Jiří Binder

A Loomis-Sikorski theorem for logics

Mathematica Slovaca, Vol. 38 (1988), No. 4, 367--371

Persistent URL: http://dml.cz/dmlcz/136475

Math. Slovaca 38, 1988, No. 4, 367—371

A LOOMIS — SIKORSKI THEOREM FOR LOGICS

JIŘÍ BINDER

Introduction and preliminaries

In the logico-algebraic approach to the foundations of quantum mechanics there is postulated that the "event structure" of a physical system be a (quantum) logic, that is an σ-orthomodular partially ordered set (see, e.g., [1], [3]). The system is classical if the corresponding logic is a Boolean σ-algebra. In this case the investigation can sometimes be reduced to "concrete" (= set representable) Boolean algebras by an application of the Loomis-Sikorski theorem. This theorem asserts that every Boolean σ-algebra is a homomorphic image of a concrete one. It is natural to search for a similar result (if there is any) in the general situation of logics. In this note we obtain a type of the "Loomis-Sikorski" theorem for logics. It should be noted that in the finitely additive case there is a Loomis-Sikorski theorem proved in [2]. Although the σ-additive setup has a different character, we use in some places some of the methods of [2].

Let us now review the basic notions as we shall use them in the sequel (see, e.g., [1] for details).

By a *logic* we mean an orthomodular poset which is σ-complete, (i.e. $\bigvee_{i\in N} a_i$ exists in L whenever $a_i \in L$ and $a_i \leq a_j'$ for any $i,j \in N$, $i \neq j$).

Let L_1, L_2 be logics. A mapping $\varphi\colon L_1 \to L_2$ is said to be a *homomorphism* if

(i) $\varphi(0) = 0$,

(ii) $\varphi(a') = \varphi(a)'$ for any $a \in L$,

(iii) $\varphi\left(\bigvee_{i\in N} a_i\right) = \bigvee_{i\in N} \varphi(a_i)$ whenever $\{a_i, i \in N\} \subset L_1$ and $a_i \leq a_j'$ for any $i,j \in N$, $i \neq j$.

When $\varphi\colon L_1 \to L_2$ is one-to-one and both φ and φ^{-1} are homomorphisms, then φ is said to be an *isomorphism*.

Let K be a subset of L such that (i) $0 \in K$, (ii) $a \in K$ implies $a' \in K$ and (iii) $\bigvee_{i\in N} a_i \in K$ whenever $a_i \in K$, $a_i \leq a_j'$ for any $i,j \in N$, $i \neq j$. Then K is called a *sublogic* of L. If in addition K forms a Boolean σ-algebra (with the lattice operations inherited from L), we call it a *Boolean sublogic* of L.

A subset of L is said to be *compatible* if there is a Boolean sublogic containing it.

An important Boolean sublogic of L is the *centre* $C(L)$ of L, i.e. the set $C(L) = \{a \in L \mid a \text{ is compatible with every } b \in L\}$.

Recall finally that a logic L is said to be *concrete* if there is a set X and a collection $\mathscr{X}$ of subsets of X such that $\mathscr{X}$ ordered by inclusion and endowed with the operation $'$ given by the complementation in X forms a logic isomorphic to L.

The following proposition gives us a simple characterization of concrete logics.

Proposition: *Let L be a logic. Then L is concrete if and only if for each $a, b \in L$ either $a \geq b$ or there exists a twovalued probability measure m such that $m(a) = 0$* and $m(b) = 1$.

The proof of the proposition uses the standard construction (see [1], [2]).

In the proof of the main theorem we shall use the following known lemma (see e.g. [1], Lemma 3.7.):

Lemma. *Let L be a logic and let $c \in L$, $\{a_i, i \in N\} \subset L$ be such that c is compatible with a_i for every $i \in N$. Suppose that the elements $\bigvee_N a_i$, $\bigvee_N (c \wedge a_i)$ exist. Then $c \wedge \left(\bigvee_{i \in N} a_i\right)$ exists end equals $\bigvee_{i \in N} (c \wedge a_i)$.*

We are going to prove that every logic is a "nice" homomorphic image of a concrete one.

Main result

Theorem. *Let L be a logic. Then there is a concrete logic K and an epimorphism $\varphi: K \to L$ such that*

(i) $\varphi(C(K)) = C(L)$,

(ii) *for any compatible subset $\{e_1, \ldots, e_n\}$, $n \in N$,* of L *there is a compatible subset $\{f_1, \ldots, f_n\}$ of K such that $\varphi(f_i) = e_i$ for any $1 \leq i \leq n$.*

Proof. We may suppose that L is not Boolean (otherwise we have the classical Loomis-Sikorski theorem). Let B_α, $\alpha \in I$ be the collection of all finite Boolean sublogics of L. Put $\mathscr{M} = \{(a, \alpha) \mid \alpha \in I, a \in L\}$ and define an equivalence relation $\sim$ on $\mathscr{M}$ as follows: we have $(a, \alpha) \sim (b, \beta)$ if either $(a, \alpha) = (b, \beta)$, or $a = b = 0$, resp. $a = b = 1$. Further we put $(a, \alpha) \leq (b, \beta)$ if either $\alpha = \beta$ and $a \leq b$, or $a = 0$, resp. $b = 1$. Then the factor set $M = \mathscr{M}/\sim$ with this ordering $\leq$ and with the orthocomplementation $'$ giving to each (a, α) the element (a', α) becomes a logic. We shall denote the equivalence class containing (x, α) by x_α. If $a \in M$, the (unique) element $x \in L$ such that $a = x_\alpha$ (for some $\alpha \in I$) will be denoted by $\bar{a}$.

By the Loomis-Sikorski theorem, there is a set X, a Boolean σ-algebra $\mathscr{X}$ of subsets of X and an epimorphism $\varphi_0 : \mathscr{X} \to C(L)$. Let K be the set of all mappings $k: X \to M$ such that $k(X)$ is at most countable and $k^{-1}(a) \in \mathscr{X}$ for each $a \in M$. Endow K canonically with the partial ordering $\leq$ and with operation $'$ (thus, $f, g \in K, f \leq g$, resp. $f' = g$, if and only if $f(x) \leq g(x)$, resp. $f'(x) = g(x)$, for any $x \in X$). We have first to prove that K is a logic. We shall show that K is σ-complete, the rest is obvious.

Let $\{k_i, i \in N\} \subset K$ and $k_i \leq k_j'$ for any $i \neq j$, $i, j \in N$. We are going to prove that the mapping $k: x \to \bigvee_{i \in N} k_i(x)$, $x \in X$, belongs to K (then, obviously, $k = \bigvee_{i \in N} k_i$). Fix an element $x \in X$ and consider the set $R(x) = \{k_i(x), i \in N\}$. Since the elements of $R(x)$ are mutually orthogonal, we see that $R(x)$ is contained in a Boolean sublogic of M. therefore $R(x)$ is finite for each $x \in X$. Let $\sim$ be an equivalence relation on X such that $x \sim y$ if and only if $R(x) = R(y)$. As the finite sets $R(x)$ are subsets of an at most countable set $\bigvee_{i \in N} k_i(X)$, the partition $\mathscr{V}$ of X associated with $\sim$ is at most countable. It follows that the set $k(X)$ is also at most countable.

To prove that for any $a \in M$ the set $k^{-1}(a) \in \mathscr{X}$, observe first that each $V \subset \mathscr{V}$ can be written as a union of an at most countable family of mutually disjoint elements of $\mathscr{X}$ in the following way ($x \in V$ is chosen arbitrarily):

if $R(x) = \{0\}$, then $V = \bigcap_{i \in N} k_i^{-1}(0) \in \mathscr{X}$;

if $R(x) = \{0, 1\}$, then $V = \bigcup_{i \in N} k_i^{-1}(1)$;

and finally if $R(x) \setminus \{0, 1\} = \{a_1, \ldots, a_m\} \neq \emptyset$, then

$$V = \bigcup_{(i_1, \ldots, i_m) \in N^m} \left[k_{i_1}^{-1}(a_1) \cap \ldots \cap k_{i_m}^{-1}(a_m) \cap \left(\bigcap_{i \in N \setminus \{i_1, \ldots, i_m\}} k_i^{-1}(0) \right) \right].$$

We have obtained the refinement $\mathscr{P}$ of the partition $\mathscr{V}$ fulfilling the following properties:

For any $P \in \mathscr{P}$

(i) $P \in \mathscr{X}$;

(ii) $x, y \in P \Rightarrow (k_i(x) = k_i(y)\ (i \in N)$ and $k(x) = k(y))$.

Denote by P_n, $n \in N$, the elements of $\mathscr{P}$, and by a_n, resp. a_{in}, $(i \in N)$, the value which k, resp. k_i, attains on P_n. We now have for any $i \in N$, $a \in M$:

$$k_i^{-1}(a) = \bigcup_{n \in N_i(a)} P_n, \text{ where } N_i(a) = \{n \in N \mid a_{in} = a\} \quad \text{and}$$

$$k^{-1}(a) = \bigcup_{n \in N(a)} P_n, \text{ where } N(a) = \{n \in N \mid a_n = a\}.$$

It follows that $k^{-1}(a) \in \mathscr{X}$. This completes the proof that K is a logic.

Using the characterization for concrete logics, we easily show that K is concrete. Indeed, if $k_1, k_2 \in K$ and $k_1 \nleq k_2$, then there is $x_0 \in X$ such that $k_1(x_0) \nleq k_2(x_0)$. Since M is evidently concrete there is a 0—1 probability measure m_0 on M such that $m_0(k_1(x_0)) = 1$ and $m_0(k_2(x_0)) = 0$. Then the function m: $K \to \{0, 1\}$ defined by the formula $m(k) = m_0(k(x_0))$ is a two-valued measure and $m(k_1) = 1$, $m(k_2) = 0$.

Let us define the mapping $\varphi\colon K \to L$. We set $\varphi(k) = \bigvee_{a \in M} (\varphi_0(k^{-1}(a)) \wedge \bar{a})$ for any $k \in K$. We have first to prove that φ is a homomorphism.

Evidently $\varphi(0) = 0$. Let $k = \bigvee_{i \in N} k_i$, where $k_i \in K$ and $k_i \leq k_j'$ for any $i, j \in N$, $i \neq j$. Since each $\varphi_0(k^{-1}(a)) \in C(L)$ we may aply the lemma. Making use of the same notation as above we obtain:

$$\begin{aligned}
\varphi(k) &= \bigvee_{a \in M} (\varphi_0(k^{-1}(a)) \wedge \bar{a}) = \bigvee_{a \in M} \left(\varphi_0 \left(\bigvee_{n \in N(a)} P_n \right) \wedge \bar{a} \right) = \\
&= \bigvee_{a \in M} \bigvee_{n \in N(a)} (\varphi_0(P_n) \wedge \bar{a}) = \bigvee_{n \in N} \left(\varphi_0(P_n) \wedge \bar{a}_n \right) = \\
&= \bigvee_{n \in N} \left(\varphi_0(P_n) \wedge \left(\bigvee_{i \in N} \bar{a}_{in} \right) \right) = \bigvee_{n \in N} \bigvee_{i \in N} (\varphi_0(P_n) \wedge \bar{a}_{in}) = \\
&= \bigvee_{i \in N} \left[\bigvee_{a \in M} \bigvee_{n \in N_i(a)} (\varphi_0(P_n) \wedge \bar{a}) \right] = \bigvee_{i \in N} \varphi(k_i).
\end{aligned}$$

Further, $\varphi(k) \leq \varphi(k')'$ and (using the additivity of φ)

$$\varphi(k) \vee \varphi(k') = \varphi(k \vee k') = 1. \text{ Thus } \varphi(k') = \varphi(k)'.$$

This completes the proof that φ is a homomorphism.

Evidently, φ is surjective and $\varphi(C(K)) = C(L)$. To verify the condition (ii), it suffices to observe that for any compatible subset $E = \{e_1, \ldots, e_m\}$ of L there is a finite Boolean sublogic B_a $(a \in I)$ of L containing E. Taking $f_i(x) = (e_i)_a$ $(x \in X,\ 1 \leq i \leq m)$ we obtain $\varphi(f_i) = e_i$. The proof of the theorem is complete.

A natural question arises whether the formulation of the theorem admits the following strengthening of the condition (ii): All compatible subsets of L have compatible preimages in K. (The condition (ii) requires this only for finite sets and the method of the proof depends on this assumption.) We have not been abled to decide if this version of the theorem is also true or not.

Acknowledgement: The author would like to express his gratitude to Prof. Pavel Pták for his encouragement during this research.

REFERENCES

[1] GUDDER, S.: Stochastic Methods in Quantum Mechanics, North Holland, New York, 1979.
[2] PTÁK, P., WRIGHT, J. D. M.: On the concreteness of quantum logics, Aplikace matematiky 30 (1985), 274—285.
[3] VARADAJARAN, V.: Geometry of Quantum Theory I., Von Nostrand, Princeton (1968).

Received October 29, 1986

Pedagogická fakulta UK
M. D. Rettigové 4
116 39 Praha 1

ТЕОРЕМА ЛЮМИСА—СИКОРСКОГО ДЛЯ КВАНТОВЫХ ЛОГИК

Jiří Binder

Резюме

Любая логока является эпиморфным образом логики множеств и соответствующий эпиморфизм учитывает определеным образом центр и одновременную наблюдаемость конечных наборов элементов логики.