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ON THE OSCILLATION OF A CLASS OF NONLINEAR DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

BOŽENA MIHALÍKOVÁ

1. Introduction

Much attention has been paid recently to the oscillatory properties of nonlinear functional differential equations with deviating arguments. However, most of the published papers dealt with scalar differential equations; comparatively little is known about the properties of systems of differential equations.

Fundamental results concerning the oscillatory properties of two-dimensional systems of differential equations have been obtained by Varech, Gritsai, Ševelo, Kitamura, Kusano. The oscillatory properties of n-dimensional systems were studied by Foltýnska, Werbowski and Marušiak.

The aim of the present paper is to extend certain results from [4, 7, 8] to a differential equation system

\[(p_i(t)\phi_i(x_i(t)))' = f_i(t, x_1(t), \ldots, x_n(t), x_1(\tau_1(t)), \ldots, x_n(\tau_n(t))) \quad i = 1, \ldots, n \quad (A)\]

under the assumption that the following conditions hold:

(a) \( p_i \in C([a; \infty), \mathbb{R}), \ p_i(t) > 0 \) and \( \int_a^\infty \frac{ds}{p_i(s)} = \infty, \ i = 1, \ldots, n; \)

(b) \( \phi_i \in C(\mathbb{R}, \mathbb{R}) \) and \( \phi_i(u) \cdot u > 0 \) for \( u \neq 0, \ |\phi_i(u)| \leq a_i |u|, \ i = 1, \ldots, n; \ a_i > 0, \ const. \)

(c) \( f_i \in C([a; \infty) \times \mathbb{R}^2, \mathbb{R}), \ i = 1, \ldots, n \) and

\[
f_i(t, u_1, \ldots, u_n, v_1, \ldots, v_n) v_{i+1} \begin{cases} > 0 & \text{if } i = 1, \ldots, n - 1 \\ < 0 & \text{if } i = n(v_{n+1} = v_i) \end{cases} \text{ for } v_i, u_i > 0; \]

(d) \( \tau_i \in C([a; \infty), \mathbb{R}) \) and \( \lim_{t \to \infty} \tau_i(t) = \infty, \ i = 1, \ldots, n. \)
The term "solution \( x(t) = (x_1(t), \ldots, x_n(t)) \) of (A)" will be understood in the sequel to refer to a solution of (A) which exists on an interval \([T_x, \infty) \subset [a; \infty)\) and satisfies the condition

\[
\sup \left\{ \sum_{i=1}^{n} |x_i(t)| ; t \geq T \right\} > 0 \text{ for every } T \geq T_x.
\]

A solution \( x(t) \) of (A) is said to be (weakly) oscillatory if each (at least one) of its components has a sequence of zeros tending to \( \infty \).

A solution \( x(t) \) of (A) is said to be (weakly) nonoscillatory if each (at least one) of its components has a constant sign for sufficiently large values of \( t \).

2. Oscillatory theorems

Lemma 1. If \( x = (x_1, x_2, \ldots, x_n) \) is a weakly nonoscillatory solution of (A), then \( x \) is nonoscillatory.

Proof. Suppose that \( x_i(t) \) is a nonoscillatory component of \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) and \( x_i(t) \neq 0 \) for \( t \geq T \geq a \).

1) Let \( 1 < i \leq n \). Owing to (c), (d) we obtain from (A)

\[
(p_i - 1(t)\varphi_i - 1(x_i - 1(t)))' \neq 0 \text{ for } t \geq T_1,
\]

with \( t_1 \) such that \( \tau_i(t) \geq T \text{ for } t \geq t_1 \). From (a) and (b) we see that \( x_{i-1}(t) \) is monotonic and therefore there exists \( t_2 \geq t_1 \) such that \( x_{i-1}(t) \neq 0 \text{ for } t \geq t_2 \). This shows that \( x_{i-1}(t) \) is a nonoscillatory component of \( x \). Analogously it can be shown that the components \( x_{i-2}(t), \ldots, x_i(t) \) are nonoscillatory.

2) Let \( i = 1 \). From the nth equation of (A) we see that

\[
(p_n(t)\varphi_n(x'_n(t)))' \neq 0 \text{ for } t \geq T_1 \geq T
\]

where \( T_1 \) is such that \( \tau_1(t) \geq T \text{ for } t \geq T_1 \). The function is monotonic and from (a) and (b) it is evident that there exists \( t_3 \geq T_1 \) such that \( x_n(t) \neq 0 \text{ for } t \geq T_3 \).

Using the same method as that we used in 1) starting with \( i = n \) we prove that all the components are nonoscillatory.

Now let us consider the system (A) assuming that

\[
f_i(t, u_1, \ldots, u_n, v_1, \ldots, v_n) \text{ sgn } v_{i+1} \geq a_i(t)q_i(v_{i+1}) \text{ sgn } v_{i+1} \geq 0 \text{ for } i = 1, \ldots, n - 1 \quad (1)
\]

\[
f_n(t, u_1, \ldots, u_n, v_1, \ldots, v_n) \text{ sgn } v_1 \leq a_n(t)q_n(v_1) \text{ sgn } v_1 \leq 0
\]

where \( a_i \in C([a; \infty), R), \ a_i(t) \geq 0, \ i = 1, \ldots, n, \ a_i \in C(R; R) \text{ and } q_i(v).v > 0, \ i = 1, \ldots, n - 1, \ q_n(v).v < 0, \ v \neq 0. \)
Lemma 2. Let the conditions (1) and
\[ \lim \inf_{|v| \to \infty} |q_i(v)| \neq 0, \; i = 1, \ldots, n - 1 \] (2)
hold. If
\[ \int_{t_1}^{\infty} a_i(t) \, dt = \infty \; \text{for} \; i = 1, \ldots, n - 1, \] (3)
then for a nonoscillatory solution \( x = (x_1, x_2, \ldots, x_n) \) of (A) we have

1) \( x_1(t) > 0 \) for \( t \geq t_0 \geq a, \; i = 1, \ldots, n; \)

2) there exists \( k \in \{1, \ldots, n\} \) and \( t_0 \geq a \) such that for \( t \geq t_0 \)
   \( x_i(t)x_k(t) > 0, \; i = 1, \ldots, k, \; x_i(t)x_k(t) < 0, \; i = k + 1, \ldots, n; \)

3) there exists a finite limit \( \lim_{t \to \infty} p_k(t)\varphi_k(x_i(t)) = c_k; \)

4) \( \lim_{t \to \infty} x_i(t) = \lim_{t \to \infty} p_i(t)\varphi_i(x_i(t)) = 0, \; i = k + 1, \ldots, n, \; k < n; \)

5) \( \lim_{t \to \infty} x_i(t) = + \infty \; (- \infty), \; i = 1, \ldots, k \) if \( c_k \neq 0, \; k > 1. \)

\( \lim_{t \to \infty} p_i(t)\varphi_i(x_i(t)) = + \infty \; (- \infty), \; i = 1, \ldots, k - 1 \)

Proof. Let \( x(t) = (x_1(t), \ldots, x_n(t)) \) be a nonoscillatory solution of (A) on \([a; \infty)\). Without loss of generality we may suppose that \( x_i(t) > 0 \) for \( t \geq t_0 \geq a \) (the proof is analogous if \( x_i(t) < 0 \)). Owing to assumption (d) there exists \( t_1 \geq t_0 \) such that \( x_1(t_1) > 0 \) for \( t \geq t_1 \). The last equation of (A) leads to the inequality
\[ p_n(t)\varphi_n(x'_n(t)) \leq p_n(t_i)\varphi_n(x'_n(t_i)), \; t \geq t_1. \] (4)

We shall show that there exists \( t_2 \geq t_1 \) such that \( x'_n(t) > 0 \) for \( t \geq t_2 \). For suppose that this were not true. This implies the existence of \( T \geq t_2 \) such that \( x'_n(T) < 0 \) and \( x'_n(t) < 0 \) for \( t \geq T_1 \). From (4)
\[ \alpha_n x_n(t) \leq \alpha_n x_n(T) + p_n(T)\varphi_n(x'_n(T)) \int_{T}^{t} \frac{ds}{p_n(s)} \to - \infty \; \text{for} \; t \to \infty, \]
and therefore \( x_n(t) \to - \infty \; \text{for} \; t \to \infty \). By condition (2) for \( i = n - 1 \) there must exist a constant \( K > 0 \) and \( T_2 \geq T_1 \) such that
\[ q_{n-1}(x_n(t_0)) < - K < 0 \; \text{for} \; t \geq T_2. \]

Using this relation and integrating the \((n - 1)\)th equation of (A), we see that

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\[ p_{n-1}(t)\varphi_{n-1}(x'_{n-1}(t)) \leq p_{n-1}(T_2)\varphi_{n-1}(x'_{n-1}(T_2)) - K \int_{T_2}^{t} a_{n-1}(s) \, ds \to -\infty \quad (5) \]

for \( t \to \infty \),

and therefore there exists \( T_3 \geq T_2 \) such that \( x'_{n-1}(t) < 0 \) for \( t \geq T_3 \). From (5), integrating and taking into consideration (b), we get

\[ a_{n-1} x_{n-1}(t) \leq a_{n-1} x_{n-1}(T_3) + p_{n-1}(T_3)\varphi_{n-1}(x'_{n-1}(T_3)) \int_{T_3}^{t} \frac{ds}{p_{n-1}(s)} \to -\infty \quad \text{for} \ t \to \infty , \]

and therefore \( x_{n-1}(t) \to -\infty \) for \( t \to \infty \). Analogously we show that \( x_i(t) \to -\infty, \ x'_i(t) < 0 \) for \( t \to \infty, \ i = n - 2, \ldots, 1 \), which contradicts the assumption that \( x_1(t) > 0 \) for \( t \geq t_0 \). Therefore \( x'_i(t) > 0 \) for \( t \geq t_2 \). Two cases may now obtain for \( x_i(t) \):

i) there exists \( t_3 \geq t_2 \) such that \( x_i(t) > 0, \ x_i(t_3(t)) > 0 \) for \( t \geq t_3 \);

ii) \( x_i(t) < 0 \) for \( t \geq t_2 \).

Suppose that i) obtains. This means that \( x_n(t) \) is a positive increasing function which either has an upper bound or is unbounded as \( t \to \infty \). In the first case there exist constant \( c > 0 \) and \( t_4 \geq t_3 \) such that \( 0 < c \leq x_n(t_3(t)) \) for \( t \geq t_4 \) and owing to the continuity of \( q_n \), this means that

\[ 0 < m \leq q_{n-1}(x_n(t_3(t))) \leq M, \ m, M \text{ - const., } t \geq t_4. \quad (6) \]

In the second case because of the condition (2) there exist a constant \( K > 0 \) and \( t_5 \geq t_4 \) such that

\[ q_{n-1}(x_n(t_3(t))) \geq K > 0 \text{ for } t \geq t_5. \quad (7) \]

Integrating the \((n - 1)\)st equation of \((A)\) and using (6) and (7), we have

\[ p_{n-1}(t)\varphi_{n-1}(x'_{n-1}(t)) \geq p_{n-1}(t_5)\varphi_{n-1}(x'_{n-1}(t_5)) + L \int_{t_5}^{t} a_{n-1}(s) \, ds \to \infty \quad \text{for} \ t \to \infty , \]

where \( L \) is a suitable positive constant. From this inequality we see that \( x'_{n-1}(t) > 0 \) for \( t \geq t_6 \geq t_2 \) and by suitably transforming and integrating we see that \( x_{n-1}(t) > 0 \) for \( t \geq t_7 \geq t_6 \) as well. Analogously it can be shown that \( x_i(t) > 0, x'_i(t) > 0 \) for \( i = n - 2, \ldots, 1 \) and a sufficiently large \( t \). This proves that 1) holds for \( i = 1, \ldots, n \) and 2) hold for \( k = n \).

Suppose now that ii) obtains. From \((n - 1)\)st equation of \((A)\),

\[ p_{n-1}(t)\varphi_{n-1}(x'_{n-1}(t)) \leq p_{n-1}(t_2)\varphi_{n-1}(x'_{n-1}(t_2)), \ t \geq t_2. \quad (8) \]
We shall show that \( x'_n(t) > 0 \) for \( t \geq t_3 \). We suppose that this is not true and that there exists \( t_4 \geq t_3 \) such that \( x'_n(t_4) < 0 \). Then by (7) \( x'_n(t) < 0 \) for \( t \geq t_4 \) and

\[
\alpha_{n-1} x_{n-1}(t) \leq \alpha_{n-1} x_{n-1}(t_4) + p_{n-1}(t_4) \phi_{n-1}(x'_{n-1}(t_4)) \int_{t_4}^{t} \frac{ds}{p_{n-1}(s)} \to -\infty
\]

so that \( x_{n-1}(t) \to -\infty \) for \( t \to \infty \). Repeating the procedure used in the first part of our proof we arrive at contradiction with the assumption that \( x_i(t) > 0 \) on \([t_0, \infty)\). Thus \( x'_n(t) > 0 \) for \( t \geq t_3 \) and two cases may obtain for \( x_{n-1}(t) \):

i) there exists \( t_4 \geq t_3 \) such that \( x_{n-1}(t) > 0, x_{n-1}(\tau_{n-1}(t)) > 0 \) for \( t \geq t_4 \);

ii) \( x_{n-1}(t) < 0 \) for \( t \geq t_3 \).

For i) we use the same method as for i) to prove that \( x_i(t) > 0, x'_i(t) > 0 \) for \( i = 1, \ldots, n-1 \) and \( t \) sufficiently large, which is exactly what statements 1) and 2) of the Lemma state for \( k = n-1 \).

For ii) we prove analogously as for ii) that \( x'_{n-2}(t) > 0 \) for \( t \geq t_4 \geq t_3 \) and that the following two possibilities exist for \( x_{n-2} \):

i) there exists \( t_5 \geq t_4 \) such that \( x_{n-2}(t) > 0, x_{n-2}(\tau_{n-2}(t)) > 0 \) for \( t \geq t_5 \);

ii) \( x_{n-2}(t) < 0 \) for \( t \geq t_4 \).

The method used in i), ii) is now used repeatedly to prove statements 1) and 2) of the Lemma for \( k = n-2, \ldots, 1 \).

By hypothesis, \( x_1(t) > 0, x_1(\tau_1(t)) > 0 \) for \( t \geq t_0 \) and therefore the function \( p_1(t) \phi_1(x'_1(t)) \) is positive and decreasing and thus has a finite limit. Statement 3) holds for \( k = n \).

If \( k \) has the property 2) then \( p_k(t) \phi_k(x'_k(t)) \) is a positive decreasing function and has a finite limit. If

\[
\lim_{t \to \infty} p_k(t) \phi_k(x'_k(t)) = c_k > 0,
\]

then there exists \( T \geq t_0 \) sufficiently large and such that

\[
\alpha_k p_k(t) x'_k(t) \geq p_k(t) \phi_k(x'_k(t)) \geq \frac{1}{2} c_k,
\]

whence we see by integrating that \( \lim_{t \to \infty} x_k(t) = \infty \). Using (3), (2) and (a) it is easy to prove from the first \( k-1 \) equations of (A) that \( \lim_{t \to \infty} x_i(t) = \infty \) for \( i = 1, \ldots, k-1 \). This proves statement 5) of the Lemma.
Statement 4) will be proved by contradiction. Assume that there exists \( j \in \{k + 1, \ldots, n\} \) such that \( \lim_{t \to \infty} p_j(t) \phi_j(x'_j(t)) = c_j > 0 \). Using the preceding part of our proof this leads to \( \lim_{t \to \infty} p_k(t) \phi_k(x'_k(t)) = \infty \) which contradicts 3). Analogously assume \( t' \) - existence of \( j \in \{k + 1, \ldots, n\} \) such that \( \lim_{t \to \infty} x_j(t) \neq 0 \). Since \( x_j(t) \) is a negative increasing function there exist constants \( c_j, d_j \) and \( T \) sufficiently large such that

\[
c_j \leq x_j(t) \leq d_j < 0, \quad t \geq T
\]

and the continuity of \( q_{j-1} \) implies that there exist constants \( m, M \) such that

\[
m \leq q_{j-1}(x_j(t)) \leq M < 0 \quad \text{for} \quad t \geq T.
\]

From the \((j - 1)\) equation of \((A)\) we have

\[
p_{j-1}(t) \phi_{j-1}(x'_{j-1}(t)) \leq p_{j-1}(T) \phi_{j-1}(x'_{j-1}(T)) + M \int_T^{t'} a_{j-1}(s) \, ds \to -\infty
\]

for \( t \to \infty \),

which again yields a contradiction to 3). This completes the proof of the Lemma.

**Theorem 1.** Suppose that, in addition to the assumptions of Lemma 2,

\[
\int a_n(t) \, dt = \infty \tag{9}
\]

and

\[
\lim_{|v| \to \infty} |q_n(v)| \neq 0, \tag{10}
\]

then every solution of \((A)\) is oscillatory.

**Proof.** Suppose that \((A)\) has a weakly nonoscillatory solution \((x_1(t), \ldots, x_n(t))\). By Lemma 1 this solution is nonoscillatory. Suppose that \( x_1(t) > 0, x_1(t_1) > 0 \) for \( t \geq t_1 \geq a \). By Lemma 2, \( x_1(t) \) is a positive increasing function and there exists \( \lim_{t \to \infty} x_1(t) = d_1 \) such that either \( d_1 < \infty \) or \( d_1 = \infty \). In both cases, owing to (10) and the continuity of \( q_n \), there exist a constant \( L > 0 \) and \( T \) sufficiently large so that

\[
q_n(x_1(t_1)) \leq -L \quad \text{for} \quad t \geq T.
\]

By Lemma 2, \( p_n(t) \phi_n(x'_n(t)) \) is a positive decreasing function. Using these properties, we see after integrating the last equation of \((A)\) that
\[ p_n(t) \varphi'_n(x'_n(t)) - p_n(T) \varphi'_n(x'_n(T)) \leq -L \int_T^t a_n(s) \, ds, \]

which contradicts (9).

Remark 1. Theorem 1 is a generalization of Theorem 2 of [8]. If \( \tau_i(t) = t \) for \( i = 1, 2, \ldots, n \), we obtain the results formulated in Theorem 1 of [7] under weaker assumptions about \( f_i \).

The following example shows that the assumption (10) of Theorem 1 is indispensable.

Example 1. The system

\[
\left( t^2 x'_i(t) \right)' = \frac{\frac{3}{2}}{t} - \frac{\frac{1}{4}}{t^4} \left( x'_2(t^2) \right)^3
\]

\[
\left( t^2 x'_i(t) \right)' = -\frac{\frac{1}{18}}{t} - \frac{\frac{5}{3}}{t^3} \left( 1 + t \right) \frac{x'_i(t^3)}{1 + \left( x'_i(t^3) \right)^2}
\]

satisfies all conditions of Theorem 1 except (10), but the system has a nonoscillatory solution \( (x_1(t), x_2(t)) = (t^3, t^3) \) for \( t > 0 \).

Remark 2. The assumptions of Theorem 1 are rather strong in the sense that the deviating arguments \( \tau_i(t) \) have no influence on the oscillatory properties of solutions of (A).

Theorem 2. Suppose that, in addition to (1) and (3),

\[
\lim \inf\limits_{|v| \to \infty} \frac{q(v)}{v} \neq 0, \quad \lim \inf\limits_{|v| \to 0} \frac{q(v)}{v} \neq 0 \quad \text{for} \quad i = 1, \ldots, n - 1 \quad (11)
\]

holds. If

\[
|q_n(v)| \leq |q_n(u)| \quad \text{for} \quad |v| \leq |u| \quad (12)
\]

and

\[
\infty = \int_T^{\infty} a_k(v_k) \int_{T + \tau_1(v_k)}^{\infty} \frac{1}{p_1(u_1)} \int_{T + \tau_2(u_1)}^{\infty} a_{k+1}(v_{k+1}) \int_{T + \tau_2(v_{k+1})}^{\infty} \frac{1}{p_2(u_2)} \int_{T + \tau_2(v_2)}^{\infty} \cdots
\]

\[
\int_{u_n}^{\infty} a_n(v)|q_n(v|c\int_T^{\tau_1(v)} \frac{1}{p_1(u_1)} \int_T^{\tau_2(u_1)} a_1(v_1) \int_T^{\tau_2(v_1)} \frac{1}{p_2(u_2)} \int_T^{\tau_2(v_2)} \cdots
\]

\[
\int_T^{\tau_1(v-1)} \int_T^{\tau_2(u-1)} a_{k-1}(v_{k-1}) \, dv_{k-1} \cdots du_1 \, dv \cdots dv_k
\]

for every \( c \neq 0 \) and \( k = 1, \ldots, n \); then every solution of (A) is oscillatory.
Proof. Suppose that (A) has a weakly nonoscillatory solution. By Lemma 1 it is nonoscillatory; assume that \( x_i(t) > 0 \) for \( t \geq t_0 \geq a \). The first part of (11) implies the validity of (2) and therefore by Lemma 2 there exist \( k \in \{1, ..., n\} \) and \( T_0 \geq t_0 \) such that for \( t \geq T_0 \) and \( i = 1, ..., k \) all \( x_i(t) \) are positive and increasing and \( \lim_{t \to \infty} x_i(t) = \infty \). Owing to (11) there exist positive constants \( K_i \) and \( T \geq T_0 \) such that

\[
q_i(x_{i+1}(\tau_{i+1}(t))) \geq K_i x_{i+1}(\tau_{i+1}(t)) \quad \text{for} \quad t \geq T, \quad i = 1, ..., k - 1 .
\]

(14)

By transforming the first \((k - 1)\) equations of (A) as follows

\[
ap_i x_i'(t) \geq p_i(t) \phi_i(x_i(t)) \geq p_i(T) \phi_i(x_i(T)) + \int_T^t a_i(s) q_i(x_{i+1}(\tau_{i+1}(s))) \, ds > 0, \quad t \geq T, \quad i = 1, ..., k - 1
\]

and integrating we obtain

\[
x_i(t) \geq \frac{K_i}{a_i} \int_T^t \frac{1}{p_i(u)} \int_T^u a_i(s) x_{i+1}(\tau_{i+1}(s)) \, ds \, du, \quad i = 1, ..., k - 2
\]

(15)

\[
x_{k-1}(t) \geq \frac{d_k K_{k-1}}{a_{k-1}} \int_T^t \frac{1}{p_{k-1}(u)} \int_T^u a_{k-1}(s) \, ds \, du.
\]

(16)

Since the \( k \)th component of the solution is an increasing function there exists a constant \( d_k > 0 \) such that \( x_k(\tau_k(t)) \geq d_k \) for \( t \geq T \). After a transformation of (15), (16) we have

\[
x_1(t) \geq c \int_T^t \frac{1}{p_1(u)} \int_T^{r_1(v_1)} a_1(v_1) \int_T^{r_2(v_2)} a_2(v_2) \int_T^{r_3(v_3)} a_3(v_3) \, dv_1 \, dv_2 \, dv_3 \]

\[
\quad \ldots \int_T^{r_{k-2}(v_{k-2})} \frac{1}{p_{k-1}(u_{k-1})} \int_T^{u_{k-1}} a_{k-1}(v_{k-1}) \, dv_{k-1} \, du_{k-1} \, \ldots \, dv_2 \, du_2 \, dv_1 \, du_1,
\]

(17)

where \( c = d_k \prod_{i=1}^{k-1} K_i/a_i \).

By Lemma 2, \( \lim_{t \to \infty} x_i(t) = 0 \), \( \lim_{t \to \infty} p_i(t) \phi_i(x_i(t)) = 0 \) for \( i = k + 1, ..., n \); therefore the \((k + 1)\)st to the nth equation of (A) yield

\[
|x_i(\tau_i(t))| \geq \frac{1}{a_i} \int_T^\tau_T \frac{1}{p_i(u)} \int_T^u a_i(s) q_i(x_{i+1}(\tau_{i+1}(s))) \, dv \, du.
\]

(18)
Further, owing to (11) there exist constants $M_i > 0$ and $T_i \geq T$ such that
\[ |q_i(x_i + 1(\tau_i + 1(t)))| \geq M_i |x_i + 1(\tau_i + 1(t))|, \quad t \geq T_i, \quad i = k + 1, \ldots, n - 1. \]

Using this property, we can transform (18) to obtain
\[ |x_{k + 1}(t)| \geq D \int_{\tau_k(u_{k + 1})}^{\infty} \frac{1}{p_{k + 1}(u_{k + 1})} \int_{u_{k + 1}}^{\infty} a_{k + 1}(v_{k + 1}) \int_{\tau_{k + 1}(v_{k + 1})}^{\infty} \frac{1}{p_{k + 2}(u_{k + 2})} \int_{u_{k + 2}}^{\infty} a_{k + 2}(v_{k + 2}) \int_{\tau_{k + 2}(v_{k + 2})}^{\infty} \ldots \]
\[ \int_{\tau_n(v_{n - 1})}^{\infty} a_n(v)|q_n(x_1(\tau_1(v)))| \, dv \, du_n \ldots dv_{k + 1} \, du_{k + 1}, \]
\[ t \geq T_1, \quad D = \frac{1}{\prod_{i=k+1}^{n-1} \alpha_i} \prod_{i=k+1}^{n-1} M_i. \]

Now by Lemma 2 there exists a finite limit $\lim_{t \to \infty} p_k(t)\varphi_k(x_k(t)) = L$. Integrating the $k$th equation of (A) we have after some manipulations
\[ |L - p_k(T_1)\varphi_k(x_k(T_1))| \geq M_k \int_{T_1}^{\infty} a_k(s)|x_{k + 1}(\tau_{k + 1}(s))| \, ds. \]

Using the fact that $|q_n(v)|$ is nondecreasing we substitute (17) into (19). The resulting expression is then substituted into (20) and this yields a contradiction to (13).

**Corollary 1.** If in addition to the assumptions of Theorem 2 with the exception of the second condition in (11)
\[ \frac{q_i(u)}{v} \geq \frac{q_i(u)}{u} \quad \text{for} \quad |u| \leq |v|, \quad i = 1, \ldots, n - 1, \]
holds, then every solution of (A) is oscillatory.

**Example 2.** The system
\[ \left(\left(\frac{3}{13} t^{-\frac{5}{6}} + \frac{1}{3} t^{-\frac{2}{3}}\right)x_1'(t)\right)' = 2t^{-\frac{1}{2}}((x_2(\tau_2(t)))^\frac{5}{3} + x_2(\tau_2(t))) \]
\[ (t^{-\frac{1}{2}}x_2'(t))' = -\frac{1}{2} t^{-3}(x_1(\tau_1(t)))^3, \quad t \geq 0 \]
with the deviating arguments $\tau_1(t) = t^{12}, \tau_2(t) = t^2$ has an nonoscillatory solution $(x_1(t), x_2(t)) = (t^4, t^2)$ for $t \geq 0$ (since for $k = 1$ the assumption (13) does not
hold), but for the deviating arguments \( \tau_1(t) = t^2, \tau_2(t) = \frac{1}{t^4} \) every solution of the system is oscillatory.

**Example 3.** For the system

\[
(t^{-3}x_1'(t))' = t^{-\frac{7}{2}} \left( \frac{15}{4} + t^2 \right) \left( t + \frac{\pi}{2} \right)^{\frac{1}{2}} x_2 \left( t + \frac{\pi}{2} \right)
\]

\[
(t^{-3}x_2'(t))' = -t^{-\frac{3}{2}} \left( t^2 - \frac{1}{4} \right) \left( t + \frac{\pi}{2} \right)^{-\frac{3}{2}} x_1 \left( t + \frac{\pi}{2} \right)
\]

all the conditions of Theorem 2 are satisfied and therefore every one of its solutions is oscillatory on \([\pi; \infty)\).

\((x_1(t), x_2(t)) = (t^3 \sin t, t^{-\frac{1}{2}} \cos t)\) is one such solution.

We shall now study the behaviour of (A) under the following assumptions:

\[
f(t, u_1, \ldots, u_n, v_1, \ldots, v_n) \sgn v_{i+1} = a_i(t)g_i(u_{i+1}) \sgn u_i \geq 0, \ i = 1, \ldots, n - 1
\]

\[
f_n(t, u_1, \ldots, u_n, v_1, \ldots, v_n) \sgn v_1 = g_n(t, v_1) \sgn v_1 \leq 0,
\]

where

\[
a_i \in C([a; \infty), \mathbb{R}), a_i(t) \geq 0, \ i = 1, \ldots, n - 1;
\]

\[
g_i \in C(\mathbb{R}; \mathbb{R}), g_i(v) > 0 \text{ for } v \neq 0, \ i = 1, \ldots, n - 1;
\]

\[
g_n \in C([a; \infty) \times \mathbb{R}; \mathbb{R}), g_n(t, v) . v < 0 \text{ for } v \neq 0.
\]

Let \( i_k \in \{1, 2, \ldots, 2n - 1\}, 1 \leq k \leq 2n - 1 \) and \( t, s \in [a; \infty) \). Define

\[
I_0(t, s) = I_0(t, s) = 1
\]

\[
I_k(t, s; y_{i_k}, \ldots, y_{i_1}) = \int_s^t y_{i_k}(x)I_{k-1}(x, s; y_{i_{k-1}}, \ldots, y_{i_1}) \, dx,
\]

\[
J_k(t, s; y_{i_k}, \ldots, y_{i_1}) = \int_s^t y_{i_{1}}(x)J_{k-1}(t, x; y_{i_k}, \ldots, y_{i_2}) \, dx.
\]

Further let us introduce the following notation

\[
R_k(t, T) = I_{2n-1} \left( t, T; \frac{1}{p_1}, a_1, \frac{1}{p_2}, a_2, \ldots, \frac{1}{p_{k-1}}, a_{k-1}, \frac{1}{p_n}, a_{n-1}, \ldots, a_k, \frac{1}{p_k} \right)
\]

\[1 \leq k \leq n.\]

\( R_k(t, a) = R_k(t) \)

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Lemma 3. Suppose that, in addition to (21),

$$\liminf_{|w| \to \infty} |g_i(u)| \neq 0 \text{ for } i = 1, \ldots, n - 1,$$

and

$$\int_{-\infty}^{\infty} a_i(t) \, dt = \infty \text{ for } i = 1, \ldots, n - 1.$$  \hfill (23)

Then for any nonoscillatory solution $x = (x_1, \ldots, x_n)$ the statements 1) to 5) of Lemma 2 hold.

The proof of the Lemma is analogous to that of Lemma 2.

Lemma 4. Suppose that, in addition to (21) and (23),

$$\frac{g_i(u)}{u} \leq \frac{g_i(v)}{v} \text{ for } |u| \leq |v|, \quad i = 1, \ldots, n - 1.$$ \hfill (24)

Then for any nonoscillatory solution $x = (x_1, \ldots, x_n)$ of (A) and $a \leq s < t$ we have

$$|p_i(t)\varphi_i(x_i(t))| \geq \prod_{i=1}^{k-1} \frac{|g_i(x_{i+k-1}(t))|}{|\alpha_{i+k-1}|} \int_{x_i(t)}^{\varphi_i(x_{i+k-1}(t))} \varphi_{i+k-1}(u) \, du, \quad 1 \leq j \leq k - 1;$$

and

$$|p_k(s)\varphi_k(x_k(s))| \geq \prod_{i=k}^{n} \frac{|g_i(x_{i+k-1}(s))|}{|\alpha_{i+k-1}|} \int_{x_i(s)}^{\varphi_i(x_{i+k-1}(s))} \varphi_{i+k-1}(u) \, du, \quad k \leq j \leq n - 1,$$

where $k = 1, \ldots, n - 1$ is determined according to Lemma 3.

Proof. Let $x = (x_1, \ldots, x_n)$ be a nonoscillatory solution of (A) defined on $[a; \infty)$ and suppose that $x_i(t) > 0$, $x_i(t) > 0$ for $t \geq t_0 \geq a$. The condition (24) implies the validity of (22). Thus by Lemma 3 there exist $T \geq t_0$ and $k \in \{1, \ldots, n\}$ such that $x_i(t) > 0$ for $i = 1, \ldots, n$, $x_j(t) > 0$ for $j = 1, \ldots, k$, $x_j(t) < 0$ for $j = k + 1, \ldots, n$ and $t \geq T$.

To prove (25), we shall use the monotonicity of the first $k$ components of the solution, the relations (21) and (24), the first $(k - 1)$ equations of (A) and integration by parts.
Suppose that \( T \leq s < t \). Then

\[
p_1(t)\varphi_1(x'_1(t)) = p_1(s)\varphi_1(x'_1(s)) + \int_s^t (p_1(z)\varphi_1(x'_1(z)))' \, dz \geq
\]

\[
\geq \int_s^t a_1(z)g_1(x_1(z)) \, dz \geq \frac{g_1(x_2(s))}{x_2(s)} \int_s^t a_1(z)x_2(z) \, dz =
\]

\[
= g_1(x_2(s)) \cdot J_1(t, s; a_1) + \frac{g_1(x_2(s))}{x_2(s)} \int_s^t x_2(z)J_1(t, z; a_1) \, dz \geq
\]

\[
\geq \frac{g_1(x_2(s))}{a_2 x_2(s)} \int_s^t \varphi_2(x'_2(z))J_1(t, z; a_1) \, dz ,
\]

which is (25) for \( j = 1 \). Integrating the last integral we have

\[
p_1(t)\varphi_1(x'_1(t)) \geq \frac{g_1(x_2(s))}{a_2 x_2(s)} \cdot p_2(s)\varphi_2(x'_2(s))J_2(t, s; a_1, \frac{1}{p_2}) +
\]

\[
+ \frac{g_1(x_2(s))}{a_2 x_2(s)} \int_s^t a_3(z)g_3(x_3(z))J_2(t, z; a_1, \frac{1}{p_2}) \, dz \geq
\]

\[
\geq \frac{g_1(x_2(s))}{a_2 x_2(s)} \cdot \frac{g_2(x_3(s))}{x_3(s)} \int_s^t a_2(z)x_3(z)J_2(t, z; a_1, \frac{1}{p_2}) \, dz .
\]

By the above transformations and (2j-2) integrations we obtain (25).

To prove (26), we use the last \((n - k + 1)\) equations of (A), the relations (21) and (24) and the properties of the last \((n - k + 1)\) components of the solution as well as the fact that they are negative increasing functions.

For \( T \leq s < t \) we have

\[
p_k(s)\varphi_k(x'_k(s)) = p_k(t)\varphi_k(x'_k(t)) - \int_s^t (p_k(u)\varphi_k(x'_k(u)))' \, du \geq
\]

\[
\geq - \int_s^t a_k(u)g_k(x_{k+1}(u)) \, du \geq - \frac{g_k(x_{k+1}(t))}{x_{k+1}(t)} \int_s^t a_k(u)x_{k+1}(u) \, du =
\]

\[
= - g_k(x_{k+1}(t))I_1(t, s; a_k) + \frac{g_k(x_{k+1}(t))}{x_{k+1}(t)} \int_s^t \varphi_{k+1}(x'_{k+1}(u))I_1(u, s; a_k) \, du \geq
\]

\[
\geq \frac{g_k(x_{k+1}(t))}{a_{k+1}x_{k+1}(t)} \int_s^t p_{k+1}(u)\varphi_{k+1}(x'_{k+1}(u)) \frac{1}{p_{k+1}(u)}I_1(u, s; a_k) \, du ,
\]

which is (26) for \( j = k \). Again integrating by parts and using the above properties \((n-1-k)\) times we obtain (26).

For \( x_1(t) < 0 \) the proof is analogous.
Theorem 3. Suppose that, in addition to the assumptions of Lemma 4, the following conditions hold:

1) \( \liminf_{|u| \to 0} \frac{g_i(u)}{u} \neq 0 \) for \( i = 1, \ldots, n - 1 \);

2) \( \frac{|g_n(t, u)|}{|u|^\beta} \leq \frac{|g_n(t, v)|}{|v|^\beta} \) for \( |u| \leq |v|, \beta > 1 \);

3) There exists a function \( h(t) \) continuous and differentiable on \([a; \infty)\), such that \( 0 < h(t) \leq \tau_i(t), h'(t) \geq 0, \lim_{t \to \infty} h(t) = \infty \).

If

\[
\int_{-\infty}^{\infty} R_k(h(t)) |g_n(t, c)| \, dt = \infty \text{ for all } c \neq 0 \text{ and } k = 1, \ldots, n, \tag{27}
\]

then all solutions of (A) are oscillatory.

Proof. Suppose that (A) has a weakly nonoscillatory solution \( x = (x_1, \ldots, x_n) \). By Lemma 1 this solution is nonoscillatory and without loss of generality we may assume that \( x_i(t) > 0, x_i(h(t)) > 0 \) for \( t \geq t_0 \geq a \). By Lemma 3, \( \lim_{t \to \infty} x_i(t) = 0 \) for \( i = k + 1, \ldots, n \) and by the assumption 1) there exist constants \( \delta_i > 0 \) and \( T \geq t_0 \) such that

\[
\frac{g_i(x_{i+1}(t))}{x_{i+1}(t)} \geq \delta_i, \quad i = k + 1, \ldots, n - 1, \quad t \geq T.
\]

Since \( p_n(t) \varphi_n(x_n'(t)) \) is decreasing, we have the following relation from (26) for \( j = n - 1 \):

\[
p_k(s) \varphi_k(x_k'(s)) \geq p_n(t) \varphi_n(x_n'(t)) \prod_{i=k}^{n-1} \delta_i \frac{1}{\alpha_{i+1}} I_{2(n-k)} \left( t, s; \frac{1}{p_n}, \frac{1}{p_k}, \ldots, \frac{1}{p_{k+1}} \right),
\]

\[
T \leq s < t. \tag{28}
\]

Substituting (28) into (25) for \( s = T, j = k - 1 \) we have

\[
p_1(t) \varphi_1(x_1'(t)) \geq \alpha p_n(t) \varphi_n(x_n'(t)) \int_{T}^{t} I_{2(n-k)} \left( t, u; \frac{1}{p_n}, \frac{1}{p_k}, \ldots, \frac{1}{p_{k+1}} \right) \times
\]

\[
\times \frac{1}{p_k(u)} \int_{T}^{t} \frac{J_{2k-3}}{p_{k-1}} \left( t, u; \frac{1}{p_2}, \ldots, \frac{1}{p_{k-1}}, \frac{1}{p_{k-1}} \right) \, du,
\]

where \( \alpha = \prod_{i=1}^{k-1} \frac{q_i(x_{i+1}(T))}{\alpha_{i+1} x_{i+1}(T)} \prod_{i=k}^{n-1} \delta_i \prod_{i=k+1}^{n} \alpha_{i+1} \).
and therefore

\[
x'_i(t) \geq \alpha p_n(t) \varphi_n(x'_n(t)) \frac{1}{p'_1(t)} \times \\
\times I_{2n-2}(t, T; a_1, \frac{1}{p_2}, \ldots, a_{k-1}, \frac{1}{p_{k-1}}, a_n, \ldots, a_k, \frac{1}{p_k}) \tag{29}
\]

Taking \( t_1 \geq T \) such that \( h(t_1) \geq T \) for \( t \geq t_1 \), calculate the following derivative using the nth equation of (A), the relation (29) and assumption 2 of the theorem:

\[
[R_k(h(t), T)p_n(t)\varphi_n(x'_n(t))x_i^{-\beta}(h(t))]' \leq \\
\leq [R_k(h(t), T)]' h'(t)p_n(t)\varphi_n(x'_n(t))x_i^{-\beta}(h(t)) + \\
+ R_k(h(t), T)x_i^{-\beta}(h(t))g_n(t, x_i(t_1(t))) \leq \\
\leq \frac{a_i}{\alpha} x_i'(h(t))h'(t)x_i^{-\beta}(h(t)) + R_k(h(t), T)g_n(t, K). K^{-\beta},
\]

where \( K = x_i(T) \).

Integrating the last inequality yields after necessary manipulations

\[
-K^{-\beta} \int_{t_1}^{t'} R_k(h(s), T)g_n(s, K) \, ds \leq a_i x_i^{-\beta}(h(t_1)) + \\
+ R_k(h(t_1), T)p_n(t_1)\varphi_n(x'_n(t_1))x_i^{-\beta}(h(t_1)).
\]

The right-hand part of this inequality is a finite positive number. Therefore the integral is convergent, which is a contradiction to (27).

Example 4. The system

\[
(t^{-\frac{1}{2}}x'_i(t))' = 4t^{-\frac{1}{2}}x_2(t), \\
(t^{-3}x'_2(t))' = -\frac{7}{4} (t^{-\frac{9}{2}} + t^{-\frac{3}{2}}) x_i^5(x_1(t)) \\
\frac{1}{1 + x_i(x_1(t))}
\]

with \( x_i(t) = x_2(t) = t \) has a nonoscillatory solution \( (x_i(t), x_2(t)) = (t^4, t^2) \) for \( t \geq 0 \). For \( x_i(t) = t^4, x_2(t) = t^2 \) every solution is oscillatory.

The following theorem presents a sufficient condition for the oscillation of all solutions of (A) if \( 0 < \beta < 1 \) in condition 2) of Theorem 3.

Let

\[
\tau_i(t) = \min (\tau_i(t), t) \\
P'_0(t, T) = 1
\]

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\[ P_{2j}(t, T) = I_{2j}\left( t, T; \frac{1}{p_1}, a_1, \frac{1}{p_2}, a_2, \ldots, \frac{1}{p_j}, a_j \right) \]

\[ P_{2j+1}(t, T) = I_{2j+1}\left( t, T; \frac{1}{p_1}, a_1, \frac{1}{p_2}, a_2, \ldots, \frac{1}{p_{j+1}}, a_{j+1} \right) \]

\[ P_k(t, a) = P_k(t), \quad 0 \leq k \leq 2n - 2. \]

**Theorem 4.** If in addition to the assumptions of Lemma 4

1) \( \lim_{|u| \to 0} \frac{g_i(u)}{u} \neq 0 \) for \( i = 1, \ldots, n - 1; \)

2) \( \frac{|g_n(t, u)|}{|u|^\beta} \leq \frac{|g_n(t, v)|}{|v|^\beta} \) for \( |u| \leq |v|, \quad 0 < \beta < 1 \)

and

\[
\int_0^\infty \left( \frac{R_k(\tau_i(t))}{P_{2k-2}(\tau_i(t))} \right)^\beta |g_n(t, cP_{2k-2}(\tau_i(t)))| \, dt = \infty \text{ for all } c \neq 0, \quad k = 1, \ldots, n. \tag{30}
\]

Then every solution of (A) is oscillatory.

**Proof.** The proof will be indirect. We start by repeating the proof of Theorem 3 up to and including the inequality (29). Integrating this inequality from \( T \) to \( t \geq T \) we have

\[ x_i(t) \geq \frac{\alpha}{a_1} p_n(t) \varphi_n(x_n'(t)) R_k(t, T). \tag{31} \]

By Lemma 3 \( x_i(t) \) is increasing and \( p_n(t) \varphi_n(x_n'(t)) \) decreasing. Using this, it is possible to transform (31) as follows:

\[
(p_n(t) \varphi_n(x_n'))^{-\beta} \geq (p_n(\tau_i(t)) \varphi_n(x_n'(\tau_i(t))))^{-\beta} 
\geq \left( \frac{\alpha}{a_1} \right)^\beta R_k^{\beta}(\tau_i(t), T)x_i^{-\beta}(\tau_i(t)) \geq \left( \frac{\alpha}{a_1} \right)^\beta R_k^{\beta}(\tau_i(t), T)x_i^{-\beta}(\tau_i(t)), \tag{32}
\]

where \( t \geq t_1 \geq T \) such that \( \tau_i(t) \geq T \) for \( t \geq t_1. \)

Starting with (25) for \( j = k - 2, \ s = T, \) integrating by parts and using the \( (k - 1) \)th equation of (A) and the monotonicity of \( x_k \) leads to

\[ p_1(t) \varphi_1(x_1(t)) \geq g_{k-1}(x_{k-1}(T)) \times \]

\[ \times \prod_{i=1}^{k-2} \frac{g_i(x_{i+1}(T))}{a_i + 1(x_{i+1}(T))} J_{2k-3} \left( t, T; \frac{1}{p_2}, a_2, \ldots, \frac{1}{p_{k-1}}, a_{k-1} \right). \]
Integrating the last inequality from $T$ to $t \geq T$ we have

$$x_i(t) \geq cP^1_{2k - 2}(t, T),$$

where $c = \frac{g_k - 1(x_k(T))k - 2}{\prod_{i=1}^{k} a_i x_i(T)}$.

(33)

Using the $n$th equation of (A), the relations (33) and (32) and condition 2) we see that

$$[(p_n(t) \varphi_n(x'_n(t)))^{1 - \beta}]' = (1 - \beta)(p_n(t) \varphi_n(x'_n(t)))^{-\beta}(p_n(t) \varphi_n(x'_n(t)))' \leq

\leq (1 - \beta) \left( \frac{\alpha}{\alpha_1} \right)^\beta R^\beta_k(\tau_1(t, \tau_1(t), T)x_1^{-\beta}(t))g_n(t, x_1(\tau_1(t))) \leq

\leq (1 - \beta) \left( \frac{\alpha}{\alpha_1} \right)^\beta R^\beta_k(\tau_4(t), T)(P^1_{2k - 2}(\tau_1(t), T))^{-\beta}g_n(t, cP^1_{2k - 2}(\tau_1(t)))].$$

Integrating the last inequality yields a contradiction to (30). This completes the proof.

Remark 4. For the case when (A) is equivalent to a differential equation with deviating arguments of order $2n$ the theorem yields a result proved in [5].

Example 5. If for some $k \in \{1, \ldots, n\}$ the assumption (30) is not satisfied, then there may exist nonoscillatory solutions of the system. The system

$$\left( \frac{-x'_1(t)}{t} \right)' = 3 \cdot t^{-2} x_1(t)$$

$$\left( \frac{-x'_2(t)}{t^2} \right)' = - \frac{2}{t^3} x_1(t)$$

does not satisfy (30) for $k = 2$ and has a nonoscillatory solution $(x_1(t), x_2(t)) = (t^3, t^2)$ for $t \geq 0$.

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О КОЛЕБЛЕМОСТИ РЕШЕНИЙ НЕЛИНЕЙНЫХ СИСТЕМ
С ОТКЛЮЧАЮЩИМСЯ АРГУМЕНТОМ

Božena Mihalíková

Резюме

В статье приведены достаточные условия колеблемости решений системы (A).