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In this note we give new proofs of the results of [2] and [3] and generalize them in two directions. Namely: we need no topology (only convergence structure) and we need no associativity (only alternativity). Here the emphasis is on the totally non-commutative case.

We start with some notations and preliminary results.

A groupoid is said to be alternative iff its every two-element generated subgroupoid is a semigroup.

In an alternative groupoid the power a^n of an element a is unambiguously defined.

A groupoid is called totally non-commutative if it has at least two idempotents and for every pair of idempotents e, f with $e \neq f$ we have $ef \neq fe$.

A convergence space is a set F with a distinguished family of sequences $\{a_n\}$ ($a_n \in F$), which we shall call convergent. We assume that for every convergent sequence $\{a_n\}$ there exists exactly one element $a \in F$, which we shall call the limit of $\{a_n\}$ (in symbol $a_n \to a$ or $\lim_n a_n = a$). We assume that the constant sequence $\{a_n\}$ ($a_n = a$) converges to the limit a, and that every subsequence of a convergent sequence converges to the same limit as the original sequence.

A convergence groupoid is a groupoid S endowed with the structure of a convergence space in which the multiplication is continuous, i.e., $a_n \to a$ and $b_n \to b$ imply $a_n b_n \to ab$.

An alternative convergence groupoid is called sequentially point compact iff every subsequence of

$$a, a^2, a^3, \dots$$

contains a convergent subsequence.

Throughout, by a groupoid we mean an alternative, sequentially point-compact groupoid. The structure of such groupoids is described in [4]. We recall some results of [4] which we shall need later.

Denote by E the set of all idempotents of the groupoid S. We say that the element a belongs to the idempotent e iff there exists a sequence $\{n_k\}$ with $a^{n_k} \to e \in E$. Let K_e be the set of all elements of S belonging to e. Then S can be written as the union of a disjoint family of sets $S = \cup \{K_e;\ e \in E\}$.

To every $e \in E$ there exists a maximal quasigroup H_e in which e is a unit. Clearly $H_e \subset K_e$ and $ae = ea$ for every a in K_e. Further, $K_e \cdot e = e \cdot K_e = H_e$ and H_e coincides with the family of all elements $a \in K_e$ with $ae = ea = a$. The sets K_e need not be groupoids in general; however, if E is in the centre of S, then K_e is always a groupoid.

First we show:

1. Theorem. *The centre of a totally non-commutative groupoid is empty.*

Proof. Let $c \in S$ commute with all $x \in S$, then clearly c^n $(n \geqq 1)$ also commutes with all elements of S. Let $c^{n_k} \to e \in E$ and let $x \in S$, then

$$\begin{aligned} ex &= (\lim_k c^{n_k}) \cdot x = \lim_k (c^{n_k} \cdot x) \\ &= \lim_k (x \cdot c^{n_k}) = x \cdot \lim_k c^{n_k} \\ &= x \cdot e. \end{aligned}$$

Let now $f \in E$ with $f \neq e$; then $ef = fe$, a contradiction.

It is interesting that a totally non-commutative groupoid has some similar properties as a commutative one. For example K_e is a groupoid. For proving this we need the following lemma.

2. Lemma. (Lemma 18 of [4]) *Let x and y be elements of S belonging to an idempotent e and let xy belong to an idempotent f. Then $ef = fe$.*

An immediate consequence of this lemma is the following:

3. Theorem. *Let S be a totally non-commutative groupoid. Then K_e is a groupoid.*

4. Theorem. *Let S be a totally non-commutative groupoid. Then K_e is sequentially closed.*

Proof. Let $a_n \in K_e$, $n = 1, 2, \ldots$ with $a_n \to b$ and let $b \in K_f$. We have to show that $e = f$. Since $a_n \in K_e$ for every $n = 1, 2, \ldots$, we have $ea_n = a_n e$. Passing to the limits we get $eb = be$. Now let

$$b^{m_k} \to f,$$

then $eb^{m_k} = b^{m_k} e$ and so $ef = fe$. So we get $e = f$.

5. Lemma. *Let S be totally non-commutative. If $e, f \in E$ $e \neq f$ and $ef = e$, then $fe = f$.*

Proof. Since $ef = e$ we get $(ef) \cdot e = e$ and $f \cdot ((ef) \cdot e) = fe$. By the alternativity of S, $(f \cdot e)^2 = f \cdot e$ and hence fe is an idempotent. Put $fe = g$, then

$$\begin{aligned} g \cdot f &= (f \cdot e) \cdot f = f \cdot (e \cdot f) = fe = g, \\ f \cdot g &= f \cdot (f \cdot e) = f \cdot e = g. \end{aligned}$$

Thus we get $g \cdot f = f \cdot g$ and so $f = g$.

6. Theorem. *Let J be a two-sided ideal of a totally non-commutative groupoid S. Then $E \subset J$.*

Proof. Let $a \in J$ and $a \in K_e$; then, since J is an ideal $e \cdot a = a \cdot e \in J$. Since $e \cdot a \in H_e$ and H_e is an alternative quasigroup there exists an element $b \in H_e$ with $b \cdot (e \cdot a) = e$ and so $e = b \cdot (e \cdot a) \in b \cdot J \subset J$. Therefore J contains at least one idempotent. Let $f \in E$; then, since J is an ideal, $e \cdot f \in J$ and also $(e \cdot f)^n \in J$. The product $e \cdot f$ need not be an idempotent, but there exists an $h \in E$ with $(e \cdot f)^{n_k} \to \to h$. Clearly $e \cdot h = h$. Since $e \in J$ we get $h = e \cdot h \in J$. As $h \cdot f = h$, by Lemma 5 we have $f \cdot h = f$. Since $h \in J$ we get $f = f \cdot h \in f \cdot J \subset J$ and so $f \in J$. This completes the proof.

Denote by N the following set

$$N = \cup \{H_e;\ e \in E\}.$$

7. Lemma. *Let S be a totally non-commutative groupoid and let J be a two-sided ideal of S. Then $N \subset J$.*

Proof. We know — by the preceding theorem — that $E \subset J$. If $a \in H_e$, then $a \cdot e = a$ and so $a = a \cdot e \in a \cdot J \subset J$.

The following result is quite surprising since we do not assume the compactness of the groupoid.

8. Theorem. *If S is a totally non-commutative alternative and sequentially point compact groupoid, then S contains the least two-sided ideal N.*

Proof. Let us prove that N is a two-sided ideal at first. Let $a \in S$ and $b \in N$; then $b \in H_e$ for some $e \in E$ and $b \cdot e = e \cdot b = b$. Let $(a \cdot b)^{n_k} \to f \in E$, i.e. $a \cdot b \in K_f$. Clearly $f \cdot e = f$ and by Lemma 5 $e \cdot f = e$. We have to show that $(a \cdot b) \cdot f = a \cdot b$. Since the closure of the groupoid generated by elements a and b contains e and f, by the alternativity of S we get

$$(a \cdot b) \cdot f = a \cdot (b \cdot e) \cdot f = (a \cdot b) \cdot (e \cdot f) = (a \cdot b) \cdot e = a \cdot (b \cdot e) = a \cdot b$$

and so $a \cdot b \in H_f \subset N$.

Similarly one can show that also $b \cdot a \in N$ and so N is a two-sided ideal. Since it is contained in every two-sided ideal it is easy to see that N is the least two-sided ideal.

To illustrate our results, we give now an example.

9. Example. Let S be the set of all pairs (t, c) where $t \in (0, 1)$ and c is a Cayley number with $|c| = 1$. Let the convergence be defined by an ordinary topology in the 9-dimension Euclidean space. If $s_1 = (t_1, c_1)$ and $s_2 = (t_2, c_2)$, we put $s_1 \cdot s_2 = (t_1, c_1 \cdot c_2)$. Clearly we get a sequentially point-compact alternative groupoid. The set of its idempotents is $E = (0, 1) \times \{1\}$. Put $H_t = \{t\} \times \{c;\ |c| = 1\}$. Then it is easy to see that H_t is a maximal quasigroup in S and that all maximal quasigroups are isomorphic.

If S is a totally non-commutative convergence semigroup then by Theorem 8, S contains the least two-sided ideal N.

In this case it is a known fact (see [1]) that N is the disjoint union of its maximal subgroups and these are isomorphic. So we have:

10. Theorem. *If S is a totally non-commutative semigroup, then its maximal subgroups are isomorphic.*

With this in mind, recalling Example 9, we add the following conjecture:

11. Conjecture. *The maximal sub-quasigroups of a totally non-commutative alternative quasigroup are isomorphic.*

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О ВПОЛНЕ НЕКОММУТАТИВНЫХ ПОЛУГРУППАХ СХОДИМОСТИ

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Резюме

Альтернативный группоид сходимости называется вполне некоммутативным, если он содержит по крайней мере два идемпотента и для яюбых двух идемпотентов $e \neq f$ имеет место $ef \neq fe$.

В работе доказываются следующие теоремы:

а) Центром такого группоида является пустое множество.

б) Множество всех элементов, принадлежащих к фиксированному идемпотенту — замкнутый группоид.

в) Существует минимальный двусторонний идеал группоида S, и он является соединением максимальных квазигрупп в S.