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1. Introduction

This paper is a contribution to the non-commutative probability theory. The folloving Theorem is known in the conventional probability theory [3].

Theorem 0. *Let* $(\Omega, \mathcal{T}, P)$ *be a probability space and let* $\{x_n\}$ *be a sequence of random variables. If* x_n *converges in probability to* x, *then* x_n *converges in distribution to* x.

We obtain in consequence of Theorem 0 that if x_n converges in the square mean to x, then x_n converges in distribution to x.

In the generalized probability theory the σ-algebra of subsets of a set is replaced by the lattice of all closed subspaces of a Hilbert space. The random variables are replaced by self-adjoint operators and the probability measures by states. Our aim is to prove a non-commutative version of Theorem 0.

2. Convergence of operators

Let us begin with some notations and preliminaries. Let H be a complex separable Hilbert space, $\dim H \geq 3$. Let $L(H)$ be the set of all closed linear subspaces of H (or equivalently, the set of all projections on H). If $e \in H$ is a unit vector (i.e. $\|e\| = 1$), then a map $m_e : L(H) \to \langle 0, 1 \rangle$ such that $m_e(P) = (Pe, e)$ is a vector state on $L(H)$. By the Gleason theorem [4], every state can be written in the form $m = \sum_{i=1}^{\infty} c_i m_{e_i}$, where $\{e_i\}$ is a complete orthonormal system with $c_i \geq 0$ $(i = 1, 2, \ldots)$ and $\sum_{i=1}^{\infty} c_i = 1$. Recall that a state m on $L(H)$ is called faithful if the equality $m(P) = 0$ implies $P = 0$. It can be easily seen that if a state $m = \sum_{i=1}^{\infty} c_i m_{e_i}$ is faithful, then $c_i > 0$ for all $i \in N$.

Let $A, A_1, A_2, \ldots$ be bounded self-adjoint operators on H. We say that the sequence $\{A_n\}$ converges on A in the measure $[m]$ if for any $\varepsilon > 0$ we have

$\lim_{n\to\infty} m(P_{A_n-A}\langle -\varepsilon, \varepsilon\rangle) = 1$, where P_{A_n-A} is the spectral resolution of $A_n - A$. We say that the sequence $\{A_n\}$ converges on A in the square mean with respect to the state m if $\lim_{n\to\infty} m((A_n - A)^2) = 0$ $\left(\text{i.e. if } \lim_{n\to\infty}\left(\sum_{i=1}^{\infty} c_i((A_n - A)^2 e_i, e_i)\right) = 0,\right.$ where $\left. m = \sum_{i=1}^{\infty} c_i m_{e_i}\right)$.

If A is a self-adjoint operator on H and m is a state on $L(H)$, then the distribution function of the operator A is the map $F_A^m\colon R \to \langle 0, 1\rangle$, where $F_A^m(t) = m(P_A(-\infty, t))$ $(t \in R)$. (We put R for the real numbers and $B(R)$ for the σ-algebra of the Borel subsets of R.) We say that the sequence $\{A_n\}$ converges to A in distribution (with respect to m) if $F_{A_n}^m(t) \to F_A^m(t)$ in every continuity point $t \in R$ of the function $F_A^m(t)$.

Finally, let us define the characteristic function of a self-adjoint operator A in the state m as the map $\Phi_A^m : R \to C$, where $\Phi_A^m(t) = \int_R e^{its} m(P_A(ds))$.

Similarly as in the conventional probability theory, we can prove the following statement.

Proposition 1. *The sequence $\{A_n\}$ of bounded self-adjoint operators on H converges on the self-adjoint operator A in distribution if and only if $\lim_{n\to\infty} \Phi_{A_n}^m(t) = \Phi_A^m(t)$ for all $t \in R$.*

The proof of the main result will be performed in several steps.

Proposition 2. *Let $\{A_n\}$ be a sequence of bounded self-adjoint operators such that the sequence $\{A_n\varphi\}$ is Cauchy for every $\varphi \in H$. Then there is a bounded self-adjoint operator A such that $A_n \to A$ in the strong topology.*

Proof. Let $\varphi \in H$. By the assumption the sequence $\{A_n\varphi\}$ is Cauchy. From the completness of H, there is a vector $\psi \in H$ such that $\lim_{i\to\infty} \|A_n\varphi - \psi\| = 0$. Put $A\varphi = \psi$. It can be easily seen that the map $A\colon \varphi \to A\varphi$ is linear. From the equality $\lim_{r\to\infty} \|A_r\varphi - A\varphi\| = \lim_{r\to\infty}\lim_{s\to\infty} \|A_r\varphi - A_s\varphi\| = 0$ it follows that $A_n \to A$ in the strong topology. The convergence in strong topology implies the convergence in weak topology and therefore

$$(\varphi, A\psi) = \lim_{n\to\infty} (\varphi, A_n\psi) = \lim_{n\to\infty} (A_n\varphi, \psi) = (A\varphi, \psi)$$

for every $\varphi, \psi \in H$, which implies that A is self-adjoint. We shall prove that the operator A is bounded. By the principle of uniform boundedness we have that the sequence $\{\|A_n\|\}$ is bounded. From this we obtain that $\|A\varphi\| =$

$= \lim_{n \to \infty} \|A_n\varphi\| \leq K\|\varphi\|$ for every $\varphi \in H$, where $\|A_n\| \leq K$ $(n = 1, 2, \ldots)$. This completes the proof.

Proposition 3. *Let m be a faithful state on $L(H)$ and let $\{A_n\}$ be a sequence of uniformly bounded self-adjoint operators such that $m((A_r - A_s)^2) \to 0$ $(r, s \to \infty)$. Then the sequence $\{A_n\varphi\}$ is Cauchy for every vector $\varphi \in H$.*

Proof. By the Gleason theorem we have that $m = \sum_{i=1}^{\infty} c_i m_{e_i}$ where $\sum_{i=1}^{\infty} c_i = 1$, $c_i > 0$ for every $i \in N$ (as m is faithful) and $\{e_i\}$ is a complete orthonormal system in H. Since

$$c_i m_{e_i}((A_r - A_s)^2) \leq \sum_{i=1}^{\infty} c_i m_{e_i}((A_r - A_s)^?) = m((A_r - A_s)^2) \to 0,$$

we have $m_{e_i}(A_r - A_s)^2) \to 0$ for every e_i $(i = 1, 2, \ldots)$. Take $\varphi \in H$. Then $\varphi = \sum_{i=1}^{\infty} (\varphi, e_i) e_i$. Let $M < \infty$ be such that $\sup_n \|A_n\| < M$. Choose $\varepsilon > 0$ and let $k \in N$ be such that $\sum_{i=k+1}^{\infty} (\varphi, e_i) e_i < \frac{\varepsilon}{4M}$. We have

$$m_{e_i}((A_r - A_s)^2) = \|(A_r - A_s) e_i\|^2 \to 0$$

for $r, s \to \infty$ and for every e_i $(i = 1, 2, \ldots)$. This implies that there is $n_0 \in N$ such that for $r, s \geq n_0$ there holds $\|(A_r - A_s) e_i\| < \frac{\varepsilon}{2D}$ for $i = 1, 2, \ldots, k$, where $D = \sum_{i=1}^{\infty} |(\varphi, e_i)|$. Then we obtain

$$\|(A_r - A_s)\varphi\| = \left\|(A_r - A_s) \sum_{i=1}^{\infty} (\varphi, e_i) e_i\right\| \leq$$

$$\leq \sum_{i=1}^{k} |\varphi, e_i)| \, \|(A_r - A_s) e_i\| + (\|A_r\| + \|A_s\|) \left\| \sum_{i=k+1}^{\infty} (\varphi, e_i) e_i \right\| < \varepsilon.$$

From this we see that the sequence $\{A_n\varphi\}$ is Cauchy for every $\varphi \in H$. This concludes the proof.

Proposition 4. *Let m be a faithful state on $L(H)$ and let $\{A_n\}$ be a sequence of uniformly bounded self-adjoint operators on H. Let $m((A_r - A_s)^2) \to 0$ for $r, s \to \infty$. Then there is a bounded self-adjoint operator A such that $m((A_n - A)^2) \to 0$ for $n \to \infty$.*

Proof. From Proposition 2 and Proposition 3 it follows that there is a bounded self-adjoint operator A such that $A_n \to A$ in the strong topology. Write $m = \sum_{i=1}^{\infty} c_i m_{e_i}$, where $c_i > 0$ for every i and $\sum_{i=1}^{\infty} c_i = 1$. Let $M < \infty$ be such that

$\sup_n \|A_n\| < M$. For every $\varepsilon > 0$ there is $k \in N$ such that $\sum_{i=k+1}^{\infty} c_i < \frac{\varepsilon}{8M^2}$. The strong convergence $A_n \to A$ implies that especially $\|(A_n - A)e_i\| \to 0$ for every e_i. Then there is $n_0 \in N$ such that for every $n \geq n_0$ there holds $\|(A_n - A)e_i\|^2 < \frac{\varepsilon}{2}$ for $i = 1, 2, \ldots, k$. For $n \geq n_0$ we have

$$m((A_n - A)^2) = \sum_{j=1}^{\infty} c_j \|(A_n - A)e_j\|^2 = \sum_{j=1}^{k} c_j \|(A_n - A)e_j\|^2 +$$

$$+ \sum_{j=k+1}^{\infty} c_j \|(A_n - A)e_j\|^2 < \sum_{j=1}^{k} c_j \frac{\varepsilon}{2} + (\|A_n\| + \|A\|)^2 \sum_{j=k+1}^{\infty} c_j < \varepsilon,$$

which implies $\lim_{n \to \infty} ((A_n - A)^2) = 0$.

Proposition 5. *Let m be a faithful state on $L(H)$ and let $\{A_n\}$ be a sequence of uniformly bounded self-adjoint operators such that $m((A_n - A)^2) \to 0$. Then $A_n \to A$ in the strong topology.*

Proof. Let $m = \sum_{i=1}^{\infty} c_i m_{e_i}$, where $c_i > 0$ for every $i \in N$. Similarly as in Proposition 3 we can show that $m_{e_i}((A_n - A)^2) \to 0$ for any $e_i (i = 1, 2, \ldots)$. Let $M < \infty$ be such that $\sup_n \|A_n\| < M$. Let $\varphi \in H$. Then $\varphi = \sum_{i=1}^{\infty} (\varphi, e_i) e_i$. Choose $\varepsilon > 0$ and $k \in N$ such that $\sum_{i=k+1}^{\infty} \|(\varphi, e_i) e_i \zeta < \frac{\varepsilon}{4M}$. Since we have $m_{e_i}((A_n - A)^2) = \|(A_n - A)e_i\|^2 \to 0$ $(n \to \infty)$ for every e_i, there is $n_0 \in N$ such that for $n \geq n_0$ there holds $\|(A_n - A)e_i\| < \frac{\varepsilon}{2D}$ for $i = 1, 2, \ldots, k$, where $D = \sum_{i=1}^{k} |(\varphi, e_i)|$. Then we have

$$\|(A_n - A)\varphi\| \leq \sum_{i=1}^{k} |(\varphi, e_i)| \|(A_n - A)e_i\| + (\|A_n\| + \|A\|) \sum_{i=k+1}^{\infty} \|(\varphi, e_i)e_i\|.$$

Therefore $A_n \to A$ in the strong topology.

We note that making use of a similar method the converse of Proposition 3 and Proposition 5 can be also proved.

Theorem 6 (Trotter). *Let A, $\{A_n\}$ be self-adjoint operators. Then $(\lambda I - A_n)^{-1} \to (\lambda I - A)^{-1}$ for every complex number λ such that $Im\,\lambda \neq 0$ if and only if $e^{itA_n} \to e^{itA}$ in the strong topology for every $t \in R$.*

The Theorem 6 is proved in [2].

Proposition 7. *Let $\{A_n\}$ be uniformly bounded self-adjoint operators. Then*

$A_n \to A$ in the strong topology if and only if $(\lambda I - A_n)^{-1} \to (\lambda I - A)^{-1}$ for every complex number λ such that $\operatorname{Im}\lambda \neq 0$.

Proof. Suppose that $A_n \to A$ in the strong topology, i.e. $\|(A_n - A)\varphi\| \to 0$ for every $\varphi \in H$. Then if $\operatorname{Im}\lambda \neq 0$, then $(A_n - A)(A - \lambda)^{-1} \to 0$ in the strong topology (since for every $\varphi \in H$ we have

$$\|(A_n - A)(A - \lambda)^{-1}\varphi\| \leq \|(A_n - A)\varphi\| \, \|(A - \lambda)^{-1}\|).$$

Using the identity

$$(A_n - \lambda)^{-1} = (A - \lambda)^{-1}(I + (A_n - A)(A - \lambda)^{-1})^{-1}$$

we obtain

$$\|(A_n - \lambda)^{-1}\varphi - (A - \lambda)^{-1}\varphi\| =$$
$$= \|(A - \lambda)^{-1}((I + (A_n - A)(A - \lambda)^{-1})^{-1} - I)\varphi\| \leq$$
$$\leq \|(A_n - A)(A - \lambda)^{-1}\varphi\| \, \|(A - \lambda)^{-1}\| \, \|(A_n - \lambda)^{-1}\| \, \|(A - \lambda)\| \leq$$
$$\leq \|(A_n - A)(A - \lambda)^{-1}\varphi\| \, (\sup_n \|A_n\| + |\lambda|)^{-1}.$$

Since $\|(A_n - A)(A - \lambda)^{-1}\varphi\| \to 0$ and $(\sup_n \|A_n\| + |\lambda|)^{-1}$ is bounded, we obtain $(\lambda I - A_n)^{-1} \to (\lambda I - A)^{-1}$ $(n \to \infty)$ for every λ such that $\operatorname{Im}\lambda \neq 0$.

Now we shall prove the converse implication. We use the identity

$$(A_n - A) = (A_n - i)((A - i)^{-1} - (A_n - i)^{-1})(A - i).$$

We obtain

$$\|(A_n - A)\varphi\| = \|(A_n - i)((A - i)^{-1} - (A_n - i)^{-1}))(A - i)\varphi\| \leq$$
$$\leq \|(A - i)^{-1}\varphi - (A_n - i)^{-1}\varphi\| \, (\sup_n \|A_n\| + 1)(\|A\| + 1).$$

From this we obviously conclude that $A_n \to A$ in the strong topology.

Theorem 8. *Let m be a faithful state on $L(H)$ and let $\{A_n\}$ be a sequence of uniformly bounded self-adjoint operators on H such that $m((A_n - A)^2) \to 0$. Then A_n converges on A in distribution (with respect to m).*

Proof. By Proposition 5, $A_n \to A$ in the strong topology. By Trotter's theorem and Proposition 7, this is equivalent to $e^{itA_n} \to e^{itA}$ in the strong topology for every t. The convergence in the strong topology implies the convergence in the weak topology. therefore for every $\varphi, \psi \in H$ we have

$$((e^{itA_n} - e^{itA})\varphi, \psi) \to 0 \ (n \to \infty).$$

Let $m = \sum_{i=1}^{\infty} c_i m_{e_i}$. Then $((e^{itA_n} - e^{itA})e_i, e_i) \to 0$ holds for every e_i. therefore we have

$$\left|\sum_{j=1}^{\infty} c_j((e^{itA_n} - e^{itA}) e_j, e_j)\right| \leq$$

$$\leq \left|\sum_{j=1}^{k} c_j((e^{itA_n} - e^{itA}) e_j, e_j)\right| + \left|\sum_{j=k+1}^{\infty} c_j((e^{itA_n} - e^{itA}) e_j, e_j)\right|.$$

As $\sum_{j=1}^{\infty} c_j = 1$, for given $\varepsilon > 0$ we can find $k \in N$ such that $\sum_{j=k+1}^{\infty} c_j < \frac{\varepsilon}{4M}$, where $\infty > M > \sup_n \|A_n\|$. The convergence $e^{itA_n} \to e^{itA}$ in the weak topology implies that for every $\varepsilon > 0$ there is $n_0 \in N$ such that for every $n \geq n_0$ there holds $|((e^{itA_n} - e^{itA}) e_j, e_j)| < \frac{\varepsilon}{2}$ $(j = 1, 2, \ldots, k)$. We have proved that the sequence of characteristic functions $\Phi_{A_n}^m$ converges on Φ_A^m, which by Proposition 1 is equivalent to the convergence $A_n \to A$ in distribution (with respect to m).

Theorem 9. *Let m be a faithful state on $L(H)$ and let $\{A_n\}$ be a sequence of uniformly bounded self-adjoint operators such that A_n converges on A in measure $[m]$. Then the sequence of operators $\{A_n\}$ converges on A in distribution.*

Proof. It follows from Theorem 8 and Lemma 5.4. [1].

Corollary 10. *Let m be a faithful state on $L(H)$ and $\{A_n\}$ be a sequence of uniformly bounded self-adjoint operators such that A_n converges on A in the square mean with respect to m and let the distribution function of A_n $(n = 1, 2, \ldots)$ be Gaussian. Then the distribution function of A (with respect to m) is Gaussian, too.*

Proof. It follows from Theorem 8 and Lemma 16.10. [3].

Finally we shall show that the assumption of uniform boundedness is necessary to prove that the convergence in the square mean with respect to m_{e_i} $\left(m = \sum_{i=1}^{\infty} c_i m_{e_i}\right)$ implies convergence in the square mean with respect to m.

Example 1. Let $\{e_n\}$ be a complete orthonormal system in H and $\{P_n\}$ be the projections on the subspaces generated by $\{e_n\}$. Let the spectra of the operators A_n $(n = 1, 2, \ldots)$ consist of 0 and $2^{n/2}$. Put $A_n\{2^{n/2}\} = P_n$ $(n = 1, 2, \ldots)$ and $m = \sum_{i=1}^{\infty} \frac{1}{2^i} m_{e_i}$. Clearly $A_n = 2^{n/2} P_n$ $(n = 1, 2, \ldots)$ and there holds $m_{e_i}(P_n) = \delta_{in}$ and $m_{e_i}(A_n^2) = 2^n \delta_{in}$ $(i, n = 1, 2, \ldots)$. Then $m_{e_i}(A_n^2) \to 0$ for every state e_i, but $m(A_n^2) = 1$.

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О СХОДИМОСТИ ОПЕРАТОРОВ

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Резюме

В этой статье доказывается, что если последовательность равномерно ограниченых самосопряженных операторов сходится по мере и если состояние точное, то сходится и по распределению.