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RÉNYI'S FORMULA WITH REMAINDER TERM ON ARITHMETICAL SEMIGROUPS

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A free commutative semigroup G with identity element 1_G generated by a countable set P is called *arithmetical* if in addition there exists a real-valued norm mapping $|\cdot|$ on G such that
i) $|ab| = |a|\cdot|b|$ for all $a, b \in G$,
ii) the total number $N_G(x)$ of elements $n \in G$ of norm $|n| \leq x$ is finite for each real x.

The elements of P, i.e. the generators of G, are called *primes*. Plainly, every element $n \neq 1_G$ in G has a unique (up to the order of factors) factorization of the form

$$n = p_1^{a_1}\cdot p_2^{a_2}\dots p_r^{a_r}, \tag{1}$$

where p_i are distinct elements of P.

In the following we shall always suppose not only the finiteness in ii), but that the following asymptotic axiom is satisfied [6, p. 75]:

Axiom A. *There exist positive constants A and δ, and a constant η with $0 \leq h < \delta$, such that*

$$N_G(x) = Ax^{\delta} + O(x^{\eta}) \quad \textit{as} \quad x \to \infty.$$

Complex valued functions defined on an arithmetical semigroup are called *arithmetical*. Generalizing the standard arithmetical functions known from the classical number theory, one can define
a) the *Möbius function* μ_G as follows

$$\mu_G(n) = \begin{cases} 1, \text{ if } n = 1_G, \\ (-1)^r, \text{ if } a_1 = a_2 = \dots = a_r = 1 \text{ in (1)}, \\ 0 \text{ otherwise}, \end{cases}$$

b) the functions ω_G and Ω_G through

$$\omega_G(1_G) = \Omega_G(1_G) = 0$$

and

$$\omega_G(n) = r, \quad \Omega_G(n) = a_1 + a_2 + \dots + a_r.$$

Similarly, the notion of the *asymptotic density* $d(T)$ of a subset T of an arithmetical semigroup G can be generalized as expected. If $T(x)$ denotes the total number of elements of T of norm at most x and if there exists

$$\lim T(x)/N_G(x), \text{ as } x \to \infty,$$

then

$$d(T) = \lim_{x \to \infty} \frac{T(x)}{N_G(x)}.$$

For the sake of simplicity we shall often omit the index G if the basic semigroup G can be deduced from the context. For further details and properties of arithmetical semigroups and arithmetical functions on them we refer the reader to Knopfmacher's book [6].

Let $\Delta_G = \Delta$ denote the pointwise difference $\Omega - \omega$, i.e.

$$\Delta(n) = \Omega(n) - \omega(n) \quad \text{for} \quad n \in G.$$

If $G = Z$, the set of positive integers, then the already classical result of Rényi [12] says that the set A_q of positive integers for which

$$\Delta_Z(n) = q$$

has asymptotic density d_q which is given by the generating series

$$\sum_{q=0}^{\infty} d_q z^q = \prod \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p - z}\right) = \frac{6}{\pi^2} \prod \frac{1 - z/(p+1)}{1 - z/p},$$

for $|z| < 2$, where the products are extended over all the rational primes. This contains as a special case for $q = 0$ the well-known fact that the asymptotical density $d(Q)$ of the set Q of all squarefree integers is $6/\pi^2$. However, since 1909 it has been known [8] that for the number $Q(x)$ of squarefree integers below x we have

(2) $$Q(x) = 6x/\pi^2 + o(x^{1/2}).$$

Landau proved this result using the fact that the sum function

$$M(x) = \sum_{n \leq x} \mu(n),$$

of the Möbius function $\mu(x)$ satisfies

(3) $$M(x) = o(x).$$

Previously he showed [9] that this result can be derived from the prime number theorem without using the method of complex integration and later [10] that also the fact that (3) implies the prime number theorem can be proved without using these analytic tools.

Unfortunately, Rényi's method does not give a possibility of obtaining an asymptotic estimate for the density of the set A_q. This gap was subsequently filled by several authors, e.g. Cohen [1], Kátai [5], W. Schwarz [13] and Delange [2—4], to mention a few. Thus Cohen proved that

$$\sum_{\substack{n \leq x \\ \Delta_Z(n) = 1}} 1 = d_1 x + O(x^{1/2} \operatorname{loglog} x).$$

and Delange using analytic means gradually improved this estimate for the general index q. However, Kátai showed that the first improvement of Delange that

$$\sum_{\substack{n \leq x \\ \Delta_Z(n) = q}} 1 = d_q x + o(x^{1/2} (\operatorname{loglog} x)^q)$$

can be deduced from Landau's estimate (2) for squarefree numbers without using analytic tools applied by Delange.

In [6, p. 151] it is proved that Rényi's original result can be extended to arithmetical semigroups:

Let G be an arithmetical semigroup satisfying Axiom A. *Then the asymptotic density $d_{q,G}$ of the set $A_{q,G}$ of those elements $n \in G$ for which $\Delta_G(n) = q$ exists for each $q = 0, 1, 2, \ldots$ and may be calculated from the power series formula*

$$\sum_{q=0}^{\infty} d_{q,G} z^q = \prod_{p \in P} (1 - |p|^{-\delta})(1 + (|p|^{\delta} - z)^{-1}).$$

From a result of [11] an analogue of the above Cohen estimate follows for $d_{1,G}$ if G satisfies Axiom A. The aim of this paper is to show that combining the ideas of [5] and [11] it is possible to prove an asymptotic estimate for the density of the set $A_{q,G}$ also in the case of arithmetical semigroups satisfying Axiom A.

Recall that the *zeta function* ζ_G of an arithmetical semigroup G is defined by the formal Dirichlet series

$$\zeta_G(z) = \sum_{a \in G} |a|^{-z}.$$

If G is an arithmetical semigroup satisfying Axiom A, the series on the right hand side is absolutely convergent for $\operatorname{Re}(z) > \delta$ and divergent for $\operatorname{Re}(z) \leqq \delta$ [6, p. 84].

First of all we shall prove an estimate for the set $Q_{k,G}$ of k-free elements in an arithmetical semigroup satisfying Axiom A. Here as usual, given an integer $k \geq 2$, an element $n \in G$ is called *k-free* if $a_i < k$ for every $i \leq r$ in (1). Then for the total number $Q_{k,G}(x)$ of k-free elements of norm at most x in G we have:

Lemma 1. *Let G be an arithmetical semigroup satisfying Axiom* A *written in the following form*

$$|N_G(x) - Ax^\delta| \leq K \cdot x^\eta.$$

Then

$$Q_{k,G}(x) - \frac{Ax^\delta}{\zeta_G(k\delta)} = \begin{cases} O(\max(A^2, AK, K^2)\, x^{\delta/k}), & \textit{if} \;\; \eta < \delta/k, \\ O(\max(A^2, AK, K^2)\, x^{\delta/k} \log x), & \textit{if} \;\; \eta = \delta/k, \\ O(\max(A^2, AK, K^2)\, x^{\eta}), & \textit{if} \;\; \eta > \delta/k, \end{cases}$$

where the O-constants do not depend on A and K.

Proof. Using standard ideas we immediately obtain the well-known formula

$$Q_{k,G}(x) = \sum_{|d|^k \leq x} \mu_G(d)\, N_G(x/|d|^k). \tag{4}$$

Consequently,

$$Q_{k,G}(x) = Ax^\delta \zeta_G^{-1}(k\delta) + O\Big(Ax^\delta \sum_{|d|^k > x} |d|^{-k\delta}\Big) + O\Big(Kx^\eta \sum_{|d|^k \leq x} |d|^{-k\eta}\Big)$$

with O-constants not depending an A and K.
Partial summation then gives

$$x^\delta \sum_{|d| > x^{1/k}} |d|^{-k\delta} = -N_G(x^{1/k}) + x^\delta \delta k \int_{x^{1/k}}^{\infty} N_G(t)\, t^{-k\delta - 1}\, \mathrm{d}t =$$
$$= O(\max(A, K)\, x^{\delta/k}).$$

Similarly,

$$x^\eta \sum_{|d| \leq x^{1/k}} |d|^{-k\eta} = N_G(x^{1/k}) + x^\eta \eta k \int_{1}^{x^{1/k}} N_G(t)\, t^{-k\eta - 1}\, \mathrm{d}t =$$
$$= \begin{cases} O(\max(A, K)\, x^{\delta/k}), & \text{if} \;\; \eta < \delta/k, \\ O(\max(A, K)\, x^{\eta} \log x), & \text{if} \;\; \eta = \delta/k, \\ O(\max(A, K)\, x^{\eta}), & \text{if} \;\; \eta > \delta/k, \end{cases}$$

and the lemma follows.

If $M_G(x)$ denotes the summation function of the Möbius function μ_G of an arithmetical semigroup G, then Lemma 1 is based on the trivial estimate $M_G(x) = O(x^\delta)$. To improve the result of Lemma 1 better estimates for $M_G(x)$ are needed. Concerning this note that if an arithmetical semigroup G satisfies Axiom A, then the prime number theorem is true in G [6, Chapter 6]. However, it can be shown on the other hand that if G satisfies Axiom A, then the prime number theorem for G and the estimate $M_G(x) = o(x^\delta)$ are equivalent. (Note that the prime number theorem and the assertion that $M(x) = o(x^\delta)$ are not equivalent in general arithmetical semigroups; see [14] for more detail.) But using more subtle techniques one can prove more:

Lemma 2. [7, Theorem 6.4] *If G satisfies Axiom* A, *then for every $a > 0$ we have*

$$M_G(x) = O(x^\delta (\log x)^{-a}).$$

Lemma 1 can be now strengthened as follows:

Theorem 1. *Let the arithmetical semigroup G satisfy Axiom* A *and let us have for a positive integer $k \geq 2$*

$$\eta k < \delta.$$

Then for the set $Q_{k,G}$ of k-free elements in G we have

$$Q_{k,G}(x) = Ax^\delta \zeta_G^{-1}(k\delta) + R(x),$$

where the estimate $M_G(x) = o(x^\delta)$ yields

$$R(x) = o(x^{\delta/k}) \tag{5}$$

and Lemma 2 *implies*

$$R(x) = O(x^{\delta/k} (\log x)^{-a}) \tag{6}$$

for every $a > 0$.

Proof. We prove first (5). Since the function $M(x) \cdot x^{-\delta}$ "increases" only in those points x which are values of the norm function, then $M_G(x) = o(x^\delta)$ implies that the function

$$\sqrt[k\delta]{\frac{|M(t)|}{t^\delta}}$$

takes its maximum on each interval $\langle x^{1/2k}, \infty)$ for every $x \geq 1$. Denote this maximum by $\eta(x)$. The function $\eta(x)$ is clearly nonincreasing and let

$$\varepsilon = \varepsilon(x) = \max\{x^{-1/2k}, \eta(x^{1/2k})\}.$$

Then $\varepsilon(x)$ is also nonicreasing and

$$\lim_{x \to \infty} \varepsilon(x) = 0. \tag{7}$$

Then for $y \geq x^{1/2k}$ we have

$$|M(y)| = \frac{|M(y)|}{y^\delta} y^\delta \lesssim (\eta(x^{1/2k}))^{k\delta} y^\delta \leq \tag{8}$$

$$\leq \varepsilon(x)^{k\delta} y^\delta.$$

Let $x^{1/k} = z$ for the sake of simplicity. Then (4) can be written in the form

$$Q_{k,G}(x) = \sum_{|n^k m| \leq x} \mu_G(n) = \sum_{|n| \leq z} \mu(n) \sum_{|m| \leq x/|n|^k} 1.$$

and consequently

$$Q_{k,G}(x) = \sum_{|n| \leq \varepsilon z} \mu(n) \sum_{|m| \leq x/|n|^k} 1 + \sum_{|m| < \varepsilon^{-k}} \sum_{|n| \leq \sqrt[k]{x/|m|}} \mu(n) -$$

$$- \sum_{|n| \leq \varepsilon \cdot z} \sum_{|m| < \varepsilon^{-k}} \mu(n) = S_1 + S_2 - S_3,$$

where $\varepsilon = \varepsilon(x)$.

For the first sum we have

$$S_1 = \sum_{|n| \leq \varepsilon z} \mu(n) N_G(x/|n|^k) = Ax^\delta \sum_{|n| \leq \varepsilon z} \mu(n) |n|^{-k\delta} +$$

$$+ O\left(x^\eta \sum_{|n| \leq \varepsilon z} |n|^{-k\eta}\right) = Ax^\delta \zeta_G^{-1}(k\delta) - Ax^\delta \sum_{|n| > \varepsilon z} \mu(n)|n|^{-k\delta} +$$

$$+ O\left(x^\eta \sum_{|n| \leq \varepsilon z} |n|^{-k\eta}\right).$$

The last term can be estimated easily as

$$O\left(x^\eta \sum_{|n| \leq \varepsilon z} |n|^{-k\eta}\right) = O(z^\delta \cdot \varepsilon^{\delta - k\eta}) = o(x^{\delta/k}),$$

Here the last equality follows from (7) and the fact that $\delta - k\eta > 0$.

Partial summation gives for the second term in S_1 that

$$\sum_{|n| > \varepsilon z} \mu(n)|n|^{k\delta} = -M(\varepsilon z) \cdot (\varepsilon z)^{-k\delta} + k\delta \int_{\varepsilon z}^{\infty} M(y) y^{-k\delta - 1} \, dy$$

Since

(9) $$\varepsilon z \geq x^{-1/2k} \cdot x^{1/k} = x^{1/2k},$$

and because the function ε is nonincreasing, the relation (8) yields

$$|M(\varepsilon z) \cdot (\varepsilon z)^{-k\delta}| \leq \varepsilon^{k\delta} \cdot (\varepsilon z)^\delta \cdot (\varepsilon z)^{-k\delta} = \varepsilon^\delta z^{\delta(1-k)} = \varepsilon^\delta x^{-\delta} x^{\delta/k}.$$

Similarly we have for $B = 1/(\delta - k\delta)$

$$\left| \int_{\varepsilon z}^{\infty} M(y) y^{-k\delta - 1} \, dy \right| \leq \varepsilon^{k\delta} \int_{\varepsilon z}^{\infty} y^{-k\delta + \delta - 1} \, dy = B\varepsilon^{k\delta} (\varepsilon z)^{\delta(1-k)} = B\varepsilon^\delta x^{-\delta} x^{\delta/k}.$$

Thus together

$$S_1 = Ax^{\delta}\zeta_G^{-1}(k\delta) + o(x^{\delta/k}).$$

For the second sum we have

$$S_2 = \sum_{|m| < \varepsilon^{-k}} \sum_{|n| \le \sqrt[k]{x/|m|}} \mu(n) = \sum_{|m| < \varepsilon^{-k}} M((x/|m|)^{1/k}).$$

If $|m| < \varepsilon^{-k}$, then owing to (9)

$$(x/|m|)^{1/k} \ge \varepsilon z \ge x^{1/2k}$$

and (8) gives

$$M((x/|m|)^{1/k}) \le \varepsilon(x)^{k\delta} \cdot (x/|m|)^{\delta/k},$$

which in turn yields that

$$|S_2| \le \varepsilon(x)^{k\delta} x^{\delta/k} \sum_{|m| < \varepsilon^{-k}} |m|^{-\delta/k} =$$

$$= O(\varepsilon^{k\delta} x^{\delta/k} (\varepsilon^{-k})^{\delta(1-1/k)}) = O(\varepsilon^{\delta} \cdot x^{\delta/k}) = o(x^{\delta/k}).$$

Finally, the relations (7), (8) and Axiom A give for the sum S_3

$$S_3 = M(\varepsilon z) \sum_{|m| < \varepsilon^{-k}} 1 = O(M(\varepsilon z) \cdot \varepsilon^{-k\delta}) =$$

$$= O(\varepsilon^{k\delta} \cdot (\varepsilon z)^{\delta} \cdot \varepsilon^{-k\delta}) = O(\varepsilon^{\delta} x^{\delta/k}) = o(x^{\delta/k}),$$

which proves (5).

In the proof of (6) let

$$\varepsilon(x) = (\log \sqrt[2k]{x})^{-a/k(\delta-\eta)}$$

and let $x_0 = x_0(a, k, \delta, \eta)$ be such that for $x > x_0^{2k}$

$$(\log \sqrt[2k]{x})^{-a/k(\delta-\eta)} \ge x^{-1/2k}.$$

Then for $x > x_0^{2k}$ we have

$$\varepsilon(x) \cdot z \ge x^{-1/2k} \cdot x^{1/k} = x^{1/2k} > x_0.$$

Therefore we obtain along the lines of the preceding part of the proof that for $x > x_0^{2k}$

$$S_1 - Ax^{\delta}\zeta_G^{-1}(k\delta) = O(\varepsilon^{\delta-k\eta} \cdot z^{\delta}(\log \varepsilon z)^{-a} + \varepsilon^{\delta-k\delta} \cdot z^{\delta}) =$$

$$= O(\varepsilon^{\delta-k\eta} \cdot z^{\delta}(1 + \varepsilon^{-k(\delta-\eta)}(\log \varepsilon z)^{-a}).$$

Since $(\log t)^{-a}$ is a decreasing function and $\varepsilon z \geq \sqrt[2k]{x}$,

$$(\log \varepsilon z)^{-a} \leq (\log \sqrt[2k]{x})^{-a} = (\varepsilon(x))^{k(\delta-\eta)},$$

and thus

$$1 + \varepsilon^{-k(\delta-\eta)}(\log \varepsilon z)^{-a} \leq 2.$$

Further

$$\varepsilon^{\delta - k\eta} = (\log \sqrt[2k]{x})^{-a(\delta-k\eta)/k(\delta-\eta)} = ((1/2k)\log x)^{-a(\delta-k\eta)/k(\delta-\eta)}.$$

With a also $r = a(\delta - k\eta)/k(\delta - \eta)$ runs through positive real numbers and therefore

$$S_1 = Ax^\delta \zeta_G^{-1}(k\delta) + O(x^{\delta/k}(\log x)^{-r})$$

for every $r > 0$ with the O-constant not depending on r.

Similarly we can prove that

$$S_2 = O\left(x^{\delta/k}(\log \varepsilon^k x)^{-a} \sum_{|m| < \varepsilon^{-k}} |m|^{-\delta/k}\right) =$$

$$= O(z^\delta \varepsilon^{\delta - k\delta}(\log \varepsilon z)^{-a}) = O(z^\delta \varepsilon^{\delta - k\eta}(\varepsilon^{-k(\delta-\eta)}(\log \varepsilon z)^{-a})) =$$

$$= O(z^\delta \varepsilon^{\delta-k\eta}) = O(x^{\delta/k}(\log x)^{-r}).$$

For S_3 we get analogically

$$S_3 = M(\varepsilon z) \sum_{|m| < \varepsilon^{-k}} 1 = O((\varepsilon z)^\delta (\log \varepsilon z)^{-a} \varepsilon^{-k\delta}) =$$

$$= O(z^\delta \varepsilon^{\delta-k\eta}(\varepsilon^{-k(\delta-\eta)}(\log \varepsilon z)^{-a})) = O(x^{\delta/k}(\log x)^{-r}),$$

and the theorem is proved.

The following analogue of a classical result will be useful in the proof of the next theorem.

Lemma 3. [6, p. 165] *If an arithmetical semigroup G satisfies axiom* A, *then*

$$\sum_{\substack{p \in P \\ |p| \leq x}} |p|^{-\delta} = \log\log x + B_G + O(1/\log x).$$

In the proof of the following theorems we shall use the "descent" technique, which is based on the following result and which shows certain advantages of the abstract approach through arithmetical semigroups:

Lemma 4. [6, p. 77] *Let a be an arbitrary element of an arithmetical semigroup G. Let $G\langle a \rangle$ denote the set of all the elements of G which are coprime to a. Then*
i) $G\langle a \rangle$ is also an arithmetical semigroup;

ii) if G satisfies Axiom A, *then also* $G\langle a\rangle$ *does and*

$$N_{G\langle a\rangle}(x) = A\varphi_\delta(a)|a|^{-\delta}x^\delta + O(x^\eta), \quad x \to \infty,$$

where for real w we define

$$\varphi_w(a) = \sum_{d|a} \mu(d)|a/d|^w.$$

This result can be proved by induction on the number of the prime divisors of a. If a has a unique prime divisor p, then

$$N_{G\langle a\rangle}(x) = N_G(x) - N_G(x/|p|) = Ax^\delta + O(x^\eta) - A(x/|p|)^\delta + \\ + O((x/|p|)^\eta) = A(1-|p|^{-\delta})x^\delta + O(x^\eta(1+|p|^{-\eta}))$$

and the lemma follows because

$$\prod_{\substack{p\in P\\ p|a}} (1-|p|^{-\delta}) = \sum_{\substack{d\in G\\ d|a}} \mu_G(d)|d|^{-\delta}.$$

An element $n\in G$ will be called *squarefull* if for every prime divisor $p\in P$ we have

$$\text{if } p|n \text{ then } p^2|n.$$

Lemma 5. *Let G be an arithmetical semigroup satisfying Axiom* A *and m a positive integer. If* $S_{2,G}^{(m)}$ *denotes the set of squarefull elements in G with at most m prime divisors, then*

$$S_{2,G}^{(m)}(x) = \sum_{\substack{n\in S_{2,G}^{(m)}\\ |n|\le x}} 1 = O(x^{\delta/2}(\log\log x)^{m-1}/\log x).$$

Proof follows the ideas of [5]. If $|p^a|\le x$, then $a = O(\log x)$, where the O-constant does not depend on p. Then the prime number theorem for arithmetical semigroups satisfing Axiom A [6, p. 154] implies

$$S_{2,G}^{(1)}(x) = \sum_{\substack{|p^a|\le x\\ a\ge 2}} 1 = \pi_G(x^{1/2}) + \pi_G(x^{1/3}) + \ldots = \pi_G(x^{1/2}) + \\ + O(\pi_G(x^{1/3})\log x) = O\left(\frac{x^{\delta/2}}{\log x}\right).$$

Thus the lemma is true for $m = 1$. For $m\ge 2$ we have

(10) $$S_{2,G}^{(m)}(x) = \sum_{\substack{n\in S_{2,G}^{(m)}\\ \omega(n)=1\\ |n|\le x}} 1 + \sum_{\substack{n\in S_{2,G}^{(m)}\\ \omega(n)\ge 2\\ |n|\le x}} 1$$

The first sum is $S^{(1)}_{2,G}$. Rearrange the terms $n \in S^{(m)}_{2,G}$ in the second sum as follows: Let q^a be one of the powers $p_i^{a_i}$ in the canonical decomposition $n = p_1^{a_1} \dots p_m^{a_m}$, where $a_i > 0$ and $p_i \neq p_j$ for which $a_i > 0$ and in which the minimum

$$\min\{|p_i^{a_i}|;\ i = 1, \dots, m\}.$$

is taken. Then $|q^a| \leq x^{1/2}$ and the second sum is

$$\sum_{\substack{n \in S^{(m)}_{2,G} \\ \omega(n) \geq 2 \\ |n| \leq x}} 1 = O\Bigg(\sum_{\substack{|q^a h| \leq x \\ a \geq 2,\ (q,h) = 1 \\ h \in S^{(m-1)}_{2,G}}} 1 \Bigg) = O\Bigg(\sum_{\substack{|q^a| \leq x^{1/2} \\ a \geq 2}} S^{(m-1)}_{2,G}(x/|q^a|) \Bigg) =$$

$$= O\Bigg(\sum_{\substack{|q^a| \leq x^{1/2} \\ a \geq 2}} \frac{x^{\delta/2} (\log\log x/|q^a|)^{m-2}}{|q^a|^{\delta/2} \cdot \log x/|q^a|} \Bigg) =$$

$$= O\Bigg(\frac{x^{\delta/2} (\log\log x)^{m-2}}{\log x} \sum_{\substack{|q^a| \leq x^{1/2} \\ a \geq 2}} \frac{1}{|q^a|^{\delta/2}} \Bigg).$$

Further,

$$\text{(11)} \qquad \sum_{\substack{|q^a| \leq x^{1/2} \\ a \geq 2}} |q^a|^{-\delta/2} = \sum_{|q| \leq x^{1/4}} |q|^{-\delta} + \sum_{\substack{|q^a| \leq x^{1/2} \\ a \geq 3}} |q^a|^{-\delta/2}.$$

Lemma 3 implies that the first sum in (11) is $O(\log\log x)$. In the second sum we have for $b(a) = [a/3]$

$$\sum_{\substack{|q^a| \leq x^{1/2} \\ a \geq 3}} |q^a|^{-\delta/2} \sum_{\substack{|q^a| \leq x^{1/2} \\ a \geq 3}} |q|^{-3\delta b(a)/2} \leq 3 \cdot \zeta_G(3\delta/2),$$

as $\zeta_G(z)$ converges for complex z with $\mathrm{Re}(z) > \delta$. Thus the left-hand side of (11) is $O(\log\log x)$. After substituting into (10) the mathematical induction finishes the proof.

In the next two theorem we shall give an estimate for densities in Rényi's result.

Theorem 2. *Let G be an arithmetical semigroup satisfying Axiom* A. *Then there exists a constant* $d_{q,G}$ *for which*

$$A_{q,G}(x) = d_{q,G} x^{\delta} + \begin{cases} O(x^{\delta/2} (\log\log x)^q), & \text{if } \eta < \delta/2, \\ O(x^{\delta/2} \log x (\log\log x)^q), & \text{if } \eta = \delta/2, \\ O(x^{\eta}), & \text{if } \eta > \delta/2. \end{cases}$$

Proof. From the remark following Lemma 4 we obtain that

$$N_{G\langle p_1 \dots p_r\rangle}(x) = A \prod_{i=1}^{r} (1 - |p_i|^{-\delta}) x^{\delta} + O\left(x^{\eta} \prod_{i=1}^{r} (1 + |p_i|^{-\eta})\right),$$

where the O-constant depends on G but not on the primes $p_1, \dots, p_r$. Since

$$\zeta_{G\langle p_1,\dots,p_r\rangle}(z) = \zeta_G(z) \prod_{i=1}^{r} (1 - |p_i|^{-z}),$$

then from Lemma 1 we have

$$Q_{2,G\langle p_1 \dots p_r\rangle}(x) = \frac{A \prod_{i=1}^{r} (1 - |p_i|^{-\delta}) x^{\delta}}{\zeta_G(2\delta) \prod_{i=1}^{r} (1 - |p_i|^{-2\delta})} +$$

(12)

$$+ \begin{cases} O\left(\prod_{i=1}^{r} (1 + |p_i|^{-\eta})^2 x^{\delta/2}\right), & \text{if } \eta < \delta/2, \\ O\left(\prod_{i=1}^{r} (1 + |p_i|^{-\eta})^2 x^{\eta} \log x\right), & \text{if } \eta = \delta/2, \\ O\left(\prod_{i=1}^{r} (1 + |p_i|^{-\eta})^2 x^{\eta}\right), & \text{if } \eta > \delta/2, \end{cases}$$

where the O-constants do not depend on $p_1, \dots, p_r$.

Let $A^*_{q,G}$ denote the set of squarefull elements from $A_{q,G}$. Every element $n \in A_{q,G}$ can be uniquely written in the form $n = km$, where $k \in A^*_{q,G}$ and m is squarefree with $(k, m) = 1$. This uniqueness and (12) yield

(13) $$A_{q,G}(x) = \sum_{\substack{|k| \le x \\ k \in A^*_{q,G}}} Q_{2,G\langle k\rangle}(x/|k|) = \frac{Ax^{\delta}}{\zeta_G(2\delta)} \sum_{\substack{|k| \le x \\ k \in A^*_{q,G}}} |k|^{-\delta} \prod_{p|k} (1 + |p|^{-\delta})^{-1} +$$

$$+ \sum_{\substack{|k| \in x \\ k \in A^*_{q,G}}} \begin{cases} O(4^{\omega(k)}(x/|k|)^{\delta/2})), & \text{if } \eta < \delta/2, \\ O(4^{\omega(k)}(k/|k|)^{\eta} \log(x/|k|)), & \text{if } \eta = \delta/2, \\ O(4^{\omega(k)}(x/|k|)^{\eta})), & \text{if } \eta > \delta/2. \end{cases}$$

Expanding the inner product in the above main term we obtain the following infinite series

$$\sum_{v} \lambda(v) |v|^{-\delta},$$

where $\lambda(v) = (-1)^{\Omega(v)}$ is the Liouville function and v runs over the set

$$S_k = \{v \in G;\ p|v \text{ implies } p|k \text{ for every } p \in P_G\}.$$

Thus the main term leads formally to the series

$$\sum_{\substack{k \in A^*_{q,G} \\ v \in S_k}} \lambda(v)\, |kv|^{-\delta}. \tag{14}$$

Let $k \in A^*_{q,G}$ and $k = p_1^{a_1} \dots p_r^{a_r}$. Then $a_i \geq 2$ for every $i = 1, \dots, r$ and simultaneously $a_1 + a_2 + \dots + a_r = r + q$. This implies firstly that $2r \leq a_1 + a_2 + \dots + a_r = r + q$, i.e. $r \leq q$. The equality $a_1 + a_2 + \dots + a_r = r + q$ gives further that for every fixed r there exist only finitely many r tuples $(a_1, \dots, a_r)$ satisfying it. Since $r \leq q$, the total number of possible exponent r tuples $(a_1, \dots, a_r)$ of elements in $A^*_{q,G}$ is finite.

Now

$$\sum_{\substack{|kv| > x \\ k \in A^*_{q,G},\, v \in S_k}} |kv|^{-\delta} = \sum_{j=0}^{\infty} \sum_{\substack{2^j x < |kv| \leqq 2^{j+1} x \\ k \in A^*_{q,G},\, v \in S_k}} |kv|^{-\delta}.$$

Fix one of the possible above mentioned exponent r tuples, say, $(a_1, \dots, a_r)$. Then every element in G can be expressed at most once in the form kv, where $k = p_1^{a_1} \dots p_r^{a_r}$ and $v \in S_k$. The product kv is squarefull and has at most $r \leq q$ prime divisors. Therefore

$$\sum_{\substack{2^j x < |kv| < 2^{j+1} x \\ k \in A^*_{q,G},\, v \in S_k}} |kv|^{-\delta} < 2^{-j\delta} \cdot x^{-\delta} \cdot S^{(q)}_{2,G}(2^{j+1} x).$$

Thus using Lemma 5 we get together

$$\sum_{|kv| > x} |kv|^{-\delta} < x^{-\delta} \sum_{j=0}^{\infty} 2^{-j\delta} \cdot S^{(q)}_{2,G}(2^{j+1} x) =$$

$$= x^{-\delta/2} \sum_{j=0}^{\infty} 2^{-j\delta} O\left(\frac{(\log((j+1)\log 2 + \log x)^{q-1}}{(j+1)\log 2 + \log x}\right) \cdot 2^{(j+1)\delta/2} =$$

$$= x^{-\delta/2} \cdot 2^{\delta/2} \cdot O((\log x)^{-1} \cdot (\log\log x)^{q-1}) \cdot \sum_{j=0}^{\infty} 2^{-j\delta/2} =$$

$$= O(x^{-\delta/2} \cdot (\log x)^{-1} \cdot (\log\log x)^{q-1}).$$

Since the number of possible exponent r tuples $(a_1, \dots, a_r)$ is finite as we have seen, we have proved that (14) is convergent and that

$$\frac{A x^{\delta}}{\zeta_G(2\delta)} \sum_{\substack{|k| \leq x \\ k \in A^*_{q,G}}} |k|^{-\delta} \prod_{p \mid k} (1 + |p|^{-\delta})^{-1} =$$

$$= d_{q,G} x^{\delta} + O(x^{\delta/2} (\log x)^{-1} \cdot (\log\log x)^{q-1}),$$

where

$$d_{q,G} = A\zeta_G^{-1}(2\delta) \sum_{\substack{k \in A_{q,G}^* \\ v \in S_k}} \lambda(v).|kv|^{-\delta}.$$

We saw that $\omega(k) \leq q$ and therefore it is enough to realise the following two estimates:
Similarly as in (11) we can prove that

$$\sum_{\substack{|k| \leq x \\ k \in A_{q,G}^*}} |k|^{-\delta/2} \leq \left(\sum_{\substack{|p^a| \leq x \\ a \geq 2}} |p^a|^{-\delta/2} \right)^q = O((\log\log x)^q),$$

which settles the case $\eta < \delta/2$.
For $\eta = \delta/2$ we have

$$\sum_{\substack{|k| \leq x \\ k \in A_{q,G}^*}} |k|^{-\delta/2} \cdot \log x/|k| \leq \log x \sum_{\substack{|k| \leq x \\ k \in A_{q,G}^*}} |k|^{-\delta/2} = O(\log x \cdot (\log\log x)^q).$$

Finally, in the case $\eta > \delta/2$ note that if $k \in A_{q,G}^*$ and

$$k = p_1^{a_1} \dots p_r^{a_r},$$

then $a_i \geq 2$, $i = 1, \dots, r$. Thus $|k|^{-\eta} \geq |p_1 \dots p_r|^{-2\eta}$. Since $2\eta > \delta$, then

$$\sum_{\substack{|k| \leq x \\ k \in A_{q,G}^*}} |k|^{-\eta} = O(1),$$

and the proof is finished.
In the next theorem we show that under certain circumstances the previous estimates can be improved.

Theorem 3. *Let G be an arithmetical semigroup satisfying Axiom* A *for which $2\eta < \delta$. Then for every $q \geq 1$ the estimate* (5) *implies*

$$A_{q,G}(x) = d_{q,G}x^\delta + o(x^{\delta/2}(\log\log x)^q),$$

whereas the estimate (6) *implies*

$$A_{q,G}(x) = d_{q,G}x^\delta + O(x^{\delta/2}(\log\log x)^{q-1}).$$

Proof. As a first step we prove an identity. Namely that for every $k \in G$ of the form

$$k = p_1^{a_1} \dots p_r^{a_r},$$

we have

$$Q_{2,G\langle k\rangle}(x) = \sum_{|v| \le x} \lambda(v)\, Q_{2,G}(x/|v|), \tag{15}$$

where v runs over all the elements of the form

$$k = p_1^{b_1} \dots p_r^{b_r}, \quad b_1, \dots, b_r = 0, 1, \dots$$

and where $\lambda(v) = (-1)^{\Omega(v)}$.

To see (15) note that $|\mu_G(n)|$ is the characteristic function of the set of squarefree elements in G and that

$$\sum_{n \in G\langle k\rangle} |\mu(n)| \cdot |n|^{-s} = \prod_{\substack{p \in P_G \\ p \nmid k}} (1 + |p|^{-s}) =$$

$$= \Bigl(\prod_{\substack{p \in P_G \\ p \mid k}} (1 + |p|^{-s})\Bigr)^{-1} \prod_{p \in P_G} (1 + |p|^{-s}) =$$

$$= \sum_v \frac{\lambda(v)}{|v|^s} \sum_{n \in G} |\mu(n)| \cdot |n|^{-s}.$$

Let

$$R(x) = Q_{2,G}(x) - A \cdot \zeta_G^{-1}(2\delta) \cdot x^{\delta}.$$

Then (13) gives

$$A_{q,G}(x) = \sum_{\substack{|k| \le x \\ k \in A_{q,G}^{*}}} Q_{2,G\langle k\rangle}(x/|k|) = \sum_{|kv| \le x} \lambda(v)\, Q_{2,G}(x/|kv|) =$$

(16)

$$= A \cdot \zeta_G^{-1}(2\delta)\, x^{\delta} \sum_{|kv| \le x} \lambda(v) \cdot |kv|^{-\delta} + \sum_{|kv| \le x} \lambda(v)\, R(x/|kv|).$$

As in the proof of the previous theorem we have

$$A\, \zeta_G^{-1}(2\delta) \sum_{|kv| \le x} \frac{\lambda(v)}{|kv|^{\delta}} = d_{q,G} + O\Bigl(\frac{(\log\log x)^{q-1}}{x^{\delta/2} \cdot \log x}\Bigr).$$

Let us estimate the second term in (16). If $M(x) = o(x^{\delta})$, then (5) implies the existence of a decreasing function $h(x)$ with $\lim_{x \to \infty} h(x) = 0$ and for which

$$|R(x)| < h(x)\, x^{\delta/2}.$$

Then for every fixed function $g(x)$, which monotonically tends to infinity, we have

$$\left|\sum_{|kv|\le x} \lambda(v) R(x/|kv|)\right| \le$$

$$\le \sum_{|kv|\le x/g(x)} |R(x/|kv|)| + \sum_{x/g(x)<|kv|\le x} |R(x/|kv|)|.$$

Moreover,

$$\sum_{|kv|\le x/g(x)} |R(x/|kv|)| < h(g(x))\, x^{\delta/2} \sum_{|kv|\le x} |kv|^{-\delta/2} <$$

$$< h(g(x))\, x^{\delta/2} O\left(\left(\sum_{\substack{|p^a|\le x\\ a\ge 2}} |p|^{-a\delta/2}\right)^q\right)$$

Similarly as in (11) we obtain that the last expression is $h(g(x))\,x^{\delta/2} O((\log\log x)^q)$. Further,

$$\sum_{x/g(x)<|kv|\le x} |R(x/|kv|)| = \sum_{x/g(x)<|kv|\le x} O((x/|kv|)^{\delta/2}) =$$

$$= (g(x))^{\delta/2} O\left(\sum_{x/g(x)<|kv|\le x} 1\right) = (g(x))^{\delta/2} O(S^{(q)}_{2,G}(x)) =$$

$$= O\left(\frac{(g(x))^{\delta/2} x^{\delta/2} (\log\log x)^{q-1}}{\log x}\right).$$

Now the first part of the theorem follows with $g(x) = (\log x)^{1/\delta}$, whereas the second part for

$$h(x) = 1/\log x, \quad g(x) = (\log x)^{2/\delta}.$$

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ФОРМУЛА РЕНЬИ С ОСТАТОЧНЫМ ЧЛЕНОМ ДЛЯ АРИФМЕТИЧЕСКИХ ПОЛУГРУПП

Štefan Porubský

Резюме

Арифметическая полугруппа G — это мультипликативно записанная свободная коммутативная полугруппа с единицей со счетной системой образующих, наделеная гомоморфизмом — нормой $|\cdot|$ в мультипликативную полугруппу положительных действительных чисел, в которой для каждого $x > 0$ найдется только конечное число элементов $n \in G$ таких, что $|n| \leq x$. Пусть для этого числа $N_G(x)$ выполняется условие $N_G(x) = Ax^\delta + O(x^\eta)$ для $x \to \infty$. Уточняются результаты об асимптотической оценке числа k-свободных элементов в G и для числа элементов

$$A_{q,G}(x) = \{n \in G;\ |n| \leqq x,\ \Omega(n) - \omega(n) = q\},$$

где k, q — натуральные числа, $k > 1$.