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LOCALLY SNAKE-LIKE GRAPHS

BOHDAN ZELINKA

This paper continues the study of local properties of graphs pursued by various authors, eg. [1]—[4]. Surveys of these investigations are in [5] and [6].

Let G be an undirected graph (without loops and multiple edges), let v be its vertex. The symbol $N_G(v)$ denotes the subgraph of G induced by the set of all vertices which are adjacent to v in G.

A snake is a graph whose vertices and edges form a path. (We distinguish a "path" as a sequence of vertices and edges and a "snake" as a graph. But a graph which is called here "snake" is often called also "path".) In other words, a snake is a connected graph in which exactly two vertices have the degree 1 and all others have the degree 2. The length of a snake is the number of its edges.

A graph G with the property that $N_G(v)$ is a snake of the length at least 1 for each vertex v of G will be called a locally snake-like graph.

Here we shall study locally snake-like graphs which are finite, planar and 3-connected. We require them to be 3-connected, because in this case the faces of a finite planar graph are uniquely determined. The number of edges forming a boundary of a face will be called the degree of that face.

Note that if a locally snake-like graph G is 3-connected and not isomorphic to K_3, then $N_G(v)$ for each v is a snake of the length at least 2; otherwise G would contain a vertex of the degree 2 and thus it would not be 3-connected.

Theorem 1. *Let G be a finite planar 3-connected graph. Then the following two assertions are equivalent:*

(i) *G is locally snake-like.*

(ii) *Each vertex of G is adjacent to exactly one face of degree greater than* 3 *and each triangle in G is the boundary of a face of G.*

Proof. (i) ⇒ (ii). First we shall prove thateach triangle in G. Consider an arbitrary triangle T in G with the vertices u, v, w. Suppose that T is not the boundary of a face of G. Then $V(G) - \{u, v, w\}$ (where $V(G)$ is the vertex set of G) is the union of two non-empty disjoint sets A, B with the property that each path in G connecting a vertex of A with a vertex of B contains at least one of the vertices u, v, w. Each of the vertices u, v, w is adjacent to a vertex of A and to a vertex of B; otherwise the remaining two of these vertices would form a two-element cutset in G and G would not be 3 connected. Consider the graph $N_G(u)$. It contains the edge vw and moreover at least one vertex of A and at least one vertex of B. As $N_G(u)$ is a snake, it contains a path connecting a vertex of A with a vertex of B; this path goes through the edge vw. Hence there exists a vertex $x_1 \in A$ adjacent to u and to one of the vertices v, w; without loss of generality let x_1 be adjacent to v. Now consider $N_G(w)$; analogously we prove that there exists a vertex $x_2 \in A$ adjacent to w and to one of the vertices u, v; without loss of generality suppose that x_2 is adjacent to u. Now return to $N_G(u)$. It contains a path connecting x_1 with x_2 and having inner vertices v, w. Both terminal vertices of this path are in A. But u is adjacent also to a vertex $x_3 \in B$. Thus $N_G(u)$ must contain a path connecting x_3 with x_1 or x_2. As $x_1 \in A$, $x_2 \in A$, $x_3 \in B$, such a path must contain v or w (the vertex u is not in $N_G(u)$). But then v or w is adjacent in $N_G(u)$ to a vertex of B and thus it has the degree at least 3 in $N_G(u)$, which is a contradiction with the assumption that $N_G(u)$ is a snake. Hence T is the boundary of a face in G; as it was chosen arbitrarily, this is true for all triangles in G.

Now consider a vertex v of G. As G is locally snake-like, the graph $N_G(v)$ is a snake. Let its vertices be $u_0, u_1, \ldots, u_k$, let its edges be $u_{i-1}u_i$ for $i = 1, \ldots, k$. Then there exist triangles $u_{i-1}u_iv$ in G for $i = 1, \ldots, k$; they are faces of G, as it was proved above. The vertices u_1, u_k are not adjacent (otherwise $N_G(v)$ would contain a circuit), thus the edges vu_1, vu_k belong to a face of G of a degree greater than 3.

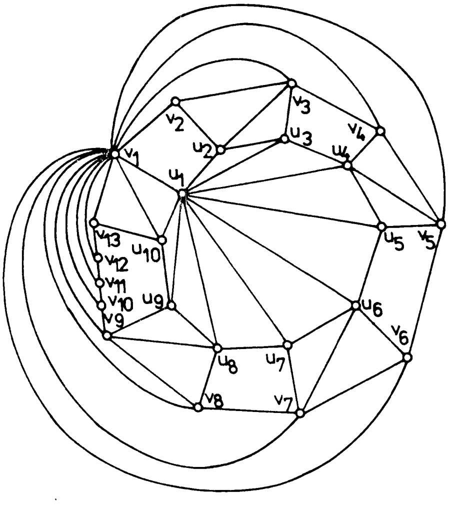
(ii) $\Rightarrow$ (i). Let $F_1, \ldots, F_k$ be the faces incident with a vertex v of G such that F_i and F_{i+1} for $i = 1, \ldots, k-1$ have a common edge vu_i and F_k, F_1 have a common edge vu_k. Let $F_2, \ldots, F_k$ be triangular, let F_1 have more than 3 edges. Then the vertex set of $N_G(v)$ is $\{u_1, \ldots, u_k\}$ and this graph contains the edges u_iu_{i+1} for $i = 1, \ldots, k-1$. Suppose that there exists an edge joining u_i, u_j, where $1 \leqq i < < i+1 < j \leqq k$. If $i = 1, j = k$, then the vertices v, u_1, u_k form a triangle and this is the boundary of F_1; the face F_1 is triangular, which is a contradiction. If $i > 1$ (or $j > k$), then the vertices v, u_i, u_j form a triangle and simultaneously they separate u_{i-1} from u_{i+1} (or u_{j-1} from u_{j+1}); thus they form a triangle which is not a face of G, again a contradiction. Hence $N_G(v)$ is a snake. As this holds for each vertex v of G, the graph G is locally snake-like. □

Corollary. *Let G be a finite planar 3-connected locally snake-like graph. Then the circuits of G which are boundaries of faces of degrees at least 4 form a spanning subragph of G, all of whose connected components are circuits.*

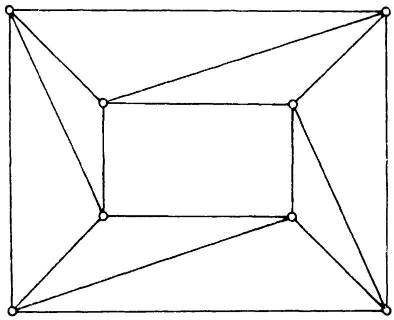
Theorem 2. *The maximum number of edges of a finite planar 3-connected locally snake-like graph with n vertices, where $n \geqq 8$, is equal to $2n + 3[n/4] - 6$.*

Proof. Let G be a planar 3-connected locally snake-like graph with $n \geqq 8$ vertices. Let G_0 be its spanning subgraph described in Corollary. Let the circuits of G_0 be $D_1, \ldots, D_k$, let d_i be the length of D_i for $i = 1, \ldots, k$. Obviously $\sum_{i=1}^{k} d_i = n$. As each D_i is the boundary of a face of G, it has no chord inside this face. It has neither a chord outside this face, because then the end vertices of this chord would form a cutset in G and G would not be 3-connected. Hence none of the circuits $D_1, \ldots, D_k$ has a chord. Thus to each D_i we may add $d_i - 3$ chords in such a way that the graph G_1 obtained from G by adding them is also planar (obviously we put them inside the face bounded by this circuit). If m denotes the number of edges of G, then G_1 has $m + \sum_{i=1}^{k} (d_i - 3) = m + n - 3k$ edges. The maximum number of edges of a planar graph with n vertices is $3n - 6$; hence $m + n - 3k \leqq 3n - 6$, which implies $m \leqq 2n + 3k - 6$. As $d_i \geqq 4$ for $i = 1, \ldots, \ldots, k$, the maximum possible k is $[n/4]$ and this gives $m \leqq 2n + 3[n/4] - 6$.

Now let $n \geqq 8$ be given. Let r be the number from the set $\{0, 1, 2, 3\}$ such that $r \equiv n \pmod 4$, let $k = [n/4]$. Consider two disjoint circuits C_1 and C_2. The vertex set of C_1 is $\{u_1, \ldots, u_{2k}\}$, the vertex set of C_2 is $\{v_1, \ldots, v_{2k+r}\}$. The edges of C_1 are $u_i u_{i+1}$ for $i = 1, \ldots, 2k - 1$ and $u_{2k} u_1$, the edges of C_2 are $v_i v_{i+1}$ for $i = 1, \ldots, \ldots, 2k + r - 1$ and $v_{2k+r} v_1$. We add the edges $u_i v_i$ for $i = 1, \ldots, 2k - 1$ and $u_{2k} v_{2k+r}$ and further the edges $u_i v_{i+1}$ for even $i \leqq 2k - 2$ and $u_{2k} v_1$. Finally, we add the edges $u_1 u_i$ for $i = 3, \ldots, 2k - 1$ and $v_1 v_i$ for $i = 3, \ldots, 2k + r - 1$. The resulting graph satisfies the conditions of Theorem 1, as the reader may verify himself. □



In Fig. 1 we see such a graph for $n = 23$. A planar 3-connected locally snake-like graph with eight vertices is shown in Fig. 2.



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ЛОКАЛЬНО ЗМЕЕОБРАЗНЫЕ ГРАФЫ

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Резюме

Символом $N_G(v)$ обозначается полграф G порожденный множством всех вершин, смежных с вершиной v в G. Если $N_G(v)$ является цепью для каждой вершины v графа G, то G называется локально змееобразным графом. Исследуются локально змееобразные графы, которые планарны и 3-связны.