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WEAK COMPACTNESS AND SUMMABILITY

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Let $T = (c_{nm})_{n, m \in \mathrm{N}}$ be a method of summability. In agreement with the terminology employed in [2], we will say that T is almost regular* if the following conditions are satisfied:

(i) $\lim\limits_{n \to \infty} \sum\limits_{m=1}^{\infty} c_{nm} = c$;

(ii) $\lim\limits_{n \to \infty} c_{nm} = c_m$ for every $m \in \mathrm{N}$;

(iii) $c \neq \sum\limits_{n=1}^{\infty} c_n$.

Here, of course, c and $c_n (n \in \mathrm{N})$ are finite, and the series are supposed to be convergent.

We will say that a subset of a Banach space has property $\mathscr{A}$ if for every sequence in the subset there is an almost regular* summability method T such that the T-means of the sequence converge weakly.

In [2] D. Waterman established a theorem which can be formulated as saying that the unit ball of a Banach space having property $\mathscr{A}$ is weakly compact. We shall prove the following generalization of this result.

Theorem. *If a bounded subset of a Banach space has property $\mathscr{A}$, then it is weakly relatively compact.*

Proof. Suppose that a bounded subset A of a Banach space E has property $\mathscr{A}$. Should not the weak closure of A be weakly compact, then, by a result of Kadec and Pełczyński [1], there is a basic sequence $(x_n)_{n \in \mathrm{N}}$ in A for which the origin is not a weak cluster point. By passing to a subsequence if necessary, we can assume that there exists x^* in E^*, the dual space of E, such that

$$\sum_{n=1}^{\infty} |x^*(x_n) - 1| < +\infty.$$

Of course, $D = \inf\{\|x_n\| : n \in \mathrm{N}\} > 0$. Let C be a positive number such that, for every choice of n, $m \in \mathbf{N}$ with $n < m$ and scalars λ_i $(1 \leqslant i \leqslant m)$, we have

$$\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \leqslant C \left\| \sum_{i=1}^{m} \lambda_i x_i \right\| .$$

Then

$$\begin{aligned} \left| \sum_{i=1}^{n} \lambda_i \right| &\leqslant \left| \sum_{i=1}^{m} \lambda_i (x^*(x_i) - 1) \right| + \left| \sum_{i=1}^{m} \lambda_i x^*(x_i) \right| \\ &\leqslant D^{-1} \max \{ \| \lambda_i x_i \| : 1 \leqslant i \leqslant m \} \sum_{i=1}^{m} |x^*(x_i) - 1| \\ &+ \| x^* \| \left\| \sum_{i=1}^{m} \lambda_i x_i \right\| \\ &\leqslant \left(2CD^{-1} \sum_{i=1}^{\infty} |x^*(x_i) - 1| + \| x^* \| \right) \left\| \sum_{i=1}^{m} \lambda_i x_i \right\| . \end{aligned}$$

This inequality jointly with the Hahn-Banach theorem shows that there is z^* in E^* such that

$$z^*(x_n) = 1 \tag{1}$$

for all $n \in \mathbf{N}$.

According to our hypothesis, there is an almost regular* summability method $T = (c_{nm})_{n, m \in \mathbf{N}}$ and a point x in the closed linear span of $\{x_n : n \in \mathbf{N}\}$ such that the T-means of $(x_n)_{n \in \mathbf{N}}$ converge weakly to x. Let $(x_n^*)_{n \in \mathbf{N}}$ be a sequence in E^* such that

$$x_n^*(x_m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} .$$

In view of (ii), for each $n \in \mathbf{N}$, the T-means of $(x_n^*(x_m))_{m \in \mathbf{N}}$ converge to c_n. Hence

$$x = \sum_{n=1}^{\infty} c_n x_n$$

and further, in view of (1),

$$z^*(x) = \sum_{n=1}^{\infty} c_n .$$

On the other hand, by virtue of (i) and (1), the T-means of $(z^*(x_n))_{n \in \mathbf{N}}$ converge to c. This and the above equality imply

$$\sum_{n=1}^{\infty} c_n = c,$$

which contradicts (iii).

The proof is complete.

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СЛАБАЯ КОМПАКТНОСТЬ И СУММИРУЕМОСТЬ

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Резюме

В работе показывается, что ограниченное подмножество банахового пространства слабо относительно компактно, если для любой последовательности элементов этого подмножества существует почти регулярный* метод суммируемости T такой, что T-суммы этой последовательности слабо сходятся.