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Mathematica Slovaca, Vol. 36 (1986), No. 3, 283--288

Persistent URL: http://dml.cz/dmlcz/136427

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ON THE A-CONTINUITY OF REAL FUNCTION II

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In the present paper two problems concerning the A-continuity to a regular matrix summability method are partially solved.

Let $A = (a_{mn})$ denote a regular summability method given by a matrix (a_{mn}). We say that a real function f is A-continuous at the point x_0 if $f(x_n)\overset{A}{\longrightarrow}f(x_0)$ whenever $x_n \overset{A}{\longrightarrow} x_0$.

R. C. Buck [2] showed that if f is a $(C, 1)$-continuous at least at one point of R, then f is a linear function. In paper [1] the existence of a regular matrix summability method A for which there exists a nonlinear function A-continuous at least at one point is given.

Professor Šalát puts the following problem:
1. To characterize regular summability methods A for which there exists a nonlinear function which is A-continuous at least at one poit.
2. To characterize C_{fA}, the set of all points of A-continuity of the function f. method is given for which only linear functions are A-continuous at least at one point.

Definition 1. *A regular matrix summability method has the property (G) if there exists sequences* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, *of zeros and ones which are A-covergent to numbers* a, b *respectively* $a\in(0, 1)$, $b\neq 0$, $b\neq 1$, $\left(\frac{a}{1-a}\right)^p \neq \left(\frac{b}{1-b}\right)^q$ *for all non-zero integers* p, q.

Lemma 1. *Let T be a regular matrix summability method which sums at least one sequence of zeros and ones to a number a, $a\neq 0$, $a\neq 1$. Let f be a T-continuos at least at one point. Then f is a continuous function.*

Proof. Let f be a T-continuos at a point z_0. Let us suppose that f is discontinuos at a point x. Thus there exists a sequence $u_n\to 0$ such that $\lim f(x+u_n) = y\neq f(x)$ (also be $y+\infty$, or $-\infty$). Let $\{\alpha_n\}_{n=1}^{\infty}$ denote a sequence of zeros and ones for which T-lim $\alpha_n = a$. The sequence $\{x_n\}_{n=1}^{\infty}$

$$x_n = \alpha_n(x + t_n) + (1 - \alpha_n)\left(\frac{z_0 - ax}{1 - a}\right)$$

is T-summable to z_0 for every sequence $\{t_n\}_{n=1}^{\infty}$, $t_n \to 0$. Especially for $x'_n = \alpha_n(x + u_n) + (1 - \alpha_n)\left(\frac{z_0 - ax}{1 - a}\right)$ we have T-lim $f(x'_n) = ay + (1 - a)f\left(\frac{z_0 - ax}{1 - a}\right)$. However, for

$$x''_n = \alpha_n x + (1 - \alpha_n)\left(\frac{z_0 - ax}{1 - a}\right) \text{ we obtain that}$$

$$\text{T-lim } f(x''_n) = af(x) + (1 - a)f\left(\frac{z_0 - ax}{1 - a}\right).$$

Since f is T-continuous at the point z_0 both above limits have the same value $(f(z_0))$. From this we can conclude that $f(x) = y$. This fact, however, is in contradiction with the assumption and the proof is finished.

Theorem 1. *Let A be a regular summability method with property (G). Let f be a A-continuous at least at one point. Then f is a linear function.*

Proof. Without restriction on generality we can suppose that f is A-continuous at the point 0 and $f(0) = 0$. The sequence $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ where $x_n = \alpha_n + (1 - \alpha_n)y$, $y_n = \beta_n u + (1 - \alpha_n)v$, $(\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}$ are sequences of the definition 1) are A-convergent to 0 if x, y, u, v satisfy equations

$$ax + (1 - a)y = 0$$
$$bu + (1 - b)v = 0.$$

The A-continuity of the function f at the point 0 and $f(0) = 0$ implies that for each x, u the following equations are valid

$$f\left(-\frac{a}{1 - a}x\right) = -\frac{a}{1 - a}f(x),$$

$$f\left(-\frac{b}{1 - b}u\right) = -\frac{b}{1 - b}f(u).$$

The last two equations can be rewriten in the form:

$$f(-k_1 x) = -k_1 f(x), \quad f(-k_2 x) = -k_2 f(x)$$

where $k_1 = \frac{a}{1 - a}$, $k_2 = \frac{b}{1 - b}$. By indukction we can verify the following equality $f(k_1^{2i}k_2^{2j}x) = k_1^{2i}f(x)$ for all x and $i, j = 0, 1, 2, \ldots$. The numbers k_1^2, k_2^2 are positive. Let $(R^+, .)$ denote the topological multiplicative group of nonnegative numbers. It is well known that a subgroup gr (c, d) generated by c, d $(c, d \in R^+)$ is dense if and only if the equality $c^p = d^q$ holds only for $p = q = 0$, p, q are integers. (See [3]

p. 27—36.) Thus we easily obtain that $f(kx)=kf(x)$ for all x and $k\in \mathrm{gr}\,(k_1^2, k_2^2)$. Since the summability method A has the property (G) the subgroup gr (k_1^2, k_2^2) is a dense subset of R^+. According to the lemma 1 f is continuos function. Thus $f(x)=f(1)x$ for $x\geqq 0$ and $f(-x)=f(-1)x$ for $x<0$, which means that f is composed of two linear parts, i. e. $f(x)=cx$ for $x\geqq 0$ and $f(x)=c'x$ for $x<0$, where c, c' are constants. The assumption a (0, 1) of property (G) gives that $k_1>0$. Computing the value of f at the point $-k_1$ in two different ways we obtain $f(-k_1)=c'(-k_1)$ and $f(-k_1)=-k_1f(1)=-k_1c$ according to (1). Thus we can conclude that $c=c'$ and f is a linear function. An example of summability method without property (G) for which there exists non-linear function A-continuous at least at one point, is given in Example 1.

Example 1. A linear transformation given by matrix $B=(b_{mn})$, where $b_{2k+1,4k+1}=b_{2k+1,4k+4}=\frac{1}{2}$, $b_{2k,4k+3}=b_{2k,4k+4}=\frac{1}{2}$, $k=0, 1, 2, \ldots$ and $b_{mn}=0$ otherwise, is a regular summability method. A sequence $\{x_n\}_{n=1}^{\infty}$ is transformed by matrix B to the sequence $\{t_n\}_{n=1}^{\infty}$, where $t_{2k+1}=\frac{1}{2}(x_{4k+1}+x_{4k+4})$ and $t_{2k}=\frac{1}{2}(x_{4k+3}+x_{4k+4})$, $k=0, 1, 2, \ldots$. Each B-summable sequence $\{z_n\}_{n=1}^{\infty}$of zeros and ones has a B-limit equal to one value of the set $\left\{0, \frac{1}{2}, 1\right\}$ as it can be easily verified. Since the terms on places of the form $4k+2$ do not have any influence on the B-limit, we have that there exist infinitely many sequences of zeros and ones, which have the B-limit equal to $\frac{1}{2}$. But for every two integers p, q the eqality $1^p=1^q$ holds. It is suffitient to take for a function f an arbitrary nonlinear odd function which is uniformly continuous and $f(0)=0$. Such a function is continuous at the point 0 and is not a linear function.

Another condition is given in the following theorem.

Theorem 2. *Let there exists for a regular summability method $A=(a_{mn})$ sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ of zeros and ones such that* A-lim $\alpha_n=a$, A-lim $\beta_n=b$, A-lim $\gamma_n=c$, $abc\neq 0$, $a\neq 1\neq b$, $c\neq 1$ and $\alpha_n+\beta_n+\gamma_n=1$ *for every n. Then f is a linear function whenever f is A-continuous at least at one point.*

Proof. Let f be A-continuous at a point x_0. Then the sequence $\{t_n\}_{n=1}^{\infty}$, $t_n=\alpha_nx+\beta_ny+\gamma_nz$, has A-lim $t_n=x_0$ whenever x, y, z satisfies the equality

$$ax+by+cz=x_0 \tag{1}$$

Since $f(t_n)=\alpha_nf(x)+\beta_nf(y)+\gamma_nf(z)$ we have that A-lim $f(t_n)=af(x)+bf(y)+cf(z)$. The A-continuity of f at poit x_0 gives the equality

$$af(x)+bf(y)+cf(z)=f(x_0) \tag{2}$$

By (1) and (2) we can conclude that the function f satisfies the following functional equality

$$f\left(-\frac{a}{c}x-\frac{b}{c}y+\frac{1}{c}x_0\right)=-\frac{a}{c}f(x)-\frac{b}{c}f(y)+\frac{1}{c}f(x_0),$$

for all x, y. According to lemma 1 f is a continuous function. Thus the well-known results about functional equalities of this type give that f is a linear function (see e. g. [5] pages 68—70).

2. The set of all points at which a given function is A-continuous strongly depends on the summability method A. Let e. g. $A=(a_{mn})$, where $a_{mn}=a_{m,\,n+1}=\frac{1}{2}$, $m=1, 2, 3, \dots$ and $a_{mn}=0$ otherwise. Then the set C_{fA} acquires one of the following posibilities:

a) the set of all real numbers
b) the empty set
c) only a one point set
d) a countable set of isolated points.

We outline a verification of this statment.

Let f be A-continuous at a poit a. Without loss of generality we can suppose that $a=0$ and $f(0)=0$ (in another case we take a function $g(x)=f(x+a)-f(a)$ which satisfies the above assumptions and $C_f=C_g+a$, $C_{f,A}=C_{gA}+a$, C_g+a is a shift of the set C_g). A-continuity a the point a implies that f necessarily satisfies the following equations

$$\begin{gathered} f(-x)+f(x+2a)=2f(a) \\ f(-x+a)+f(x+a)=2f(a) \end{gathered} \tag{3}$$

for all $x\in R$. Especially for $a=0$ we obtain $f(-x)=-f(x)$. This condition allows us to give an example of a function for which $C_{fA}=0$ (e. g. $f(x)=x^2$). An example of a function for which $C_{fA}=0$ is the function defined by

$$f(x)=\begin{cases} 1 & \text{for } x\geqslant 1 \\ x & \text{for } x\in(-1,1) \\ -1 & \text{for } x\leqslant -1 \end{cases}$$

(for more detail see [1]).

We prove that C_{fA} is a finite set if and only if C_{fA} contains exactly one point. If 0 and a are points of A-continuity of a function f, then using (3) we obtain that $f(na)=nf(a)$ for $n=0, \pm1, \pm2, \dots$ and all points na are also points of A-continuity of the function f. An example of such a function is the function sinus.

Let C_{fA} have a finite limit point. Without loss of generality we can suppose that this limit point is 0 and $f(0)=0$. Then f is an odd function and for all $a\in C_{fA}$ $f(na)=nf(a)$ $n=0, \pm1, \pm2, \dots$. Since there exists a sequence $a_n\to 0$, $a_n\in C_{fA}$ then

C_{fA} is a dense set in R. Since according to lemma 1 f is continuous on R and $f(na_i) = nf(a_i)$ $(i = 1, 2, 3, \ldots, n = 0, \pm 1, \pm 2, \ldots)$, f is a linear function.

The fact that every linear function is A-continuous on R is evident.

The following theorem tells us more about the possibilities for C_{fA}.

Theorem 3. *Let B be a G_δ set. Then there exist a regular summability method T stronger than the convergence and real function f for which $C_{fT} = B$.*

Proof. It is well known that to any G_δ set B there exists a function f for which $C_f = B$. Let $\{n_1 < n_2 < \ldots\}$ be an infinite set of positive integers whose complementary set (in N) is also an infinite set.

Let us define T in the following way: $T = (a_{mn})$, where $a_{mn_m} = 1$ and $a_{mn} = 0$ for $n \neq n_m$, $m = 1, 2, 3, \ldots$. The regularity of such a method is evident. For regular method A such that $\{f(x_n)\}_{n=1}^{\infty}$ is A-summable whenever $\{x_n\}_{n=1}^{\infty}$ converges the lemma of [4] gives that f is continuous. It is sufficient to prove that $C_{fT} \supset C_f$.

The convergence field of T consists of all sequences $\{y_n\}_{n=1}^{\infty}$ for which the subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ is convergent. Let $x_0 \in C_f$. Let $x_n \xrightarrow{T} x_0$. Then $x_{n_k} \to x_0$. Since f is continuous at x_0, the sequence $f(x_{n_k}) \to f(x_0)$. However, this fact means that $f(x_n) \xrightarrow{T} f(x_0)$ and so we have that $x_0 \in C_{fT}$. Thus $C_{fT} = C_f = B$ and the proof is complete.

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Received June 14, 1984

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О А-НЕПРЕРЫВНОСТИ ВЕЩЕСТВЕННЫХ ФУНКЦИЙ

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Резюме

Пусть A — регуларная матрица. Функция $f\colon R \to R$ называется А-непрерывной в точке x_0, если из А-lim $x_n = x_0$ вытекает А-lim $f(x_n) = f(x_0)$. В работе даны достаточные условия для того, чтобы из А-непрерывности функции вытекала линейность функции. Тоже доказаню, что дла любого множества B типа G_δ существует матрица A и функция f такие, что множество всех точек А-непрерывности функции f равно B.