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GRAPH ISOMORPHISMS OF PARTIALLY ORDERED SETS

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All partially ordered sets considered in this paper are assumed to be of locally finite length. Each such partially ordered set is a multilattice in the sense of [1]. In the formulation of results and proofs below the multilattice terminology will be useful.

In the present paper a theorem on graph isomorphisms of lattices established in [2] will be generalized for the case of directed partially ordered sets.

Let us recall some basic concepts and denotations.

A partially ordered set $\mathcal{P} = (P; \leqq)$ *is said to be of a locally finite length if each bounded chain in* $\mathcal{P}$ *is finite. For elements* $a, b \in P$ *we write* $a \prec b$ *(a is covered by b) if* $a < b$ *and there does not exist any element* $c \in P$ *such that* $a < c < b$. *If* $a, b \in P$, $a \prec b$, *then the ordered pair* $(a, b) \in P \times P$ *is said to be the prime interval* $[a, b]$. *We denote by* $\mathcal{P}^{\sim}$ *the partially ordered set dual to* $\mathcal{P}$.

A multilattice (cf. Benado [1]) *is a poset* $\mathcal{M} = (M; \leqq)$ *in which the condition (i) and its dual are satisfied: (i) If* $a, b, h \in M$, $a \leqq h$, $b \leqq h$, *then there exists* $v \in M$ *such that (a)* $a \leqq v$, $b \leqq v$ and $v \leqq h$ *(b)* $z \in M$, $a \leqq z$, $b \leqq z$, $v \geqq z$ *implies* $z = v$.

Let $a, b, c \in M$ and let $a \leqq c$, $b \leqq c$, the symbol $(a \vee b)_c$ designates the set of all elements $v \in M$ satisfying (*i*) and $v \leqq c$, the symbol $(a \wedge b)_d$ has a dual meaning.

We denote $a \vee b = \bigcup_{\substack{a \leqq c \\ b \leqq c}} (a \vee b)_c$, $a \wedge b = \bigcup_{\substack{a \geqq d \\ b \geqq d}} (a \wedge b)_d$.

In what follows all multilattices are supposed to be directed.

By a graph $G(\mathcal{M})$ *of a multilattice* $\mathcal{M}$ *is meant an unoriented graph whose vertices are elements of* M; *two vertices* a, b *are joined by the edge* (a, b) *iff either* $a \prec b$ *or* $b \prec a$.

Let $\mathcal{M}_1 = (M_1, \leqq)$, $\mathcal{M}_2 = (M_2; \leqq)$ *be multilattices. If* $g\colon M_1 \to M_2$ *is a bijection such that* (x, y) *is an edge in* $G(\mathcal{M}_1)$ *iff* $(g(x), g(y))$ *is an edge in* $G(\mathcal{M}_2)$, *then g will be called a graph isomorphism of the multilattice* M_1 *onto* M_2.

Let $u, v, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ *be distinct elements of* M *such that* (*i*) $u \prec x_1 \prec x_2 \ldots \prec x_m \prec v$, $u \prec y_1 \ldots \prec y_n \prec v$ *and* (*ii*) *either* $v \in x_1 \vee y_1$ *or* $u \in x_m \wedge y_n$. *Then the set* $C = \{u, v, x_1, \ldots, x_m, y_1, \ldots, y_n\}$ *is said to be a cell in* $\mathcal{M}$.

A cell C *in* $\mathcal{M}$ *is called proper if either* $m \geqq 2$, *or* $n \geqq 2$.

A cell C *in* $\mathcal{M}$ *such that* $m = 1$, $n = 1$ *will be called elementary square.*

Let g *be a graph isomorphism of the multilattice* $\mathcal{M}_1 = (M_1, \leqq)$ *onto the multilattice* $\mathcal{M}_2 = (M_2; \leqq)$. *We shall say that an elementary square* $C = \{u, v, x_1, y_1\}$ *in* M_1 *is broken by* g *if either* $g(u) \prec g(x_1)$, $g(u) \prec g(y_1)$, $g(v) \prec g(x_1)$, $g(v) \prec g(y_1)$ *or* $g(x_1) \prec g(u)$, $g(y_1) \prec g(u)$, $g(x_1) \prec g(v)$, $g(y_1)$, $\prec g(v)$.

A cell C *in* M_1 *is called regular under* g *if either for each prime interval* $[x_i, x_{i+1}]$ *of* C *the relation* $g(x_i) \prec g(x_{i+1})$ *is satisfied, or for each prime interval* $[x_i, x_{i+1}]$ *of* C *the relation* $g(x_{i+1}) \prec g(x_i)$ *holds.*

Let us assume in the sequel that $\mathcal{M} = (M; \leqq)$, $\mathcal{M}_1 = (M; \leqq_1)$ are multilattices such that the identity mapping h on M is a graph isomorphism of $\mathcal{M}$ onto $\mathcal{M}_1$. If $c, d \in M$ and $c \leqq_1 d$, then the interval of $\mathcal{M}_1$ with the endpoints c, d will be denoted by $[c, d]_1$. Let P and P_1 be the set of all prime intervals in $\mathcal{M}$ and in $\mathcal{M}_1$, respectively.

We denote $Q = P \cap P_1$, $Q' = P \backslash Q$. The above notions can be applied for $g = h$. Thus a cell C in $\mathcal{M}$ is regular if each prime interval of C belongs to Q or if each prime interval of C belongs to Q'.

An elementary square $C = \{u, v, x_1, y_1\}$ *in* $\mathcal{M}$ *is broken iff either* $u \prec_1 x_1$, $u \prec_1 y_1$, $v \prec_1 x_1$, $v \prec_1 y_1$ *or* $x_1 \prec_1 u$, $y_1 \prec_1 u$, $x_1 \prec_1 v$, $y_1 \prec_1 v$.

Let us consider the following conditions:

(*a*) *There exist multilattices* $\mathcal{A} = (A; \leqq)$, $\mathcal{B} = (B; \leqq)$ *and a bijection* f: $M \to A \times B$ *such that* f *is an isomorphism of* $\mathcal{M}$ *onto* $\mathcal{A} \times \mathcal{B}$ *and at the same time* f *is an isomorphism of* $\mathcal{M}_1$ *onto* $\mathcal{A} \times \mathcal{B}^{\sim}$.

(*b*) *The identity mapping* h *on* M *is a graph isomorphism* $\mathcal{M}$ *onto* $\mathcal{M}_1$ *such that no elementary square of* $\mathcal{M}$ *and* $\mathcal{M}_1$ *is broken.*

(*c*) *All proper cells of* $\mathcal{M}$ *and all proper cells of* $\mathcal{M}_1$ *are regular under the identity mapping* h.

Lemma 1. *Let* (*a*) *be valid. Then* (*b*) *and* (*c*) *hold.*

Proof. The implication $(a) \Rightarrow (b)$ was proved in [6, Theorem 1]. The implication $(a) \Rightarrow (c)$ can be proved analogously to the lemma 2.1 in [2].

Lemma 2. (*Cf.* [6] *lemma* 1.) *Let* $C = \{u, v, x_1, y_1\}$ *be an elementary square in* M. *Let* (*b*) *hold. Then* $[u, x_1] \in Q$ $([u, x_1] \in Q')$ *iff* $[y_1, v] \in Q$ $([y_1, v]\ Q')$.

Now let us assume that the conditions (*b*) and (*c*) are valid.

By the same method as in the proofs of lemmas 2.3—2.6 in [2] the following lemmas can be proved:

Lemma 3. *Let $u, v, x_1, \ldots, x_m, y_1, \ldots, y_n$ be distinct elements of M such that (i) $u \prec x_1 \prec x_2 \prec \ldots \prec x_m \prec v$, $u \prec_1 x_1 \prec_1 \ldots \prec_1 x_m \prec_1 v$, (ii) $u \prec y_1 \prec y_2 \prec \ldots \prec y_n \prec v$. Then $u \prec_1 y_1 \prec_1 \ldots \prec_1 y_n \prec_1 v$.*

Lemma 4. *Let $u, v, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ be distinct elements of M such that (i) $u \prec x_1 \prec x_2 \prec \ldots \prec x_m \prec v$, $u \prec y_1 \prec y_2 \prec \ldots \prec y_n \prec v$, (ii) there are $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$ such that $u \in x_i \wedge y_j$, $v \in x_i \vee y_j$, $u \prec_1 x_1 \prec_1 \ldots \prec_1 x_i$, $u \prec_1 y_1 \prec_1 \ldots \prec_1 y_j$. Then we have $x_i \prec_1 x_{i+1} \prec_1 \ldots \prec_1 x_m \prec_1 v$ and $y_j \prec_1 y_{j+1} \prec_1 \prec_1 \ldots \prec_1 y_n \prec_1 v$.*

Lemma 5. *Let $u, v, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ be distinct elements of M such that the condition (i) from lemma 4 is valid. Assume that there are $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$ such that $u \in x_i \wedge y_j$, $v \in x_i \vee y_j$, $u \succ_1 x_1 \succ_1 \ldots \succ_1 x_i$, $u \succ_1 \ldots \succ_1 y_j$. Then all prime intervals of $[x_i, v]$ and of $[y_j, v]$ belong to Q'.*

Lemma 6. *Let $u, v, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ be distinct elements of M such that the condition (i) from lemma 4 is valid. Assume that there are $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$ such that $u \in x_i \wedge y_j$, $v \in x_i \vee y_j$, all prime intervals of $[u, x_i]$ belong to Q and all prime intervals of $[u, y_j]$ belong to Q'. The all prime intervals of $[y_j, v]$ belong to Q and all prime intervals of $[x_i, v]$ belong to Q'.*

(The main idea of the modification of the proofs is as follows: the assertion $x_1 \vee y_1 < v$ is replaced by the assertion there exists $v_1 \in (x_1 \vee y_1)_r$ such that $v_1 < v$.)

Let $x, y \in M$. We put xR_1y (xR_2y) if there exists an element $v \in x \vee y$ such that all prime intervals of $[x, v]$, $[y, v]$ belong to Q (Q').

The relations R'_1, R'_2 on M are defined analogously (by taking the relation $\leqq_1$).

From lemmas 2,4 and the dual of lemma 4 it follows that xR_1y (xR_2y) is equivalent to each of the following conditions:

(α_1) *If $u \in x \wedge y$, then all prime intervals of $[u, x]$ and $[u, y]$ belong to Q (Q').*

(α_2) *If $v \in x \vee y$, then all prime intervals of $[x, v]$ and $[y, v]$ belong to Q (Q').*

A similar equivalence is valid for the relation R'_i $(i = 1, 2)$.

It is easy to verify that R_1 coincides with R'_1 and R_2 coincides with R'_2.

Lemma 7. *R_1 and R_2 are equivalence relations on M.*

Proof. Let $i \in \{1, 2\}$. Evidently R_i is reflexive and symmetric. The transitivity of R_i can be verified by the same method as in [3, lemma 7].

Lemma 8. *The relations R_1, R_2 satisfy the following conditions:*

(i) $R_1 R_2 = R_2 R_1$

(ii) *$R_1 \cap R_2 = O$, $R_1 \cup R_2 = I$ (O, I is the least (the greatest) element of the lattice of all equivalence relations on M).*

(iii) *If $a, b, c, d \in M$, $a \leqq c$, $a R_1 b$, $b R_2 c$, then $a \leqq b \leqslant c$.*

(iv) *Let $a, b, c, d \in M$, $a R_1 b$, $c R_1 d$, $a R_2 c$, $b R_2 d$. Then from $a \leqq b$ it follows that $c \leqq d$ and from $a \leqq c$ it follows that $b \leqq d$.*

Proof: (*i*) This condition can be proved in the same way as in [3, lemma 10].

(*ii*) Since for each $a, b \in M$ such that $[a, b]$ is a prime interval one of the cases $a R_1 b$, $a R_2 b$ is valid we get $R_1 \cap R_2 = 0$, $R_1 \cup R_2 = I$.

(*iii*) Let $u \in a \vee b$, $v \in b \vee c$. From $a R_1 b$, $b R_2 c$ it follows that all prime intervals of $[a, u]$, $[b, u]$ belong to Q and all prime intervals of $[b, v]$, $[c, v]$ belong to Q'. Let $w \in a \wedge b$. If $r \in (b \wedge c)_w$, then all prime intervals of $[r, b]$ belong to Q by lemma 3 and according to the assertion dual to lemma 5 they belong to Q'. Hence $r = b$ and $b \leqq c$. If $s \in (a \vee b)_v$, then we get $s = b$ and $b \geqq a$.

(*iv*) Let us choose $u \in a \vee c$, $v \in b \vee d$, $w \in c \wedge d$ and let $a R_1 b$, $c R_1 d$, $a R_2 c$, $b R_2 d$. Then all prime intervals of $[w, c]$, $[w, d]$ belong to Q in view of (α_1) and all prime intervals of $[a, u]$, $[c, u]$, $[b, v]$, $[d, v]$ belong to Q' by (α_2). Let $a \leqq b$ and $p \in u \vee b$. Then all prime intervals of $[a, b]$ belong to Q. It follows that $a \in u \vee b$. In view of lemma 6 all prime intervals of $[b, p]$ belong to Q'. Choose $t \in p \vee v$. According to lemma 5 all prime intervals of $[v, t]$ belong to Q'. If $s \in (c \vee d)_t$, then all prime intervals of $[d, s]$ belong simultaneously to Q and Q'. Hence $s = d$. Thus $c \leqq d$. Similarly we can get $b \leqq d$ in view of $a \leqq c$.

The following assertion (K) was proved in [4, Thm. 3.4.2].

(K). *Let M be a quasiordered set. There exists a one-one correspondence between the nontrivial direct decompositions of the quasiordered set M into two factors and pairs (R_1, R_2) of nontrivial equivalence relations R_1, R_2 on M satisfying the properties (i), (ii), (iii), (iv) from lemma* 8. *To each couple (R_1, R_2) with the mentioned properties there corresponds the decomposition $M \sim M/R_1 \times M\, R_2$ and to each element $a \in M$ there corresponds the element (a_1, a_2) where a_i is the equivalence class under R_i, $(i = 1, 2)$ containing a.*

Let R_1, R_2 be the equivalence from lemma 7. From lemma 8 it follows that the equivalences R_1, R_2 and R_1', R_2' satisfy the conditions of the Theorem (K). Denote $\mathscr{M}/R_2 = (A, \leqq) = \mathscr{A}$, $\mathscr{M}/R_1 = (B, \leqq) = \mathscr{B}$, $\mathscr{M}_1/R_2' = (A, \leqq_1) = \mathscr{A}'$, $\mathscr{M}_1/R_1' = (B, \leqq_1) = \mathscr{B}'$. Then there exist isomorphisms:

$$\psi : \mathscr{M} \to \mathscr{A} \times \mathscr{B}$$
$$\psi_1 : \mathscr{M}_1 \to \mathscr{A}' \times \mathscr{B}'$$

defined in the same way as in (K).

Since $\mathscr{M}$, $\mathscr{M}_1$ are multilattices of locally finite length then the partially ordered sets $\mathscr{A}$, $\mathscr{B}$, $\mathscr{A}'$, $\mathscr{B}'$ must be of locally finite length as well and thus $\mathscr{A}$, $\mathscr{B}$, $\mathscr{A}'$, $\mathscr{B}'$ are multilattices.

Lemma 9. *Let $u, v \in A$. Then $u \prec v$ in $\mathscr{A}$ iff $u \prec_1 v$ in $\mathscr{A}'$.*

Proof. Let $b_0 \in B$. Denote $\mathscr{B}_0 = \{b_0\}$.

Let f_1: A $\to \mathscr{A} \times \mathscr{B}_0$
f_1': A$' \to \mathscr{A}' \times \mathscr{B}_0$

be such mappings that to each $a \in A$ there coresponds the element (a, b_0). Then

f_1, f_1' are isomorphisms. Let $u, v \in A$. Then $u \prec v$ in $\mathscr{A}$ iff $\psi^{-1}f_1(u) \prec \psi^{-1}f_1(v)$ in $\mathscr{M}$. It follows that in $\mathscr{M}_1$ either $\psi^{-1}f_1(u) \prec_1 \psi^{-1}f_1(v)$ or $\psi^{-1}f_1(v) \prec_1 \psi^{-1}f_1(u)$. In the second case we have $\psi^{-1}f_1(u)\, R_2\, \psi^{-1}f_1(v)$. Since ψ is an isomorphism and $f_1(u) = (u, b_0)$, $f_1(v) = (v, b_0)$, we obtain $(u, b_0)\, R_2 (v, b_0)$ in $A \times B$. Therefore $u = v$, because $(a, b)\, R_2 (a_1, b_1)$ iff $a = a_1$. But $u = v$ is impossible. Hence $\psi^{-1}f_1(u) \prec_1 \psi^{-1}f_1(v)$ in $\mathscr{M}_1$ and $f_1'^{-1}\psi_1\psi^{-1}f_1(u) = u \prec_1 v = f_1'\psi_1\psi^{-1}f_1(v)$ in $\mathscr{A}'$. The converse implication follows by symmetry.

Analogously we can prove the following lemma:

Lemma 10. *Let $u, v \in B$. Then $u \prec v$ in $\mathscr{B}$ iff $v \prec_1 u$ in $\mathscr{B}'$.*

From lemmas 9 and 10 the following lemma follows immediately.

Lemma 11. *The multilattice $\mathscr{A}$ is isomorphic to $\mathscr{A}'$ and the multilattice $\mathscr{B}$ is dually isomorphic to $\mathscr{B}'$.*

Corollary. *Let $\mathscr{M} = (M, \leqq)$, $\mathscr{M}_1 = (M, \leqq_1)$ be multilattices fulfilling the conditions* (*b*) and (*c*). *Then $\mathscr{M}$ and $\mathscr{M}_1$ satisfy the condition* (*a*).

Now if we consider the multilattices $\mathscr{M} = (M, \leqq)$, $\mathscr{M}_1 = (M_1, \leqq_1)$ and the bijection $h\colon M \to M_1$, then according to lemma 1 and lemma 11 the following assertion is valid.

Theorem. *Let $\mathscr{M} = (M, \leqq)$, $\mathscr{M}_1 = (M_1, \leqq)$ be directed multilattices of locally finite length and let $h\colon M \to M_1$ be a bijection. Then the following conditions are equivalent.*

(β_1) *h is a graph isomorphism of the multilattice $\mathscr{M}$ onto $\mathscr{M}_1$ such that no elementary square of $\mathscr{M}$, $\mathscr{M}_1$ is broken under h or h^{-1}, respectively and all proper cells o $\mathscr{M}$, M_1 are regular under h or h^{-1}, respectively.*

(β_2) *There exist multilattices $\mathscr{A} = (A, \leqq)$, $\mathscr{B} = (B, \leqq)$ and direct representations $f\colon \mathscr{M} \to \mathscr{A} \times \mathscr{B}$, $g\colon \mathscr{M}_1 \to \mathscr{A} \times \mathscr{B}^{\sim}$ such that $h = g^{-1}f$.*

This theorem generalizes the theorem of [2].

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ГРАФОВЫЕ ИЗОМОРФИЗМЫ ЧАСТИЧНО УПОРЯДОЧЕННЫХ МНОЖЕСТВ

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Резюме

Й. Якубик [2] доказал несколько утверждений относительно графовых изоморфизмов решеток, которые не должны быть модулярные. В статье обобщена одна из этих теорем для случая частично упорядоченных множеств локально конечной длины.