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THE GREATEST ARCHIMEDEAN IDEAL
IN A SEMIGROUP

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Let $S$ be a semigroup. By an ideal we mean a non-empty two-sided ideal of $S$. An ideal $Q \subseteq S$ is prime if for any ideals $A$, $B$ of $S$, $AB \subseteq Q$ implies $A \subseteq Q$ or $B \subseteq Q$. An ideal $P \subseteq S$ is completely prime if for any $a, b \in S$, $ab \in P$ implies $a \in P$ or $b \in P$. An ideal $A \subseteq S$ is called completely semiprime if $a^n \in A$ for any positive integer $n$ implies $a \in A$.

Denote by $Q^*$ the intersection of all prime ideals of $S$ and by $P^*$ the intersection of all completely prime ideals of $S$ (see [3]). It is known (see [4]) that $P^*$ and $Q^*$ may be empty.

An element $x \in S$ is nilpotent with respect to an ideal $J$ if $x^n \in J$ for some positive integer $n$. An ideal $I$ is called a nilideal with respect to an ideal $J$ if any $x$ of $I$ is nilpotent with respect to $J$. Denote by $R^*(J)$ the Clifford radical with respect to $J$, i.e. the union of all nilideals of $S$ with respect to $J$ (see [5]).

A semigroup $S$ is archimedean (see [2]) if for any $a, b \in S$ there exists a positive integer $n$ such that $a^n \in SbS$. An ideal $I \subseteq S$ is called archimedean if the sub-semigroup $I$ is archimedean.

Denote by $U^*$ the intersection of all prime ideals $Q \subseteq S$ with the property $R^*(Q) = Q$.

M. Satyanarayana [3, Theorem 10] proved that if $Q^*$ is a completely semiprime ideal, then $Q^*$ is the greatest archimedean ideal of the semigroup $S$.

By M. Satyanarayana [3, p. 291] “It is an open problem whether this result is true in arbitrary case”.

In this note we have solved this problem. The result is as follows: If $U^* \neq \emptyset$, then it is the greatest archimedean ideal of $S$ (Theorems 2 and 3). Concluding this result we give a new characterization of the Clifford radical $R^*(J)$ as an intersection of some (in general not all) prime ideals.

**Theorem 1.** Let $S$ be a semigroup and $U^* \neq \emptyset$. Then $R^*(U^*) = U^*$.

**Proof.** Let $U^* = \bigcap_{A \in A} Q_A$. Evidently $U^* \subseteq R^*(U^*)$. Suppose that $U^* \subseteq R^*(U^*)$, $U^* \neq R^*(U^*)$. Let $x \in R^*(U^*) - U^*$. Since $x \notin U^*$, then $x \notin Q_a$ for
some \( \alpha \in A \). Then the principal ideal \( (x) \not\subseteq Q_a = R^*(Q_a) \), hence there exists \( y \in (x) \) such that \( y^n \notin Q_a \) for all positive integers \( n \). Since \( Q_a \supseteq U^* \), \( y \) is not nilpotent with respect to \( U^* \), a contradiction to \( y \in (x) \subseteq R^*(U^*) \). Therefore \( U^* = R^*(U^*) \) holds.

**Lemma 1.** Let \( S \) be a semigroup, \( J \) an ideal, \( H = \{ x, x^2, x^3, \ldots \} \) a cyclic subsemigroup of \( S \) and let \( H \cap J = \emptyset \). Then there exists a prime ideal \( Q \) containing \( J \) such that \( H \cap Q = \emptyset \) and \( R^*(Q) = Q \).

**Proof.** Denote by \( T \) the set of all ideals which contain \( J \) and do not meet \( H \). The set \( T \) is non-empty since it contains \( J \). By Zorn’s lemma there exists a maximal element \( Q \in T \).

We prove that \( Q \) is a prime ideal. Suppose that for some ideals \( A \subseteq Q \) and \( B \subseteq Q \), we have \( AB \subseteq Q \). Then \( x^r \in Q \cup A, x^s \in Q \cup B \) for some positive integers \( r, s \). Since \( x^r, x^s \notin Q \), we have \( x^r \in A, x^s \in B \), thus \( x^r + x^s \in AB \subseteq Q \), which contradicts \( H \cap Q = \emptyset \). Therefore \( Q \) is a prime ideal.

We prove that \( R^*(Q) = Q \). Evidently \( Q \subseteq R^*(Q) \). Suppose that \( Q \subset R^*(Q) \), \( Q \neq R^*(Q) \). Then \( x^m \in H \cap R^*(Q) \) for some positive integer \( m \). However, \( x^m \in R^*(Q) \) implies \( (x^m)^n \in Q \) for some positive integer \( n \). This is a contradiction to \( Q \cap H = \emptyset \). Therefore \( R^*(Q) = Q \).

**Lemma 2.** Let \( S \) be a semigroup, \( x \in U^* \) and \( A \) be any ideal of \( S \). Then \( x^n \in A \) for some positive integer \( n \).

**Proof.** If \( A = S \), then the statement holds. Suppose therefore that \( A \) is a proper ideal of \( S \) and \( x \notin A \) for all positive integers \( n \). By Lemma 1 there exists a prime ideal \( Q = R^*(Q) \) such that \( x \notin Q \). This contradicts \( x \in U^* \). Thus for any proper ideal \( A \) we have \( x^n \in A \) for some positive integer \( n \).

**Theorem 2.** Let \( S \) be a semigroup and \( U^* \neq \emptyset \). Then \( U^* \) is an archimedean ideal.

**Proof.** Let \( x, y \in U^* \). Then by Lemma 2, \( x^n \in (y) = S^1yS^1 \) for some positive integer \( n \). From this we have obtained that \( x^n \in xS^1yS^1x \subseteq U^*yU^* \). Thus \( U^* \) is an archimedean semigroup.

**Theorem 3.** Let \( S \) be a semigroup \( U^* \neq \emptyset \), and let \( A \) be an ideal of \( S \). Then \( A \) in an archimedean ideal if and only if \( A \subseteq U^* \).

**Proof.** Let \( A \) be an archimedean ideal of \( S \). Suppose \( A \not\subseteq U^* \). By Theorem 1, \( U^* = R^*(U^*) \) therefore \( A \not\subseteq R^*(U^*) \). Then we obtain that \( A \) is not a nilideal with respect to the ideal \( U^* \). Therefore there exists an element \( y \in A \) such that \( y^n \notin U^* \) for all positive integers \( n \). Let \( a \in A \cap U^* \). Then \( AaA \subseteq (a) \subseteq U^* \) and \( y^n \notin AaA \) for any positive integer \( n \). This contradicts the assumption that \( A \) is an archimedean semigroup. Therefore for any archimedean ideal \( A \) of \( S \) we have \( A \subseteq U^* \).

Conversely, suppose that \( A \) is an ideal of \( S \) and \( A \subseteq U^* \). Then for any \( x, y \in A \) we have by Theorem 2, \( x^n \in U^*yU^* \) for some positive integer \( n \). Then \( x^n + 2 \in U^*yU^*x \subseteq AyA \), which means that \( A \) is an archimedean semigroup.
We now give a new characterization of the Clifford radical $R^*(J)$.

**Theorem 4.** Let $S$ be a semigroup, $J$ an ideal, $\{Q_\lambda \mid \lambda \in \Lambda\}$ be the set of all prime ideals of $S$ containing $J$ with the property $R^*(Q_\lambda) = Q_\lambda$. Then $R^*(J) = \bigcap_{\lambda \in \Lambda} Q_\lambda$.

**Proof.** By the assumption $J \subseteq Q_\lambda$ implies $R^*(J) \subseteq R^*(Q_\lambda) = Q_\lambda$, hence

$$R^*(J) \subseteq \bigcap_{\lambda \in \Lambda} Q_\lambda.$$ 

Conversely, we show that $\bigcap_{\lambda \in \Lambda} Q_\lambda \subseteq R^*(J)$. If $R^*(J) = S$, then $\bigcap_{\lambda \in \Lambda} Q_\lambda \subseteq S$ holds. Suppose therefore that $R^*(J) \neq S$. It is sufficient to show that for any $x \notin R^*(J)$ there exists an $a \in \Lambda$ such that $x \notin Q_a$. Let $x \notin R^*(J)$. Then the principal ideal $(x) \nsubseteq R^*(J)$ and so there exists $y \in (x)$ such that $y^n \notin J$ for all positive integers $n$. Denote $H = \{y, y^2, y^3, \ldots\}$. We have $H \cap J = \emptyset$. By Lemma 1 there exists a prime ideal $Q_a$ such that $H \cap Q_a = \emptyset$, $Q_a \supseteq J$ and $Q_a = R^*(Q_a)$. We have $x \notin Q_a$ since $x \in Q_a$ would imply $(x) \subseteq Q_a$, hence $y \in Q_a$ a contradiction with $H \cap Q_a = \emptyset$.

Next we shall show some relations concerning radicals and the sets $Q^*$, $U^*$, $P^*$.

Let $S$ be a semigroup with an ideal $J$. The McCoy radical $M(J)$ with respect to $J$ is the intersection of all prime ideals of $S$ containing $J$. The Luh radical $C(J)$ with respect to $J$ is the intersection of all completely prime ideals containing $J$.

If $S$ is a semigroup with a zero $0$, then $M(0) = Q^*$, $C(0) = P^*$ and by Theorem 4, $R^*(0) = U^*$.

The following examples show that there are semigroups with $Q^* \neq U^*$ and $U^* \neq P^*$.

**Example 1.** Let $S_1$ be a semigroup generated by a set $\{0, a_1, a_2, \ldots, a_n, \ldots\}$ subject to the generating relations $0 = x \cdot 0 = x^2$ for any $x \in S$. Then $M(0) = 0$ and $R^*(0) = S_1$ (see [1, p. 232]). Thus in $S_1$ we have $0 = Q^* \neq U^* = S_1$.

**Example 2.** Let $S_2 = \{0, e_{11}, e_{12}, e_{21}, e_{22}\}$ be a semigroup with the multiplication $e_{ik} \cdot e_{kn} = e_{in}$, $e_{ik} \cdot e_{jm} = 0$, $e_{ik} = e_{ik} \cdot 0 = 0$ for $i, j, k, n \in \{1, 2\}, j \neq k$. Then $U^* = R^*(0) = 0$, $P^* = C(0) = S_2$, thus $U^* \neq P^*$. Evidently $U^*$ is not a completely semiprime ideal of $S_2$ since $e_{12}^2 \in U^*$, $e_{12} \notin U^*$.

We note that if $P^*$ is non-empty then it is a completely semiprime ideal of $S$.

**Theorem 5.** Let $S$ be a semigroup. Then $Q^* \subseteq U^* \subseteq P^*$. If $U^* \neq \emptyset$ and $U^* \neq P^*$, then $U^*$ is not a completely semiprime ideal of $S$.

**Proof.** Let $U = \{Q_\lambda \mid \lambda \in \Lambda\}$ be the set of all prime ideals of $S$ with the
property $R^*(Q_\lambda) = Q_\lambda$. Since $U$ is a subset of the set of all prime ideals of $S$ we have $Q^* \subseteq U^*$.

Let $P = \{P_\lambda | \lambda \in \Lambda_1\}$ be the set of all completely prime ideals of $S$. Evidently a completely prime ideal is prime. The inclusions $P_\lambda \subseteq R^*(P_\lambda) \subseteq C(P_\lambda)$ (see [5, Lemma 19]) and the equality $C(P_\lambda) = P_\lambda$ imply $R^*(P_\lambda) = P_\lambda$, for any $\lambda \in \Lambda_1$. Therefore $P$ is the set of all such prime ideals of $S$ which have the property $R^*(P_\lambda) = P_\lambda$ and are completely prime. Hence $P \subseteq U$ and so $U^* \subseteq P^*$.

An ideal $A \subseteq P^*$, $A \neq P^*$ cannot be completely semiprime, since the assumption that $A$ is completely semiprime would imply $A = \bigcap_{\lambda \in \Lambda_2} P_\lambda$ (see [2, Theorem II. 3.7]) where $\Lambda_2 \subseteq \Lambda_1$ hence $P^* \subseteq A$, a contradiction.

Thus if $U^* \neq \emptyset$ and $U^* \neq P^*$, then $U^*$ is not a completely semiprime ideal of $S$.

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НАИБОЛШИЙ АРХИМЕДОВ ИДЕАЛ В ПОЛУГРУППЕ

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Резюме

Пусть $U^*$-пересечение всех простых идеалов $Q$ полугруппы $S$, обладающих свойством $R^*(Q) = Q$, где $R^*(Q)$-радикал Клиффорда относительно $Q$.

Доказано, что если $U^* \neq \emptyset$, то $U^*$ является наибольшим архимедовым идеалом полугруппы.