from $\{ com \ F : F \subset \mathcal{A}, F \text{ finite} \}$, then the following conditions are equivalent: (a) $\mathcal{A}_1, ..., \mathcal{A}_n$ are jointly measurable in a state $m$; (b) $m(com \ F) = 1$ for any finite $F \subset \mathcal{A}$; (c) $m(a^0(\mathcal{A}_1, ..., \mathcal{A}_n)) = 1$.

(ii) Let $\{ A_s : s \in S \}$ be a system of Boolean subalgebras of a separable logic $L$. Then the following conditions are equivalent: (a) $\{ A_s : s \in S \}$ is jointly measurable in a state $m$; (b) $m(a^0(\mathcal{A}_{i_1}, ..., \mathcal{A}_{i_n})) = 1$ for any finite subset $\{ s_{i_1}, ..., s_{i_n} \} \subset S$; (c) $m \left( com \bigcup_{s \in S} A_s \right) = 1$.

Proof. To prove Theorem 2.3 it suffices to show that if $\{ a_i^k \}_{k=1}^\infty$ is a sequence of elements from $\mathcal{A}_i$, $i = 1, ..., n$, then

$$m \left( \bigwedge_{k=1}^\infty a^0(a_1^k, ..., a_n^k) \right) = 1 \text{ for any } j = 1, 2, ... .$$

Indeed, let $j$ be given. For any $i = 1, ..., n$ there is an observable $x_i$ so that $\mathcal{R}(x_i) = A_i$, where $A_i$ is the minimal subalgebra of $L$ generated by $\{ a_1, ..., a_n \}$. For the observables $x_1, ..., x_n$ there is a joint distribution in a state $m$ [3, 4]. Therefore
\[ m \left( \bigwedge_{k=1}^{m} a^0(a^k, \ldots, a^k) \right) = 1. \] The continuity of a state \( m \) from above implies that
\[ m \left( \bigwedge_{k=1}^{m} a^0(a^k, \ldots, a^k) = \lim_{j \to \infty} m \left( \bigwedge_{k=1}^{m} a^0(a^k, \ldots, a^k) \right) = 1. \] Hence \( m(a^0) = 1 \). This observation and (2.7) complete the proof of the first part of Theorem 2.3. Analogically we proceed for the second part, too.

Q.E.D.

Remark 2. If a state \( m \) on a logic \( L \) has the property: "if \( m(a_t) = 1 \) for any \( t \in T \), and \( a = \bigwedge_{t \in T} a_t \) exists, then
\[ m(a) = 1, \quad (2.9) \]
then the part (i) in Theorem 2.3 holds without the assumption on countable obtainability of \( a^0(\mathcal{A}_t, \ldots, \mathcal{A}_t) \) provided \( \text{com} (\mathcal{A}_1, \ldots, \mathcal{A}_n) \) exists.

From the results of [12] it follows that the commutator of a sequence of separable subalgebras of \( L \) exists on every logic.

3. Independence and strong independence

We say that observables \( x_1, \ldots, x_n \) are independent in a state \( m \) if
\[ m \left( \bigwedge_{i=1}^{n} x_i(E_i) \right) = \prod_{i=1}^{n} m(x_i(E_i)), \quad (3.1) \]
where \( E_i \in B(R_1), \quad i = 1, \ldots, n \).

Following S. Gudder [8] we say that bounded observables \( x_1, \ldots, x_n \) on a sum logic [7] are strongly independent in a state \( m \) if for any bounded Borel functions \( f_1, \ldots, f_n \)
\[ m_{f_1 \star x_1 + \ldots + f_n \star x_n} = m_{f_1 \star x_1 \star \ldots \star m_{f_n \star x_n}}, \quad (3.2) \]
where the sign \( \star \) denotes the convolution.

The system of observables \( \{ x_s: s \in S \} \) is independent (strongly independent) in a state \( m \) if any finite subsystem of \( \{ x_s: s \in S \} \) is independent (strongly independent) in a state \( m \).

For compatible observables these notions are equivalent. Gudder [8, Theorem 4.5] has proved that the strong independence of observables in a state \( m \) implies the independence. The converse proposition has been proved only for special question observables on the logic \( L(H) \) of all closed subspaces of a separable Hilbert space \( H \) in a pure state [8, Theorem 4.6]. We extend the validity of the above converse implication on a logic \( L(H) \) for all states and observables.
Comparing (3.1) and (1.1) we see that the observables \( x_1, \ldots, x_n \) are independent in a state \( m \) iff there is a joint distribution in a state \( m, m_{x_1 \ldots x_n} \), so that \( m_{x_1 \ldots x_n}(E_1 \times \ldots \times E_n) = \prod_{i=1}^{n} m(x_i(E_i)), \ E_i \in B(R_i), \ i = 1, \ldots, n. \)

**Theorem 3.1.** Let \( x_1, \ldots, x_n \) be purely atomic observables on a logic \( L \) and let \( m \) be a state. Let \( \sigma(x_i) = \{ \lambda_i^1, \lambda_i^1, \ldots \} \). Put

\[
M(x_1, \ldots, x_n) = \{ a \in L: \text{there are } i_1, \ldots, i_n \text{ so that} \ a = x_1(\{\lambda_{i_1}^1\}) = \ldots = x_n(\{\lambda_{i_n}^n\}) \},
\]

\[
M_m(x_1, \ldots, x_n) = \{ a: a \in M(x_1, \ldots, x_n): m(a) \neq 0 \},
\]

\[
a^\sigma = \bigvee \{ a: a \in M(x_1, \ldots, x_n) \},
\]

\[
a_m = \bigvee \{ a: a \in M_m(x_1, \ldots, x_n) \}.
\]

Then the following conditions: (a) \( x_1, \ldots, x_n \) have a joint distribution in a state \( m \); (b) \( m(a^\sigma) = 1 \); (c) \( m(a_m) = 1 \) are equivalent.

**Proof.** From the results of paper [12, Corollary 2.1] we have that \( a^\sigma = a^\sigma(\mathcal{H}(x_1), \ldots, \mathcal{H}(x_n)) \), and Theorem 2.3 yields the statement of Theorem 3.1.

Q.E.D.

**Theorem 3.2.** Let \( \{ x_s: s \in S \} \) be a system of purely atomic observables on a logic \( L \). Then

(i) \( \{ x_s: s \in S \} \) has a joint distribution in a state \( m \) iff any pair of observables \( \{ x_s, x_t \}, \ s, t \in T, \) has a joint distribution.

(ii) \( \{ x_s: s \in S \} \) are independent in a state \( m \) iff any pair of observables \( \{ x_s, x_t \}, \ s \neq t, \) is independent in \( m \).

**Proof.** For any finite system \( x_1, \ldots, x_n \) we may verify that \( M_m(x_1, \ldots, x_n) = \bigcap_{i=1}^{n} M_m(x_i) \).

Theorem 3.1 proves the part (i).

For (ii) we may observe that for any \( i \) there is a unique \( \lambda_i \) such that \( m(x_i(\{\lambda_i\})) = 1 \). Therefore the independence of any pair of the observables implies independence of the entire system \( \{ x_s: s \in S \} \). Q.E.D.

If \( H \) is a Hilbert space over real or complex numbers, not necessarily separable, and \( W \) is a von Neumann operator, that is, \( W \) is a Hermitean, nonnegative operator of trace class with \( trW = 1 \) (for details of the trace class see [14]), then \( m_w(M) = tr(WP^M), \ M \in L(H), \) is a state on \( L(H) \). Here \( P^M \) denotes the orthoprojector onto the subspace \( M \). It is known that to any bounded observable \( x \) there is a unique Hermitean operator \( A_x \) and conversely. For any \( f \in H, \| f \| = 1 \), we define an operator \( f \otimes \hat{f}: x \mapsto (f, x)f, \ x \in H \). Using results of [9, 3], we have:
Theorem 3.3. (i) The Boolean subalgebras $A_1, \ldots, A_n$ of $L(H)$ are jointly measurable in a state $m_w$, where $W = \sum_{a \in I} \lambda_a f_a \otimes \tilde{f}_a$, $f_a \perp f_b$, $a \neq b$, $\|f_a\| = 1$, $a, b \in I$, is a von Neumann operator on $H$, iff one of the following conditions is satisfied:
(a) for any $f_a$ with $\lambda_a \neq 0$ we have
$$P_{m_1} \ldots P_{m_n} f_a = P_{m_1} \ldots P_{m_n} f_a,$$
where $(i_1, \ldots, i_n)$ is any permutation of $(1, \ldots, n)$ and $M_i \in A_i$, $i = 1, \ldots, n$;
(b) $P_{m_1} \ldots P_{m_n} W = P_{m_1} \ldots P_{m_n} W$ for any $M_i \in A_i$, $i = 1, \ldots, n$.

(ii) The bounded observables $x_1, \ldots, x_n$ have a joint distribution in a state $m_w$ iff one condition is satisfied: (a) for any $f_a$ with $\lambda_a \neq 0$ we have
$$A_{x_1} \ldots A_{x_n} f_a = A_{x_1} \ldots A_{x_n} f_a,$$
where $(i_1, \ldots, i_n)$ is any permutation of $(1, \ldots, n)$;
(b) $A_{x_1} \ldots A_{x_n} W = A_{x_1} \ldots A_{x_n} W$
for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$.

Moreover, if $H_0$ is the commutator of $A_1, \ldots, A_n$, then $A_1, \ldots, A_n$ are jointly measurable in $m_w$ iff $f_a \in H_0$ for any $a \in I$ with $\lambda_a \neq 0$.

Proof. The proof follows from the observation that if a von Neumann operator is of the form $W = \sum_{a \in I} \lambda_a f_a \otimes \tilde{f}_a$, then there is a countable subset $D \subset I$ such that $\lambda_a = 0$ if $a \in I - D$. The rest of the proof follows from results of papers [9, Theorem 3.7] and [3, Theorem 14]. Q.E.D.

We shall say that a subspace $H_0$ reduces an observable $x$ on $L(H)$ if $P_{x(E)} P_{H_0} = P_{H_0} P_{x(E)}$ for any $E \in B(R_1)$.

We recall that a cardinal $I$ is said to be non-real measurable if there exists a positive measure $\mu \neq 0$ on the power set of $I$ with $\mu(\{a\}) = 0$ for each $a \in I$.

Theorem 3.4. Let $L(H)$ be a logic of a Hilbert space $H$ (real or complex) whose dimension is a non-real measurable cardinal number. Then the independence and strong independence of bounded observables in any state are equivalent.

Proof. Let a finite system of bounded observables $x_1, \ldots, x_n$ be independent in a state $m$ on $L(H)$. The completeness of the lattice $L(H)$ provides an existence of a commutator $H_0 = \text{com} (\mathcal{R}(x_1), \ldots, \mathcal{R}(x_n))$. If $H$ is a separable Hilbert space, then the commutator is countable obtainable, so that, due to Theorem 2.3 (i), $m(H_0) = 1$. If $H$ is nonseparably, then according to a generalization of the Gleason theorem to a nonseparable Hilbert space logic with a non-real measurable cardinal $I$ we have ([1, 5]) that there is a unique von Neumann operator $W = \sum_{a \in I} \lambda_a f_a \otimes \tilde{f}_a$, 260
The commutator $H_0$ reduces the observables $x_1, \ldots, x_n$ so that $x_0(E) = x_i(E) \wedge H_0, \; E \in B(R_1), \; i = 1, \ldots, n$, are mutually compatible observables on a logic $L(H_0)$. Moreover, $m \left( \bigvee_{i=1}^n x_i(E_i) \right) = m_0 \left( \bigvee_{i=1}^n x_0(E_i) \right), \; E_i \in B(R_1)$.

If $H_0$ reduces $x$, then it reduces $f \circ x$ for any Borel function $f$. Let bounded observable $x$ correspond to a Hermitean operator $A_x$. Then $H_0$ reduces $x$ iff $A_x P^{H_0} = P^{H_0} A_x$. So that $H_0$ reduces $f_1 \circ x_1 + \ldots + f_n \circ x_n$ for any bounded functions $f_1, \ldots, f_n$ and $(f_1 \circ x_1 + \ldots + f_n \circ x_n)_0 = f_1 \circ x_{10} + \ldots + f_n \circ x_{n0}$.

It can be easily checked that

$$m((f_1 \circ x_1 + \ldots + f_n \circ x_n)(E)) = m((f_1 \circ x_1 + \ldots + f_n \circ x_n)(E \wedge H_0)) =$$

$$= m_0((f_1 \circ x_{10} + \ldots + f_n \circ x_{n0})(E)).$$

Hence

$$m_{f_1 \circ x_1 + \ldots + f_n \circ x_n} = m_{0(f_1 \circ x_{10} + \ldots + f_n \circ x_{n0})} =$$

$$= m_{0(f_1 \circ x_{10}) \ast \cdots \ast m_{0(f_n \circ x_{n0})}} = m_{f_1 \circ x_1 \ast \cdots \ast f_n \circ x_n}.$$

Q.E.D.

This Theorem is a consequence of the correspondence between the joint distribution of observables defined by (1.1) and the so-called type II joint distribution on the logic $L(H)$.

We recall that the bounded observables $x_1, \ldots, x_n$ on a sum logic have a type II joint distribution in a state $m$ [15] if for any $a_1, \ldots, a_n \in R_1$ there is a probability measure $\mu^{a_1 \cdots a_n}$ on $B(R_n)$ such that

$$\mu^{a_1 \cdots a_n}((t_1, \ldots, t_n): \; a_1 t_1 + \ldots + a_n t_n \in E)) =$$

$$= m((a_1 x_1 + \ldots + a_n x_n)(E)), \; E \in B(R_1).$$

By analogical reasonings as those used in the proof of Theorem 3.4 we may prove the following theorem.

**Theorem 3.5.** Let $L(H)$ be a logic of a Hilbert space $H$ (real or complex) whose dimension is a non-real measurable cardinal. Then if the bounded observables $x_1, \ldots, x_n$ have a joint distribution in a state $m$, they have a type II joint distribution, and they are identical.

Remark 3. Theorem 3.5 may be proved for the case of unbounded observables, too, see [3, Theorem 20].
The general relationship between the joint distribution of observables in a state by (1.1) (type I joint distribution, too) and a type II joint distribution is still unknown. A very special case of independence may be given for logics with a distributive Segal product $x \circ y$ of two bounded observables $x$ and $y$ defined in [7]:

$$x \circ y = 1/2((x + y)^2 - x^2 - y^2).$$

(3.3)

For any $a \in L$ we define a question observable $x_a$: $x_a(\{0\}) = a^\perp$, $x_a(\{1\}) = a$.

**Lemma 3.6.** If the Segal product is distributive, then

$$x_a + x_b)(\{1\}) = (a \land b^\perp) \lor (a^\perp \land b).$$

(3.4)

**Proof.** The distributivity of the product (3.3) implies

$$(x_a + x_b)^2 + (x_a - x_b)^2 = 2x_a^2 + 2x_b^2 = 2x_a + 2x_b,$$

$$(x_a - x_b)^2 = 2(x_a + x_b) - (x_a + x_b)^2.$$  

But

$$(x_a - x_b)^2(\{1\}) = [2(x_a + x_b) - (x_a + x_b)^2](\{1\}).$$

Hence

$$(x_a - x_b)((-1,1)) = f \circ (x_a + x_b)(\{1\}) = (a^\perp \lor b) \lor (b^\perp \lor a) =

= (x_a + x_b)(\{1\}),$$

where $f(t) = 2t - t^2$.

The latter equality follows from [7, Corollary 6.3].

Q.E.D.

**Corollary 3.6.1.** Two observables $x$ and $y$ on a sum logic with the distributive Segal product are independent in a state $m$ iff

$$m_{x \circ y} = m_x \land m_y$$

(3.5)

for any $E, F \in B(R_1)$. ($\chi_A$ is a characteristic function of a set $A \in B(R_1)$).

**Proof.** In [7, Theorem 6.2, Corollary 6.3] one proves that

$$(x_a + x_b)(\{0\}) = a^\perp \land b^\perp,$$

$$(x_a + x_b)(\{2\}) = a \land b,$$

for any $a, b \in L$. If for $E, F \in B(R_1)$ we put $a = x(E), b = y(F)$, then $x_a = \chi_E \circ x$, $x_b = \chi_F \circ y$. An easy calculation shows that

$$m_{x_a} \land m_{x_b}(\{0\}) = m(a^\perp \land b^\perp),$$

$$m_{x_a} \land m_{x_b}(\{1\}) = m(a^\perp \land b) + m(a \land b^\perp),$$

$$m_{x_a} \land m_{x_b}(\{2\}) = m(a \land b).$$

Using these equalities and Lemma 3.6 we prove (3.5). Q.E.D.
4. Valuation property

In [2, p. 349] the author posed the following question: If \( x_1, \ldots, x_n \) have a joint distribution in a state \( m \), does \( m \) have the property of a valuation on the minimal sublogic \( L_0 \) generated by all ranges \( \mathcal{A}(x_i) \)? The question is whether we have

\[
m(a \lor b) + m(a \land b) = m(a) + m(b)
\]

(4.1)

for any \( a, b \in L_0 \).

Here we show that the answer is in the affirmative.

Let \( a \in L \) and \( \emptyset \neq M \subset L \) be p.c. \( a \). By Zorn’s lemma there is a maximal set \( Q_a \) p.c. \( a \) and such that \( M \subset Q_a \subset L \). Denote by \( L_0(M) \) the minimal sublogic of \( L \) generated by \( M \).

**Lemma 4.1.** Let \( M \) be p.c. \( a \) and let \( \text{com} \ M \) exist. Then

(i) \( a \prec \text{com} \ M \).

(ii) If \( a = \text{com} \ M \), then

\[
a = \text{com} \ Q_a = \text{com} \ L_0(M) = \text{com} \ M.
\]

Proof. Since for any finite subset \( F \subset M \) we have \( \text{com} \ F \prec a \) we obtain (4.2) immediately.

According to Theorem 3.8 from [11], we see that \( Q_a \) is a sublogic of \( L \), so that \( M \subset L_0(M) \subset Q_a(M) \). From (4.2) we have that for any finite \( F \subset Q_a(M) \), \( a \prec \text{com} \ F \). Let now \( c \prec \text{com} \ F \), then \( c \prec \text{com} \ M \). Therefore (4.3) holds.

Q.E.D.

**Theorem 4.2.** Let \( \{ \mathcal{A}_s : s \in S \} \) be a system of Boolean subalgebras of \( L \) jointly measurable in a state \( m \). Let either \( \mathcal{A} \) be countably obtainable from \( \{ \text{com} \ F : F \subset \mathcal{A} \} \) or let \( \text{com} \ \mathcal{A} \) exist and for the state \( m \) the condition (2.9) hold. Then the restrictions of \( m \) to \( Q_{\text{com} \wedge} \) and \( L_0(\mathcal{A}) \), respectively, have the valuation property (4.1).

Proof. Let \( a = \text{com} \ \mathcal{A} \). Using Remark 3.9 from [11], we have that \( Q_a \wedge a \) is a Boolean sub-\( \sigma \)-algebra of logic \( L_{(0,a)} = \{ b \in L : b < a \} \), and the restriction of \( m \) to \( L_{(0,a)} \), \( m_0 \), is a state. Theorem 2.3 and Remark 2 imply that \( m(b) = m(b \wedge a) = m_0(b \wedge a) \) for any \( b \in Q_a \). For mutually compatible elements from \( Q_a \wedge a \) the property (4.1) holds, and therefore it holds for all elements from \( L_0(\mathcal{A}) \) and \( Q_a \), respectively.

Q.E.D.

**Remark 4.** In the process of investigating the properties of the above constructions some open problem arose:

1. Does pairwise joint measurability in a state of Boolean subalgebras imply a joint measurability of a given system of Boolean subalgebras?
2. Especially, have we
\[ \text{com} (a, b, c) = \text{com} (a, b) \land \text{com} (a, c) \land \text{com} (b, c) \]
for any \( a, b, c \in L \)?

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О ДВУХ ПРОБЛЕМАХ КВАНТОВЫХ ЛОГИК

Anatolij Dvorečenskih

Резюме

Дается решение предположения Гаддера о равносильности независимости и сильной независимости наблюдаемых на логике сепарабельного пространства Гильберта, а также обобщение предположения на логику пространства Гильберта, размерность которого неизмеримое кардинальное число.

Кроме того решается одна проблема автора заметки: Если система наблюдаемых имеет совместное распределение в состоянии, будет-ли состояние оценкой на минимальной логике, генерированной всеми областями значений наблюдаемых?