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ON A CLASS OF UNIFORMLY DISTRIBUTED SEQUENCES

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In [4] Knapowski investigated the question of the uniform distribution of sequences of rational numbers of the form

$$\omega(A) = \left\{\frac{1}{a_1}, \frac{2}{a_1}, \ldots, \frac{a_1 - 1}{a_1}, \frac{1}{a_2}, \frac{2}{a_2}, \ldots, \frac{1}{a_n}, \ldots, \frac{a_n - 1}{a_n}, \ldots\right\},$$

where

$$A = \{a_1 < a_2 < a_3 < \ldots\}$$

is a given sequence of positive integers. He proved that this sequence is uniformly distributed in [0, 1] if

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_1 + a_2 + \ldots + a_n} = 0, \tag{1}$$

Knapowski's paper also contains a condition on a_n's guaranteeing that $\omega(A)$ is not uniformly distributed in [0, 1].

In this paper we shall investigate sequences

$$X = \{x(1), x(2), x(3), \ldots, x(n), \ldots\}$$

of real numbers from the unit interval [0, 1] composed of blocks X_n, $n = 1, 2, \ldots$, where the nth block X_n contains a_n terms of X. In other words, if $A = \{a_n\}_{n=1}^{\infty}$ is a given sequence of positive integers, not necessarily increasing and

$$N_n = a_1 + a_2 + \ldots + a_n \quad \text{for} \quad n = 1, 2, \ldots \quad \text{with} \quad N_0 = 0,$$

then

$$\left.\begin{aligned} X &= \{x(i)\}_{i=1}^{\infty},\ x(i) \in [0, 1] \\ X_n &= \{x(i)\}_{N_{n-1} < i \leqq N_n}. \end{aligned}\right\} \tag{2}$$

We show that (1) is a sufficient and necessary condition for a block sequence (2) to be uniformly distributed in [0, 1] provided X is uniformly distributed in blocks and the terms of blocks are ordered according to their magnitude, i.e. if

$$\lim_{n\to\infty} \frac{A(I, X_n)}{a_n} = |I| \tag{3}$$

for every interval $I \subset [0, 1]$ where as usual $A(I, X_n)$ denotes the number of terms of X_n which belong to I and $|I|$ is the length of I, and

$$X_n = \{x(N_{n-1}+1) \leqq x(N_{n-1}+2) \leqq \ldots \leqq x(N_n)\}. \tag{4}$$

This implies, among others, that (1) is a sufficient and necessary condition also for Knapowski's sequence $\omega(A)$.

In the first part of the paper we prove a general result concerning block sequences (2) satisfying (3). In the second part we shall investigate in detail the sequences $A = \{a_n\}_{n=1}^{\infty}$ of positive integers which satisfy condition (1). In part three we show some metrical and topological properties of the system of all such sequences.

1. Basic properties of block sequences.

We shall use the following notation and definitions from [5]:

$A(I, N, X)$ for the number of terms $x(n) \in X$, $1 \leqq n \leqq N$ for which $x(n) \in I$, $R_N(x)$ for the remainder function

$$R_N(x) = A([0, x), N, X) - Nx$$

if $0 \leqq x < 1$, while $R_N(1) = 0$,
D_N^* for the discrepancy,

$$D_N^* = \sup_{0 \leqq x \leqq 1} \frac{|R_N(x)|}{N}$$

$D_N^{(2)}$ for the L^2 discrepancy,

$$D_N^{(2)} = \left(\int_0^1 \left(\frac{R_N(x)}{N}\right)^2 \mathrm{d}x\right)^{1/2}.$$

Given a sequence $M_1, M_2, \ldots$ of positive integers we say that a sequence X is $\{M_n\}_{n=1}^{\infty}$ almost uniformly distributed in [0, 1] if

$$\lim_{n\to\infty} \frac{A(I, M_n, X)}{M_n} = |I|$$

for every interval $I \subset [0, 1]$.

Our first result reflects some relations between the uniform distribution of the sequence (2) and the order properties of its blocks X_k, $k = 1, 2, 3, \ldots$.

Proposition 1. *Let a block sequence X of real numbers from interval* $[0, 1]$ *be composed of blocks X_n, $n = 1, 2, \ldots$ as in* (2) *satisfy* (3) *for every interval $I \subset [0, 1]$. Then we have*

(i) *The sequence X is $\{N_n\}_{n=1}^{\infty}$ almost uniformly distributed in* $[0, 1]$.

(ii) *If* (1) *holds then X is uniformly distributed in* $[0, 1]$ *independently of the ordering in which the terms of the blocks X_k, $k = 1, 2, \ldots$ are given.*

(iii) *If*

$$\limsup_{n \to \infty} \frac{a_{n+1}}{a_1 + a_2 + \ldots + a_n} > 0, \tag{5}$$

then it is possible to rearrange the terms of the blocks X_k for every $k = 1, 2, \ldots$ in such a way that the corresponding sequence X is not u.d.

(iv) *If the sequence X is not u.d., then the terms of the blocks X_k, $k = 1, 2, \ldots$ can be so rearranged that the corresponding sequence X is u.d.. Moreover, if the terms of the blocks X_k, $k = 1, 2, \ldots$ are originally ordered according to their magnitude, then there exists such a rearrangement which depends only on the number of terms in X_k and not on the terms of X_k, $k = 1, 2, \ldots$.*

(v) *If the sequence X corresponding to a given ordering of terms of the blocks X_k, $k = 1, 2, \ldots$ is not u.d., then there exists a sequence $\{m_k\}_{k=1}^{\infty}$ of positive integers with the property that the sequence X' corresponding to the sequence of blocks constructed by listing successively m_k copies of X_k for each $k = 1, 2, \ldots$ is u.d..*

Proof. The proposition can be deduced directly from the definition. However, we shall use the L^2 discrepancy to prove it.

Every positive integer N can be written in the form

$$N = N_n + k, \quad \text{where} \quad 0 \leqq k < a_{n+1}. \tag{6}$$

Let further

$$R_a(x, X_i) = A([0, x), a, X_i) - ax \quad \text{for} \quad 0 \leqq x < 1$$

and

$$R_a(1, X_i) = 0 \quad \text{for} \quad x = 1.$$

Then

$$R_N(x) = \sum_{i=1}^{n} R_{a_i}(x, X_i) + R_k(x, X_{n+1})$$

and consequently

$$N^{-2} \int_0^1 R_N^2(x)\,\mathrm{d}x = (N_n + k)^{-2} \int_0^1 \left(\sum_{i=1}^{n} R_{a_i}(x, X_i) \right)^2 \mathrm{d}x +$$

$$+ (N_n + k)^{-2} \int_0^1 R_k^2(x, X_{n+1})\,\mathrm{d}x +$$

$$+ 2(N_n + k)^{-2} \int_0^1 R_k(x, X_{n+1}) \left(\sum_{i=1}^{n} R_{a_i}(x, X_i) \right) dx. \tag{7}$$

Condition (3) is equivalent to one of the following three relations

$$\left.\begin{aligned} \lim_{n\to\infty} D^*_{a_n} &= \lim_{n\to\infty} \sup_{0\le x\le 1} \frac{|R_{a_n}(x, X_n)|}{a_n} = 0 \\ \lim_{n\to\infty} (D^{(2)}_{a_n})^2 &= \lim_{n\to\infty} a_n^{-2} \int_0^1 R^2_{a_n}(x, X_n)\, dx = 0 \\ \lim_{n\to\infty} a_n^{-1} &\int_0^1 |R_{a_n}(x, X_n)|\, dx = 0. \end{aligned}\right\} \tag{8}$$

Moreover, (3) implies that

$$\lim_{n\to\infty} a_n = \infty. \tag{9}$$

The relations (8) and (9) imply using the so-called Cauchy—Stolz theorem [4, p. 78, Ex. 5] that the first and the third term on the right-hand side of (7) converge to zero if $N \to \infty$ and k, n are arbitrary integers satisfying (6), i.e.

$$N^{-2} \int_0^1 R_N^2(x)\, dx = N^{-2} \int_0^1 R_k^2(x, X_{n+1})\, dx + o(1). \tag{10}$$

Part (i) now follows easily for $k = 0$.

Now write

$$\frac{1}{(N_n + k)^2} \int_0^1 R_k^2(x, X_{n+1})\, dx = \frac{a^2_{n+1}}{(N_n + k)^2} \cdot \frac{k^2}{a^2_{n+1}} \cdot \frac{1}{k^2} \int_0^1 R_k^2(x, X_{n+1})\, dx. \tag{11}$$

Note that

$$|R_k(x)/k| \le 1 \quad \text{and} \quad k/a_{n+1} \le 1.$$

Then (1) implies that the right hand side of (11) converges to zero for $n \to \infty$ and k in the range $0 \le k < a_{n+1}$ which proves (ii).

(iii) We prove more than stated in the theorem. Namely, given a subinterval $I \subset [0, 1]$ with $0 < |I| < 1$, there exists such a rearrangement of terms in X_n for every $n = 1, 2, \dots$ that $A(I, N, X)/N$ does not converge to $|I|$ for $N \to \infty$.

Fix $I \subset [0, 1]$ with $0 < |I| < 1$. Let

$$M_{n+1} = a_{n+1} - A(I, X_{n+1}) \quad \text{for} \quad n = 0, 1, \dots$$

denote the number of terms of the block X_{n+1} which do not belong to I. Then relabel the terms of the block X_{n+1} in such a way that its first M_{n+1} terms

$$x(N_n+1),\ x(N_n+2),\ \ldots,\ x(N_n+M_{n+1})$$

will lie outside I.

The integration over [0, 1] in (10) can be replaced by one over any interval $I \subset [0, 1]$. Further, the following estimation from below is true over any interval not containing the terms of the sequence

$$\frac{1}{N^2}\int_I R^2_{M_{n+1}}(x, X_{n+1})\,\mathrm{d}x \geqq \frac{M^2_{n+1}}{12N^2}\cdot|I|^3 = \left(\frac{a_{n+1}}{N_n+M_{n+1}}\right)^2\cdot\left(\frac{M_{n+1}}{a_{n+1}}\right)^2\cdot\frac{1}{12}|I|^3.$$

Since the right-hand side does not converge to zero, $A(N, I, X)/N$ does not converge to the length of I for $N = N_n + M_{n+1}$ with $n \to \infty$, as claimed.

Note that if we want only to show that X is not u.d. in the case when the terms of blocks X_n, $n = 1, 2, \ldots$ are originally ordered according to their magnitude, then it suffices to take $I = [1/2, 1]$ and the above reasoning works without the necessity to relabel the terms of X_n for every $n = 1, 2, \ldots$.

(iv) Without loss of generality we can suppose that the terms of blocks X_n, $n = 1, 2, \ldots$ of the sequence X are ordered according to their (non-decreasing) magnitude. Put

$$s_1 = a_1$$

and

$$s_n = [a_n/\sqrt{N_{n-1}}] + 1 \quad \text{for} \quad n \geqq 2.$$

Now split each of the blocks X_n, $n = 1, 2, \ldots$ in the following subblocks ordered according to their (non-decreasing) magnitude

$$X_{n,i} = \{x(N_{n-1}+j) : j \equiv i (\operatorname{mod} s_n),\ 1 \leqq j \leqq a_n\} \quad \text{for} \quad i = 1, \ldots, s_n.$$

It is not difficult to see that the blocks $X_{n,i}$ of the sequence

$$X' = \{X_{1,1}, \ldots, X_{1,s_1}, X_{2,1}, \ldots, X_{2,s_2}, \ldots\}$$

satisfy the relations (3), (1) and by (ii), X' is u.d..

(v) To prove this statement it is sufficient to show that the condition (1) is fulfilled in the form

$$\lim_{n\to\infty} \frac{a_{n+1}}{m_1a_1 + m_2a_2 + \ldots + m_na_n} = 0.$$

One of the possibilities is

$$m_n = n \,.\, a_{n+1},$$

and the Proposition is proved.

Parts of Proposition 1 are certainly used at least implicitly in literature, e.g. a result of the type (v) is used in [2]. However, in spite of this fact we have not found these results explicitly stated elsewhere and we hope that these simple but useful properties can be of some independent interest. For instance, (iv) shows that the von Neumann theorem can be proved without transfinite rearrangement provided that (3) is satisfied.

For our purposes the main consequence of Proposition 1 is the following result:

Theorem 1. *Let X be a block sequence of real numbers from the interval* [0, 1] *composed of blocks X_n, $n = 1, 2, \ldots$ which satisfy condition* (3). *Let the terms of each block X_n, $n = 1, 2, \ldots$ be ordered according to their non-decreasing magnitude. Then X is uniformly distributed if and only if the sequence $\{a_n\}_{n=1}^{\infty}$ of the length of blocks X_n fulfils the condition* (1).

Let now $A = \{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence (not necessarily increasing) of positive integers for which $\lim_{n \to \infty} a_n = \infty$. Let

$$X_n = \left\{\frac{1}{a_n}, \frac{2}{a_n}, \ldots, \frac{a_n - 1}{a_n}\right\}$$

for every $n = 1, 2, \ldots$. Then

$$A(I, a_n, X_n) = |I|/(1/a_n) + O(1) = (a_n - 1)|I| + O(1)$$

and Theorem 1 implies that the Knapowski type sequence $X = \omega(A)$ is uniformly distributed if and only if

$$\lim_{n \to \infty} \frac{a_{n+1} - 1}{(a_1 - 1) + (a_2 - 1) + \ldots + (a_n - 1)} = 0,$$

which is obviously equivalent to (1).

We conclude this section with the determination of the L^2 discrepancy of a slightly modified Knapowski sequence $X = \omega'(A)$ composed of blocks

$$X_n = \left\{\frac{1}{a_n}, \frac{2}{a_n}, \ldots, \frac{a_n}{a_n}\right\}$$

for $n = 1, 2, \ldots$.

In the next theorem (a, b) denotes the g.c.d of a and b, and $\{x\}$ the fractional part of x.

Theorem 2. *Let $A = \{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Let $N_n = \sum_{i=1}^{n} a_i$ and $N = N_n + k$ with $0 \leq k < a_{n+1}$. Then for the L^2 discrepancy of the sequence $\omega'(A)$ we have*

$$(ND_N^{(2)})^2 = \frac{1}{4} n^2 + \frac{1}{12} \sum_{i,j=1}^{n} \frac{(a_i, a_j)^2}{a_i a_j} + \left(\frac{k}{a_{n+1}}\right)^2 \left(\frac{1}{3} k^2 + \frac{1}{2} k + \frac{1}{6}\right) +$$

$$+ \left(\frac{k}{a_{n+1}}\right)\left(-\frac{2}{3} k^2 - \frac{1}{2} k + \frac{1}{6}\right) + \frac{1}{3} k^2 + \frac{1}{2} kn + \frac{1}{6} k \sum_{i=1}^{n} \frac{1}{a_i} +$$

$$+ 2 \int_0^{k/a_{n+1}} \{xa_{n+1}\} \left(\sum_{i=1}^{n} \{xa_i\}\right) \mathrm{d}x - 2a_{n+1} \int_0^{k/a_{n+1}} x \left(\sum_{i=1}^{n} \{xa_i\}\right) \mathrm{d}x -$$

$$- 2k \int_{k/a_{n+1}}^{1} \left(\sum_{i=1}^{n} \{xa_i\}\right) \mathrm{d}x. \qquad (12)$$

To compute the integrals one can use the following formulae

$$\int_0^{k/b} \{xb\} \{xa\} \mathrm{d}x = \frac{1}{b}\left(\frac{1}{3} \cdot \frac{ak}{b} - \frac{1}{2} \cdot \frac{k}{ba} (a-1) \left(\frac{2}{3} a + \frac{1}{6}\right) +\right.$$

$$\left. + \sum_{s=0}^{a-1} \sum_{i=0}^{k-1} \frac{2s+1}{2a^2} \left\{\frac{s+ia}{b}\right\}\right), \qquad 0 \leqq k \leqq b, \qquad (13)$$

$$\int_0^t x\{xa\} \mathrm{d}x = \frac{1}{4} t^2 + \frac{1}{12} \cdot \frac{t}{a} - \frac{1}{2} \cdot \frac{\{ta\}^3}{a^2} + \frac{1}{2} \cdot \frac{t\{ta\}^2}{a} - \frac{1}{2} \cdot \frac{t\{ta\}}{a} +$$

$$+ \frac{1}{4} \cdot \frac{\{ta\}^2}{a^2} - \frac{1}{12} \cdot \frac{\{ta\}}{a^2}, \qquad (14)$$

$$\int_0^t \{xa\} \mathrm{d}x = \frac{1}{2} t + \frac{1}{2} \cdot \frac{\{ta\}^2}{a} + \frac{1}{2} \cdot \frac{\{ta\}}{a}. \qquad (15)$$

Proof. We have

$$A([0, x], a_i, X_i) = [xa_i],$$
$$A([0, x], k, X_{n+1}) = \min\{k, [xa_{n+1}]\}.$$

Then using [5, p. 163] we obtain

$$\int_0^1 \left(\sum_{i=0}^{n} R_{a_i}(x, X_i)\right)^2 \mathrm{d}x = \frac{1}{4} n^2 + \frac{1}{12} \sum_{i,j=1}^{n} \frac{(a_i, a_j)^2}{a_i a_j}$$

and from [5, p. 145]

$$\int_0^1 R_k(x, X_{n+1})^2 \mathrm{d}x = \frac{1}{3} k^2 + k \sum_{i=1}^{k} \left(\frac{i}{a_{n+1}}\right)^2 + \sum_{i=1}^{k} \frac{i}{a_{n+1}} - 2 \sum_{i=1}^{k} i \frac{i}{a_{n+1}} \cdot$$

Further

$$\int_0^1 R_k(x, X_{n+1}) \cdot \left(\sum_{i=1}^{n} R_{a_i}(x, X_i)\right) dx =$$

$$= \int_0^1 (\min\{k, [xa_{n+1}]\} - kx) \cdot \left(-\sum_{i=1}^{n} \{xa_i\}\right) dx =$$

$$= \frac{1}{4} kn + \frac{1}{12} k \sum_{i=1}^{n} \frac{1}{a_i} - a_{n+1} \int_0^{k/a_{n+1}} x \left(\sum_{i=1}^{n} \{xa_i\}\right) dx +$$

$$+ \int_0^{k/a_{n+1}} \{xa_{n+1}\} \left(\sum_{i=1}^{n} \{xa_i\}\right) dx - k \int_{k/a_{n+1}}^{1} \left(\sum_{i=1}^{n} \{xa_i\}\right) dx.$$

Relation (12) now follows using (7).

To prove relation (13) note that

$$\int_{i/b}^{(i+1)/b} \{xb\} . \{xa\} \, dx = \frac{1}{b} \int_0^1 y \left\{ y \frac{a}{b} + i \frac{a}{b} \right\} dy.$$

If $s/a < y < (s+1)/a$ then

$$\left\{ y \frac{a}{b} + i \frac{a}{b} \right\} = y \frac{a}{b} - \frac{s}{b} + \left\{ \frac{s + ia}{b} \right\}$$

and thus

$$\int_{s/a}^{(s+1)/a} y \left\{ y \frac{a}{b} + i \frac{a}{b} \right\} dy =$$

$$= \frac{1}{3ba^2} ((s+1)^3 - s^3) + \left(\left\{ \frac{s+ia}{b} \right\} - \frac{s}{b} \right) \cdot \frac{1}{2a^2} \cdot ((s+1)^2 - s^2).$$

The summation of the last integral over $i = 0, 1, \ldots, k-1$, $s = 0, 1, \ldots, a-1$ gives (13). The integrals (14) and (15) can be evaluated directly.

2. Basic properties of sequences $A = \{a_n\}_{n=1}^{\infty}$ satisfying condition (1) and $\lim_{n \to \infty} a_n = \infty$.

In this section we shall investigate on the one hand the relationship between property (1) of a sequence $A = \{a_n\}_{n=1}^{\infty}$ of positive integers and the behaviour of a_n, N_n, a_n/a_{n+1} and the asymptotic density of A on the other hand.

Note that condition (1) is plainly equivalent to each of the following three ones

$$\lim_{n \to \infty} \frac{a_n}{a_1 + a_2 + \ldots + a_n} = 0, \tag{16}$$

$$\lim_{n \to \infty} \frac{a_n}{a_1 + a_2 + \ldots + a_{n-k}} = 0, \tag{17}$$

$$\lim_{n \to \infty} \frac{a_1 + a_2 + \ldots + a_{n-k}}{a_1 + a_2 + \ldots + a_n} = 1. \tag{18}$$

where k is an arbitrary but fixed positive integer.

Having in mind the application of the proved results to the Knapowski type sequences we shall assume in the next results that investigated sequences will be monotone. The reader can easily decide where such assumptions are not necessary from the point of view of the fulfilment of relation (1).

2.1. Estimates.

Theorem 3. *Let $A = \{a_n\}_{n=1}^{\infty}$ be an increasing sequence of positive integers. Then A satisfies condition* (1) *if one of the following four conditions is fulfilled*:

(a) *There exists $\alpha \geqq 1$ such that*

$$\liminf_{n \to \infty} \frac{a_n}{n^\alpha} > 0 \quad \text{and} \quad a_n = o(n^{\alpha+1}).$$

(b) *There holds $a_n = o(n^2)$.*

(c) *There exists $\beta > 1/2$ such that*

$$\liminf_{n \to \infty} \frac{A(n)}{n^\beta} > 0,$$

where $A(n) = \sum_{a_i \leqq n} 1.$

(d) *The sequence A has positive lower asymptotic density.*

Proof. It is not difficult to see that it is enough to prove that (a) implies (1). But this can be easily done. The condition on the limes inferior implies

$$\frac{a_n}{a_1 + a_2 + \ldots + a_n} < c \frac{a_n}{1^\alpha + 2^\alpha + \ldots + n^\alpha} < c' \frac{a_n}{n^{\alpha+1}}$$

and the condition on a_n implies (16) and thus finishes the proof.

If $A = \boldsymbol{P}$, the set of primes, then Theorem 3(b) shows that $\boldsymbol{P}$ satisfies (1). A refined version of this result was shown by Knapowski [3]. The sequence $\boldsymbol{P}$ shows at the same time that (d) does not give a necessary condition for (1) and that not even the positivity of the upper asymptotic density is necessary for this conclusion.

Although conditions (a) and (b) of Theorem 3 are not eqivalent (take $a_n = p_n^2$, p_n the nth prime), condition (b) is best possible in the sense that the o-symbol cannot be replaced by the O-symbol without further additional assumptions. Namely, there exists sequence $A = \{a_1 < a_2 < \dots\}$ with $a_n = O(n^2)$ for which (1) does not hold. To construct such a sequence we first construct a sequence of indices $n_1 < n_2 < \dots$ inductively in the following way:

a) $n_1 = 2$

b) if n_k is already known, let n_{k+1} be a positive integer n satisfying the following three inequalities

$$n^2 - n(4n_k^2 - 2n_k - 1) - 4n_k^4 + 4n_k^3 + n_k^2 - n_k > 0, \tag{19}$$

$$2n^2 - n - 2n_k^2 + n_k + 1 > 0, \qquad n > n_k, \tag{20}$$

then the sequence A in question is defined as follows: $a_1 = 1$, $a_2 = 2$, ..., and

$$a_n = 2n_k^2 + n - n_k - 1 \quad \text{for} \quad n_k + 1 \leqq n \leqq n_{k+1}.$$

The inequality (20) guarantees that A is increasing and (19) implies that

$$\frac{a_{n_{k+1}} + 1}{a_1 + a_2 + \dots + a_{n_{k+1}}} \geqq \frac{2n_{k+1}^2}{1 + 2 + 3 + \dots + a_{n_{k+1}} - 1 + a_{n_{k+1}}} \geqq 2.$$

Thus the condition (17) is not satisfied.

Choosing the smallest possible n_k, $k = 1, 2, \dots$, the initial segment of the sequence A is

$$a_1 = 1,\ a_2 = 2,\ a_3 = 8,\ a_4 = 9,\ \dots,\ a_{14} = 19,\ a_{15} = 392,\ a_{16} = 393,$$
$$\dots,\ a_{912} = 1289,\ a_{913} = 1\,663\,488,\ a_{914} = 1\,663\,489,\ \dots,\ a_{4013813} = 5\,676\,388,$$
$$a_{4013814} = 32\,221\,389\,597\,938,\ a_{4013815} = 32\,221\,389\,597\,939,\ \dots$$

The next result contains an information about the speed of divergence of N_n if (1) is satisfied.

Theorem 4. *If a sequence $A = \{a_n\}_{n=1}^{\infty}$ of positive integers satisfies* (1), *then*

$$\lim_{n \to \infty} \frac{1}{n} \log(a_1 + a_2 + \dots + a_n) = 0. \tag{21}$$

Proof. Niederreiter [7] proved that if $\{x_n\}_{n=1}^{\infty}$ is a monotone uniformly distributed sequence mod 1, then

$$\lim_{n \to \infty} |x_n|/\log n = \infty. \tag{22}$$

Let

$$x_n = k + \frac{i}{a_k},$$

where

$$n = \sum_{j=1}^{k-1} a_j + i, \qquad 0 \leqq i < a_k.$$

If (1) is satisfied and if $\lim_{n \to \infty} a_n = \infty$, then $\{x_n\}_{n=1}^{\infty}$ is uniformly distributed mod 1 and (22) implies that

$$\lim_{k \to \infty} k/\log N_k = \infty$$

and the result follows.

Another proof of the previous result, which does not depend on the assumption that $\lim_{n \to \infty} a_n = \infty$, follows using upper and lower Riemann's sums of the function $1/x$ over the interval $[1, N_n]$. These sums over the subdivision on the intervals $[1, N_1], [N_1, N_2], \ldots, [N_{n-1}, N_n]$ and the integral (after division by n) give the following inequalities

$$\frac{1}{n} \sum_{i=1}^{n} \frac{a_i}{N_i} < \frac{1}{n} \log N_n < \frac{1}{n} \sum_{i=1}^{n} \frac{a_i}{N_{i-1}}. \tag{23}$$

The relation (21) follows now immediately from (23) and (1).

Note that from (23) one obtains Abel's theorem [1] stating that the divergence of the series $\sum_{n=1}^{\infty} a_n$ is equivalent to the divergence of each of the series

$$\sum_{n=1}^{\infty} \frac{a_n}{N_n}, \quad \sum_{n=1}^{\infty} \frac{a_{n+1}}{N_n},$$

which shows that the convergence in (1) cannot be too quick.

The next two simple results can be useful. Their proofs follow from the Toeplitz theorem for $x_n = a_{n-1}/a_n$. Then $a_1 + \ldots + a_{n-1} = a_2 x_2 + \ldots + a_n x_n$.

Theorem 5. *Let $A = \{a_1 < a_2 < \ldots\}$ be a sequence of positive integers.*

i) If A satisfies

$$\lim_{n \to \infty} \frac{a_{n-1}}{a_n} = 1 \tag{24}$$

then (1) *is true for* A.

ii) *If* A *satisfies* (1) *then*

$$\limsup_{n\to\infty} \frac{a_{n-1}}{a_n} = 1$$

Theorem 5 ii) indicates that the condition (24) cannot be necessary for a sequence A satisfying (1). In fact, given any α, $0 < \alpha < 1$ there exists a sequence $A = \{a_1 < a_2 < \ldots\}$ of positive integers with

$$\liminf_{n\to\infty} \frac{a_{n-1}}{a_n} \leqq \alpha$$

and satisfying (1). To see this let c be a positive integer with $c \geqq 1/\alpha$. Then the sequence A, which we obtain from the sequence of positive integers after deleting all the segments of the form

$$c^{2k+1},\ c^{2k+1}+1,\ \ldots,\ c^{2k+2}-1,\ c^{2k+2}, \qquad k = 1, 2, \ldots$$

has the required properties.

The analysis of the points of accumulation of the sequence $\{a_{n-1}/a_n\}_{n=2}^{\infty}$ for the just constructed sequence $A = \{a_1 < a_2 < \ldots\}$ suggests the conjecture that for "almost all" n, a_{n-1}/a_n converges to 1 provided that (1) is true. This is really so, as the next result shows.

Theorem 6. *Given a real number* δ *and a sequence* $A = \{a_1 < a_2 < \ldots\}$ *of positive integers, let*

$$A(\delta) = \left\{n;\ \frac{a_n}{a_{n+1}} \leqq \delta\right\}.$$

If the sequence A *satisfies* (1) *then for every* $0 < \delta < 1$ *the upper asymptotic density* $\bar{\mathrm{d}}(A(\delta)) = 0$.

Proof. Let $A(\delta) = \{n_1 < n_2 < \ldots\}$. Since A is monotone, $a_{n_i}/a_{n_{i-1}} \geqq 1/\delta$ for every $i = 2, 3, \ldots$. Thus

$$a_{n_i} = \frac{a_{n_i}}{a_{n_{i-1}}} \cdots \frac{a_{n_2}}{a_{n_1}} \cdot a_{n_1} \geqq (1/\delta)^{i-1}.$$

Then

$$\frac{1}{n_i} \log N_{n_i} \geqq \frac{1}{n_i} \log a_{n_i} \geqq \frac{i-1}{n_i} \log(1/\delta)$$

If A satisfies (1), then (21) implies that $\lim_{i \to \infty} i/n_i = 0$, which is equivalent to $d(A(\delta)) = 0$.

Note that the condition of Theorem 6 is not sufficient. We shall see that there exist sequences $A = \{a_n\}_{n=1}^{\infty}$ which satisfy (1) but for which the sequence $B = \{a_n^2\}_{n=1}^{\infty}$ does not satisfy (1). On the other hand

$$\{A(\delta);\ 0 < \delta < 1\} = \{B(\delta);\ 0 < \delta < 1\}$$

here.

2.2. Subsequences.

The next result may be instrumental in many proofs of results of the investigated type.

Theorem 7. *If $B \subset A$ are two increasing sequences of positive integers and if the sequence B satisfies* (1), *then also the sequence A satisfies* (1).

Proof. Let

$$A = \{a_1 < a_2 < \ldots\}$$

and

$$B = \{a_{n_1} < a_{n_2} < \ldots\}.$$

Given k, define n_s by the inequalities

$$n_{s-1} < k \leqq n_s.$$

Then

$$\frac{a_k}{a_1 + a_2 + \ldots + a_k} < \frac{a_{n_s}}{a_{n_1} + a_{n_2} + \ldots + a_{n_{s-1}}}$$

Since B satisfies (1), the right hand side tends to 0 as $s \to \infty$ according to (17). We may therefore apply (16) to the sequence A and the result follows at once.

We have seen in the previous result that whenever a subsequence $\{a_{b_1} < a_{b_2} < \ldots\}$ of A satisfies (16) then A does also. The set of all positive integers shows that the converse is not true. However, if for $\{b_n\}$ we take any arithmetic sequence, then this converse statement is true. For if, say,

$$b_n = b \,.\, n + c,$$

then

$$\frac{a_{nb+c}}{a_{b+c} + a_{2b+c} + \ldots + a_{nb+c}} \leqq \frac{a_{nb+c}}{(a_{c+1} + a_{c+2} + \ldots + a_{nb+c})/b}$$

and the conclusion follows at once.

We could ask for more general conditions on $\{b_n\}$ which imply that the preporty (16) is satisfied for $\{a_{b_n}\}$, provided that for $\{a_n\}$ it is too. The next example shows — contrary to an eventual expectation raised by Theorem 3(d) — that the arithmetical sequences of indices cannot be replaced by sequences with positive lower asymptotic density in general. We shall construct a sequence $\{a_n\}$ satisfying (16) having a subsesquence $\{a_{b_n}\}$ not satisfying (16) with the property that the sequence $\{b_n\}$ of indices has asymptotic density 1.

Suppose that we already have constructed the first N terms

$$a_1 < a_2 < \ldots < a_N$$

of the sequence $\{a_n\}$ and that a_N belongs to the subsequence $\{a_{b_n}\}$.

The next $y = Na_N$ terms of the form

$$a_{N+i} = a_N + i, \quad 1 \leqq i \leqq y$$

will all belong to $\{a_{b_n}\}$. Now comes a gap of length $z - 1$, where

$$z = [(Na_N)^{3/4}]. \tag{25}$$

with the terms

$$a_{N+y+i} = [(Na_N)^{3/2}] + i, \quad 1 \leqq i \leqq z - 1$$

not belonging to $\{a_{b_n}\}$. Finally put

$$a_{N+y+z} = (Na_N)^2$$

As the next term of $\{a_n\}$ belonging to $\{a_{b_n}\}$ and repeat the previous construction with $N + y + z$ instead of N, etc.

The sequence $\{a_n\}$ thus constructed is a strictly increasing sequence of positive integers. This sequence satisfies (16) as can be seen from the following facts:

a) the quotient on the left hand side of (16) is decreasing in the range $n = N$, $N + 1 \ldots N + y$.

b) for $n = N + y + 1$ this quotient is small because $\sum_{i=1}^{y} a_{N+i}$ is asymptotically equal to $(Na_N)^2/2$ and $a_{N+y+1} = [(Na_N)^{3/2}] + 1$. To the values $n = N + y + 2$, $N + y + 3, \ldots, N + y + z - 1$ we can apply the same argument as in a).

c) for $n = N + y + z$ this quotient is also small because $\sum_{i=1}^{z-1} a_{n+y+i}$ is asymptotically equal to $(Na_N)^{3/2}(Na_N)^{3/4}$ and $a_{N+y+z} = (Na_N)^2$.

On the other hand the subsequence $\{a_{b_n}\}$ does not fulfil (16). To see this put $b_n = N + y + z$, then

$$\frac{a_{b_n}}{a_{b_1} + a_{b_2} + \dots + a_{b_n}} \geqq \frac{(Na_N)^2}{\sum_{i=1}^{N} a_i + \sum_{i=1}^{y} a_{N+i}} \geqq 2 - \varepsilon_N,$$

where $\varepsilon_N \to 0$ with $N \to \infty$. Relation (25) implies that the density of the sequence $\{b_n\}$ is $\lim_{N \to \infty} y/(y + z) = 1$ as stated.

Note further that Theorem 3(d) implies that $\{b_n\}$ also satisfies (16). The just constructed example shows, by the way, that if two increasing sequences $\{a_n\}$,, $\{b_n\}$ of positive integers satisfy (16) then their superposition $\{a_{b_n}\}$ does not satisfy (16) in general.

2.3. Arithmetical properties.

Theorem 8. *If the sequences* $A = \{a_1 < a_2 < \dots\}$, $B = \{b_1 < b_2 < \dots\}$ *satisfy* (1), *then also* $C = \{c_n\}_{n=1}^{\infty}$ *and* $D = \{d_n\}_{n=1}^{\infty}$ *are increasing and satisfy* (1), *where* $c_n = a_n + b_n$ *and* $d_n = a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1$ *for* $n = 1, 2, \dots$.

Proof. The proof of the statement concerning the sequence C follows from the general inequality

$$\frac{a+b}{c+d} \leqq \frac{a}{c} + \frac{b}{d}$$

(a, b, c, d positive) for $a = a_n$, $b = b_n$, $c = a_1 + a_n$ and $d = b_1 + \dots + b_n$.

The termwise addition of two increasing sequences is increasing and so C is trivially increasing. The fact that D is also increasing follows from

$$a_1 b_n < a_1 b_{n+1},\ a_2 b_{n-1} < a_2 b_n,\ \dots,\ a_n b_1 < a_n b_2.$$

A well-known fact from the theory of Riesz or Norlund means [9, vol. I. Absch. 1, Aufg. 73] says that if

$$\lim_{n \to \infty} \frac{a_n}{a_1 + a_2 + \dots + a_n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n}{b_1 + b_2 + \dots + b_n} = 0$$

then also

$$\lim_{n \to \infty} \frac{d_n}{d_1 + d_2 + \dots + d_n} = 0,$$

where $d_n = a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1$ and the proof is finished.

The question about the termwise multiplication of sequences satisfying (1) is more delicate. In this connection the next result follows from theorem 3(a). Note only that if $a_n = O(n^{3/2})$, then $\{a_n\}_{n=1}^{\infty}$ satisfies (1) owing to Theorem 3(b).

Theorem 9. *If $a_n = O(n^{3/2})$ and $b_n = o(n^{3/2})$, then the sequence $\{a_n b_n\}_{n=1}^{\infty}$ satisfies* (1).

The assumptions of the preceding theorem cannot be extended without additional requirements. We shall construct a sequence $A = \{a_1 < a_2 < \ldots\}$ with $a_n = o(n^{3/2})$ for which the sequence $\{a_1^2 < a_2^2 < \ldots\}$ does not satisfy (1).

We shall proceed by induction. Let $a_1 = 1$, $a_2 = 2$ and suppose that the first N terms $a_1 < a_2 < \ldots < a_N$ with $a_n \leqq n^{3/2}$ for $n = 1, 2, \ldots, N$ have already been found.

Now let k be any (e.g. the least) positive integer satisfying the next two inequalities

$$(N + k)^{3/2} > a_N + k - 1 \tag{26}$$

$$\frac{[(N + k)^{3/2}]^2}{a_1^2 + a_2^2 + \ldots + a_{N-1}^2 + k a_N^2 + a_N k(k - 1) + (k - 1)k(2k - 1)/6} \geqq 1. \tag{27}$$

(Here [ ] denotes the integral part.)

The for the next k terms of the constructed sequence take

$$a_{N+i} = a_N + i \quad \text{for} \quad i = 1, 2, \ldots, k - 1$$

and

$$a_{N+k} = [(N + k)^{3/2}].$$

Then $a_n \leqq n^{3/2}$ for $n = N + 1, n + 2, \ldots, N + k$. The inequality (26) implies that the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing and (27) implies that

$$\limsup_{n \to \infty} \frac{a_n^2}{a_1^2 + a_2^2 + \ldots + a_n^2} \geqq 1$$

This implies that $\{a_1^2 < a_2^2 < \ldots\}$ does not satisfy (1).

If we always take the least positive value of k we obtain the following initial segment

$$a_1 = 1,\ a_2 = 2,\ a_3 = 5,\ a_4 = 7,\ a_5 = 11,\ a_6 = 12,\ \ldots,\ a_9 = 15,\ a_{10} = 31,$$

$$a_{11} = 32,\ \ldots,\ a_{41} = 62,\ a_{42} = 272,\ a_{43} = 273,\ \ldots,\ a_{489} = 719,\ a_{490} = 10\,846,$$

$$a_{491} = 10\,847,\ \ldots.$$

Finally note that if $\{a_1^2 < a_2^2 < \ldots\}$ satisfies (1), then $\{a_1 < a_2 < \ldots\}$ always satisfies (1), for

$$\left(\frac{a_n}{a_1 + a_2 + \ldots + a_n}\right)^2 \leqq \frac{a_n^2}{a_1^2 + a_2^2 + \ldots + a_n^2}$$

Theorem 10. *If $A = \{a_1 < a_2 < \ldots\}$ satisfies* (1), *then also* $S = \left\{\sum_{k=1}^{n} a_k\right\}_{n=1}^{\infty}$ *does.*

Proof. Let $N_n = \sum_{k=1}^{n} a_k$. It follows from (18) that if A satisfies (1), then $\lim_{n \to \infty} N_{n-1}/N_n = 1$ and Theorem 5 i) immediately implies that S satisfies (1) too.

Theorem 11. *Let $p(x)$ be a polynomial with integral coefficients and a positive leading coefficient. Then a sequence $A = \{a_1 < a_2 < \ldots\}$ satisfies* (1) *if and only if the sequence of positive terms of $\{p(n)\,a_n\}_{n=1}^{\infty}$ does.*

Proof. With $b_i = i^k$ in the so-called Čebyšev inequality [11, p. 21 and 119] we have

$$\frac{a_n n^k}{a_1 1^k + a_2 2^k + \ldots + a_n n^k} \leqq n \cdot \frac{a_n}{a_1 + a_2 + \ldots + a_n} \cdot \frac{n^k}{1^k + 2^k + \ldots + n^k}.$$

This implies that if A satisfies (1), then also $\{n^k a_n\}_{n=1}^{\infty}$ does. On the other hand, with $\{a_n\}_{n=1}^{\infty}$ the sequences $\{ca_n\}_{n=1}^{\infty}$ and $\{a_n + c\}_{n=1}^{\infty}$ for any integer c satisfy (16), too. Since a finite number of terms does not influence the truth of (16) we can immediately conclude that if $\{a_n\}_{n=1}^{\infty}$ satisfies (16), then also the sequence of positive terms of $\{p(n)\,a_n\}_{n=1}^{\infty}$ satisfies (16), which is the first part of our theorem.

Conversely, if a sequence $\{c_n\}_{n=1}^{\infty}$ satisfies (16), then also the sequence of rational numbers $\{c_n/n^k\}_{n=1}^{\infty}$, k a fixed positive integer, satisfies (16), which can be seen from the inequality

$$\frac{\dfrac{c_n}{n^k}}{\dfrac{c_1}{1^k} + \dfrac{c_2}{2^k} + \ldots + \dfrac{c_n}{n^k}} = \frac{c_n}{c_1(n/1)^k + c_2(n/2)^k + \ldots + c_n(n/n)^k} \leqq \frac{c_n}{c_1 + c_2 + \ldots + c_n}.$$

In particular, if $k = \deg(p(x))$, the fact that $\{p(n)\,a_n\}_{n=1}^{\infty}$ satisfies (16) implies that also the sequence

$$\left\{\frac{p(n)}{n^k}\,a_n\right\}_{n=1}^{\infty}$$

does.

Finally, if $\{b_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ are two sequences of not necessarily integral positive real numbers with

$$\lim_{n \to \infty} \frac{b_n}{d_n} = c > 0,$$

then both $\{b_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ simultaneously fulfil or do not fulfil condition (16). Since

$$\lim_{n \to \infty} \left(\frac{p(n)}{n^k} a_n\right) \Big/ a_n$$

is positive, the conclusion follows.

2.4. Linear recurrent sequences.

Throughout this section we adopt a new convention in the notation. For the sake of the notational simplicity we shell label the sequences beginning with the index 0 instead of with 1.

Thus, let $A = \{a_n\}_{n=0}^{\infty}$ be a linear recurrence of positive integers, that is

$$a_n = -b_1 a_{n-1} - b_2 a_{n-2} - \dots - b_s a_{n-s}, \qquad n \geqq s \tag{28}$$

with integral coefficients $b_1, b_2, \dots, b_s$. If we assume that A is of order s, that is s is the least possible value in a relation of the type (28), then

$$Q(z) = z^s + b_1 z^{s-1} + b_2 z^{s-2} + \dots + b_s$$

is the so-called characteristic polynomial of the linear recurrence A.

The next theorem solves our problem for linear recurrences.

Theorem 12. *Let $A = \{0 < a_0 < a_1 < a_2 < \dots\}$ be an increasing linear recurrence. Then A satisfies* (1) *if and only if the characteristic polynomial $Q(z)$ of A has the following two properties*:
(a) *all roots of $Q(z)$ are roots of unity,*
(b) $Q(1) = 0$ *and the multiplicity of* 1 *is at least* 2 *and it is* (*strongly*) *greater than the multiplicity of any other root of $Q(z)$.*

Proof. Let $z_1, \dots, z_p$ be all the distinct roots of $Q(z)$ with multiplicity $k_1, \dots, k_p$, respectively. Assume that $|z_1| \geqq \dots \geqq |z_p|$ and $k_j \geqq k_{j+1}$ whenever $|z_j| = |z_{j+1}|$, $j = 1, \dots, p-1$. Let $t = \max\{j;\ |z_j| = |z_1|\}$ and $k_j = k_1$. Then

$$a_n = |z_1|^n . n^{k_1 - 1}(C_1 e^{i\alpha_1 n} + \dots + C_t e^{i\alpha_t n} + \varepsilon_n), \qquad \mathrm{n} > 0,$$

where $C_j \neq 0$, $z_j = |z_1| e^{i\alpha_j}$, $j = 1, \dots, t$ and $\lim_{n \to \infty} \varepsilon_n = 0$.

The sufficiency of (a) and (b) follows immediately from Theorem 3(a). To prove the necessity of the conditions suppose that A satisfies (1).

Let $0 < \delta < 1$ be arbitrary. By Theorem 6 we may choose an integer $r(\delta) \geqq 1$ such that

$$a_n/a_{n+1} \geqq 1 - \delta, \ ((n+1)/n)^{k_1 - 1} \leqq 1 + \delta, \quad \text{and} \quad |\varepsilon_n| \leqq \delta$$

for $n = r(\delta), r(\delta) + 1, \ldots, r(\delta) + t$. A routine calculation gives that

$$\sum_{j=1}^{t} C_j(1 - |z_1| e^{i\alpha_j}) e^{i\alpha_j r(\delta)} e^{i\alpha_j k} = \beta_k(\delta), \qquad k = 0, 1, \ldots, t-1,$$

where $|\beta_k(\delta)| \leqq C\delta$, $k = 0, 1, \ldots, t-1$. Letting $\delta \to 0$, we conclude that $1 - |z_1| e^{i\alpha_j} = 0, j = 0, 1, \ldots, t$. Hence $\alpha_1 = \ldots = \alpha_t = 0$, so that $t = 1$ and $z_1 = 1$. Thus, condition (a) holds. Since $t = 1$, we have $k_1 > k_2 \geqq \ldots \geqq k_p$. From $a_n = n^{k_1 - 1}(C_1 + \varepsilon_n) > n$ we infer that $k_1 \geqq 2$ and Theorem 12 is proved.

Corollary. *Let $A = \{0 < a_0 < a_1 < a_2 < \ldots\}$ be an increasing linear recurrence. Then A satisfies* (1) *if and only if* $\lim_{n \to \infty} a_n/a_{n+1} = 1$.

Let $A = \{a_n\}_{n=1}^{\infty}$ be a linear recurrence of positive integers. Then the Laurent series

$$\mathscr{Z}(\{a_n\}_{n=0}^{\infty}) = \sum_{n=0}^{\infty} a_n z^{-n},$$

called the $\mathscr{Z}$-transform of A, defines a rational function, say, of the form

$$\mathscr{Z}(\{a_n\}_{n=0}^{\infty}) = \frac{P(z)}{Q(z)}.$$

Without loss of generality we can assume that the polynomials $P(z)$ and $Q(z)$ are coprime. It follows from the known results [9, II, Absch. 8, Aufg. 156] that we can also assume that P and Q have integral coefficients and that the polynomial Q is the characteristic polynomial of $\{a_n\}_{n=0}^{\infty}$.

Some of the results proved for the general sequences A follow immediately from Theorem 12 if A is a strongly increasing linear recurrence. Thus, for instance, if $A = \{0 < a_0 < a_1 < a_2 < \ldots\}$ satisfies (1), then also $B = \{b_n\}_{n=0}^{\infty}$ does, where

a) $b_n = \sum_{i=0}^{n-1} a_i$, $b_0 = 0$

b) $b_n = n \, . \, a_n$

c) $b_n = a_{n.m}$ with fixed m.

For the proof it is enough to write the $\mathscr{Z}$-transform of the sequence $\{b_n\}_{n=0}^{\infty}$ which is

$$\mathscr{Z}\left(\left\{\sum_{i=0}^{n-1} a_i\right\}_{n=0}^{\infty}\right) = \frac{F(z)}{z-1}, \quad \mathscr{Z}(\{na_n\}_{n=0}^{\infty}) = -z \cdot \frac{\mathrm{d}F(z)}{\mathrm{d}z},$$

$$\mathscr{Z}(\{a_{n.m}\}_{n=0}^{\infty}) = m^{-1} \cdot \sum_{i=0}^{m-1} F(z/\zeta_i)$$

where ζ_i, $i = 0, 1, \ldots, m-1$ are all mth roots of unity. In all cases $F(z)$ stands for the $\mathscr{Z}$-transform of the given sequence A.

d) Let $A = \{a_0 < a_1 < a_2 < \ldots\}$ and $C = \{c_0 < c_1 < c_2 < \ldots\}$ be two linear recurrences of positive integers. Then their convolution

$$\{a_n\}_{n=0}^{\infty} * \{c_n\}_{n=0}^{\infty} = \left\{\sum_{i=0}^{n} a_i c_{n-i}\right\}_{n=0}^{\infty}$$

satisfies (1) if and only if both A and C satisfy (1). This extends the corresponding part of Theorem 8 for linear recurrences. To the proof note that

$$\mathscr{Z}(\{a_n\}_{n=0}^{\infty} * \{c_n\}_{n=0}^{\infty}) = \mathscr{Z}(\{a_n\}_{n=0}^{\infty}) * \mathscr{Z}(\{c_n\}_{n=0}^{\infty}).$$

e) A similar extension of Theorem 9 is also true for the strongly increasing recurrences A and C. The sequence $\{a_n c_n\}_{n=0}^{\infty}$ satisfies (1) if and only if both A and C satisfy (1). The proof follows immediately from the last Corollary.

Finally, Theorem 12 can be useful for many well-known linear recurrences. For instance:

f) the Fibonacci sequence $\{f_n\}_{n=0}^{\infty}$ does not satisfy (1),

g) the sequence of binomial coefficients $\left\{\binom{n}{k}\right\}_{n=k}^{\infty}$ satifies (1),

h) the sequence $\{(a^{n+1} - b^{n+1})/(a-b)\}_{n=0}^{\infty}$ does not satisfy (1) if $a > b$ are positive integers, etc. To see this note that the $\mathscr{Z}$-transforms of these three sequences are

$$z^2/(z^2 - z - 1),\ z/(z-1)^{k+1},\ z^2/(z-a)(z-b).$$

3. Metrical and topological properties.

Let $\mathscr{U}$ denote the system of all the increasing infinite sequences of positive integers, where we return to the original convention to label the terms of sequences from 1 onwards. Let $\mathscr{U}_1$ denote the system of all the sequences from $\mathscr{U}$ which satisfy (1).

If $A \in \mathscr{U}$ and $A = \{a_1 < a_2 < \ldots\}$, define

$$\varrho(A) = \sum_{k=1}^{\infty} 2^{-a_k}.$$

it is known [8, p. 17—18] that this so-called dyadic transform is a bijection of $\mathcal{U}$ onto the interval (0, 1]. Following a standard convention we shall attribute the metrical and topological properties of the set

$$\varrho(\mathcal{S}) = \{\varrho(A);\ A \in \mathcal{S}\}$$

in the interval (0, 1] to a given subset $\mathcal{S}$ of $\mathcal{U}$.

The next theorem shows that from the metrical point of view allmost all sequences from $\mathcal{U}$ satisfy (1) although one can readily deduce from Theorem 5 that the complement comp($\mathcal{U}_1$) in $\mathcal{U}$ is uncountable. Indead, all the sequences of the type

$$\{2^{k_n}\}_{n=1}^{\infty}, \qquad 0 < k_1 < k_2 < \ldots$$

lie in comp ($\mathcal{U}_1$).

Theotem 13. *We have that*
(i) *the set* $\varrho(\mathcal{U}_1)$ *has the full Lebesgue measure in* (0, 1]
(ii) *the Hausdorff dimension* $\dim \varrho(\text{comp}(\mathcal{U}_1)) = 0$.

Proof. Given a real $\eta \geqq 0$, let $\Gamma(\eta)$ denote the set of all the sequences $A \in \mathcal{U}$ with lower asymptotic density equal to η. Theorem 3(d) implies that

$$\Gamma(\eta) \subset \mathcal{U}_1 \tag{29}$$

for every $0 < \eta \leqq 1$.

Borel's theorem [8, p. 1901] implies that $\varrho(\Gamma(1/2))$ has Lebesgue messure 1 in (0, 1] and the statement (i) follows from (29) for $\eta = 1/2$. From (29) we obtain

$$\bigcup_{0<\eta\leqq 1} \Gamma(\eta) \subset \mathcal{U}_1$$

and consequently

$$\dim \varrho(\text{comp}(\mathcal{U}_1)) \leqq \dim \varrho(\Gamma(0)).$$

However, $\dim \varrho(\Gamma(0)) = 0$, [8, p. 195] and the Theorem is proved.

Theorem 3 gives an impetus to find an estimate for the Hausdorff dimension of $\varrho(\text{comp}(\mathcal{U}_1))$ with respect to a finer system of meassure functions than the system of the standard messure functions t^{α}, $0 \leqq t \leqq 1$, $\alpha \in (0, 1]$. Namely, the system $\mathcal{F}$ of the measure functions $\mu^{(\alpha)}$ defined through ([10], [12])

$$\mu^{(\alpha)}(0) = 0,$$

$$\mu^{(\alpha)}(t) = \left(-\frac{\log 2}{\log t}\right)^{\left(-\frac{\log t}{\log 2}\right)^{\alpha}}, \qquad 0 < t \leqq 1/2 \tag{30}$$

for $\alpha \in (0, 1]$.

If $\dim_{\mathscr{F}} M$ denotes the Hausdorff dimension of M with respect to the system $\mathscr{F}$, then

$$\dim_{\mathscr{F}} M \geqq \dim M$$

for every $M \subset (0, 1]$.

Theorem 14. *With $\mathscr{F}$ defined in* (30) *we have*

$$\dim_{\mathscr{F}} \varrho(\operatorname{comp}(\mathscr{U}_1)) \leqq 1/2.$$

Proof. Take $\alpha > 1/2$ and let $\mathscr{P}_\alpha$ denote the set of all the sequences $A \in \mathscr{U}$ for which

$$\liminf_{x \to \infty} \frac{A(x)}{x^\alpha} > 0.$$

Theorem 3 asserts that $\mathscr{P}_\alpha \subset U_1$ and therefore

$$\dim_{\mathscr{F}} \varrho(\operatorname{comp}(\mathscr{U}_1)) \leqq \dim_{\mathscr{F}} \varrho(\operatorname{comp}(\mathscr{P}_\alpha)).$$

owing to the definition of $\mathscr{P}_\alpha$ the complement $\operatorname{comp}(\mathscr{P}_\alpha)$ consists of all such sequences $A = \{a_1 < a_2 < \ldots\}$ for which

$$\liminf_{n \to \infty} \frac{A(n)}{n^\alpha} = 0.$$

If $G_2(\beta, \xi)$ denotes the set of all the $A \in \mathscr{U}$ with

$$\liminf_{n \to \infty} \frac{A(n)}{n^\beta} = \xi,$$

then it is proved in [12] that for $\xi > 0$ we have

$$\dim_{\mathscr{F}} \varrho(G_2(\beta, \xi)) = \beta.$$

But this result can be extended to the case of $\xi = 0$ in the form

$$\dim_{\mathscr{F}} \varrho(G_2(\beta, \xi)) \leqq \beta. \tag{31}$$

Suppose we have proved it. Since $G_2(\alpha, 0) = \operatorname{comp}(\mathscr{P}_\alpha)$,

$$\dim_{\mathscr{F}} \varrho(\operatorname{comp}(\mathscr{U}_1)) \leqq \alpha$$

for every $\alpha > 1/2$ and the theorem follows.

To prove (31) take a [illegible] such that $\gamma > \beta + \delta$.

Let $A \in G_2(\beta, 0)$. Then there exists an infinite sequence $\{N_k\}_{k=1}^{\infty}$ such that

$$\mathscr{A}(N_k)/N_k^{\beta} < 1. \tag{32}$$

On the other hand, the dyadic transform $\varrho(A)$ can be written in the form

$$\varrho(A) = \sum_{n=1}^{\infty} 2^{-a_k} = \sum_{n=1}^{\infty} \varepsilon_n 2^{-n},$$

where

$$\varepsilon_n = \begin{cases} 1, & \text{if } \; n \in A \\ 0, & \text{if } \; n \notin A. \end{cases}$$

Then (32) says that

$$\sum_{n \leqq N_k} \varepsilon_n < N_k$$

for every $k = 1, 2, \ldots$.

Given a positive integer m, let $\mathscr{S}_m$ be the set of the all zero-one sequences $\{\varepsilon_n\}_{n=1}^{\infty}$ for which

$$\sum_{n \leqq m} \varepsilon_n < m^{\beta}.$$

to this m, let S_m be the union of all such intervals of the mth order

$$\left[0, \frac{1}{2^m}\right), \left[\frac{1}{2^m}, \frac{2}{2^m}\right), \ldots, \left[\frac{2^m - 1}{2^m}, 1\right)$$

which contain at least one point of the form

$$\sum_{n=1}^{\infty} \varepsilon_n \,.\, 2^{-n}, \qquad \text{where} \qquad \{\varepsilon_n\} \in \mathscr{S}_m.$$

Therefore $A \in G_2(\beta, 0)$ implies that

$$\sum_{n=1}^{\infty} 2^{-a_n} \in S_m$$

for infinitely many m. In other words

$$\varrho(G_2(\beta, 0)) \subset \bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} S_m \tag{33}$$

However,

$$\operatorname{card}(S_m) \leqq 1 + \binom{m}{1} + \binom{m}{2} + \ldots + \binom{m}{[m^{\beta}]}$$

and following the ideas of [12] we obtain that the sum on the right-hand side is

$$(1 + o(1)) \cdot \binom{m}{[m^\beta]}.$$

Now, we have seen is (33) that $\bigcup_{m=k}^{\infty} S_m$ is a 2^{-k}-covering of the set $H = \varrho(G_2(\beta, 0))$ for every k. Therefore if k is such that $2^{-k} \leqq \eta$, then

$$\mu_\eta^{(\gamma)}(H) \leqq \sum_{m=k}^{\infty} \operatorname{card}(S_m) \cdot \mu^{(\gamma)}(2^{-m}) \leqq K \sum_{m=k}^{\infty} \binom{m}{[m^\beta]} \cdot \mu^{(\gamma)}(2^{-m}) = K \sum_{m=k}^{\infty} \frac{\binom{m}{[m^\beta]}}{m^{m^\gamma}}.$$

There follows from Lemma 1 of [12] that to $\delta > 0$ there exists m_0 such that for

$$m > m_0 \text{ we have } \binom{m}{[m^\beta]} < m^{m^{\beta+\delta}}.$$

For $k > m_0$ we thus obtain

$$\mu_\eta^{(\gamma)}(H) \leqq K \sum_{m=k}^{\infty} \frac{m^{m^{\beta+\delta}}}{m^{m^\gamma}} = K \sum_{m=k}^{\infty} m^{m^{\beta+\delta} - m^\gamma}.$$

Since $\gamma > \beta + \delta$, we have on the right-hand side a remainder of a convergent series. Thus for $k \to \infty$ we obtain $\mu_\eta^{(\gamma)}(H) = 0$, which means that $\mu^{(\gamma)}(H) = 0$ and the proof of Theorem is complete.

Note that the same result can be proved for the followsing simpler system of measure functions

$$g^{(\alpha)}(0) = 0$$
$$g^{(\alpha)}(t) = \exp(-\log t)^\alpha$$

for $0 < t \leqq 1/2$ and $\alpha \in (0, 1]$ (c.f. [10, p. 46]).

We now prove two topological results concerning the system $\mathcal{U}_1$. To do this we shall use the Baire's metric [6, p. 115]. This is defined on the system $\mathcal{U}$ of all the monotone sequences of positive integers as follows. If $A = \{a_n\} \in \mathcal{U}$ and $B = \{b_n\} \in \mathcal{U}$, then $d(A, B) = 0$ if $A = B$ and $d(A, B) = 1/k$ if $A \neq B$, where $k = \min\{n;\ a_n \neq b_n\}$. One sees immediately that $(\mathcal{U}, d)$ is a complete metric space.

The next theorem describes the position of the set $\mathcal{U}_1$ in the space $(\mathcal{U}, d)$.

Theorem 15. *The set $\mathcal{U}_1$ is of the type $F_{\sigma\delta}$ of the first Baire category in $\mathcal{U}$.*

Proof. We know that $A = \{a_k\}_{k=1}^{\infty}$ belongs to $\mathcal{U}_1$ if and only if

$$\lim_{k \to \infty} \frac{a_k}{a_1 + \ldots + a_{k-1}} = 0.$$

This implies that $A \in \mathcal{U}_1$ if and only if to every positive integer v there exists a positive integer p such that for every integer $n > p$ we have

$$\frac{a_k}{a_1 + \ldots + a_{k-1}} \leqq \frac{1}{v} .$$

If we therefore denote

$$R(v, n) = \left\{A \in \mathcal{U};\ \frac{a_k}{a_1 + \ldots + a_{k-1}} \leqq \frac{1}{v}\right\},$$

then

$$\mathcal{U}_1 = \bigcap_{v=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{n=p+1}^{\infty} R(v, n). \tag{34}$$

We finish the proof in three steps.

1. First of all note that $R(v, n)$ is a closed set in $\mathcal{U}$. Since, if $A^{(j)} = \{a_k^{(j)}\}_{k=1}^{\infty}$ is a sequence from $R(v, n)$ converging to $A = \{a_k\}_{k=1}^{\infty}$, then the definition of the Baire's metric implies that

$$a_k^{(j)} = a_k \qquad \text{for} \qquad k = 1, 2, \ldots, n$$

for all sufficiently large j. Thus if all $A^{(j)}$ belong to $R(v, n)$, then also A must.

2. The previous part and (34) give in turn that $\mathcal{U}_1$ is of type $F_{\sigma\delta}$ in $\mathcal{U}$.

3. To finish the proof it is sufficient to show that the set

$$R^*(v, n) = \bigcap_{n=p+1}^{\infty} R(v, n)$$

is nowhere dense in $\mathcal{U}$ for all p. It follows from the first part that $R^*(v, p)$ is closed and therefore it is enough to show that its complement in $\mathcal{U}$ is dense in $\mathcal{U}$.

Let $B = \{b_k\}$ be an arbitrary element of $\mathcal{U}$ and δ a positive real number. We shall construct a monotone integral sequence $A = \{a_n\}$ not belonging to $R^*(v, p)$ but lying in the δ-neighbourhood of B. To do this let $r > p$ be an integer with $1/r < \delta$. Define A as follows

$$a_k = b_k \qquad \text{for} \qquad k = 1, 2, \ldots, r$$

$$a_{r+1} \geqq (1 + 1/v) . (a_1 + \ldots + a_r)$$

$$a_{r+j} = a_{r+1} \qquad \text{for} \qquad j = 2, 3, \ldots .$$

The sequence A is clearly monotone. The definition of r and of the first r members of A ensures that A lies in the δ-neighbourhood of B whereas the definition of a_{r+1} implies that A does not belong to $R^*(v, p)$ and Theorem is proved.

Note that $\mathscr{U}_1$ is a dense set in U. In a given δ-neighbourhood of a given sequence $\boldsymbol{B} = \{b_k\}$ from $\mathscr{U}$ there lies, for instance, the sequence $C = \{c_k\}$ for which

$$c_k = b_k,\ k = 1, \ldots, r \quad \text{with} \quad 1/r < \delta$$

$$c_{r+j} = c_r + j, \qquad j = 1, 2, \ldots.$$

Plainly

$$\lim_{j \to \infty} \frac{c_{r+j}}{c_1 + \ldots + c_{r+j-1}} = 0,$$

which shows that $\boldsymbol{C} \in \mathscr{U}_1$.

We use the Theorem 15 to the set $\varrho(\mathscr{U}_1)$ in the interval $(0, 1]$, where ϱ is the dyadic transform defined above. The mapping $\varrho\colon \mathscr{U} \to (0, 1]$ is a continuous one. This follows immediately from the fact that if a sequence $\boldsymbol{B} = \{b_n\}$ lies in a $1/m$-neighbourhood of a sequence $A = \{a_n\}$ then $a_k = b_k$ for $k = 1, 2, \ldots, m$. On the other hand ϱ is a bijection of $\mathscr{U}$ onto $(0, 1]$. Thus there exists its inverse mapping $\varrho^{-1}\colon (0, 1] \to \mathscr{U}$. However, ϱ^{-1} is not continuous in every point of the interval $(0, 1]$. To see this take, for instance, point $1/2$. Then

$$\varrho^{-1}(1/2) = \{2, 3, 4, \ldots\},$$

but for $t > 1/2$ we have

$$\varrho^{-1}(t) = \{n_1 = 1 < n_2 < n_3 \ldots\}.$$

Thus $d(\varrho^{-1}(1/2), \varrho^{-1}(t)) = 1$ for every $t > 1/2$. Similarly we can see that ϱ^{-1} is not continuous (from the right) in every dyadicaly rational number of the interval $(0, 1]$, that is in every rational number with denominator of the form 2^n, $n = 1, 2, \ldots$.

Let $\mathscr{U}_0$ denote the subset of those sequences $A = \{a_k\}$ from $\mathscr{U}$ for which there exists a positive integer m such that $a_{m+j} = a_{m+j-1} + 1$ for $j = 1, 2, \ldots$. If, on the other hand, D denotes the set of all dyadicaly rational numbers in $(0, 1]$, then ϱ maps $\mathscr{U}$ onto D. The restriction $\varrho|(U - U_0)$ of ϱ onto $U - U_0$ has the following properties:

1) it is a bijection of $\mathscr{U} - \mathscr{U}_0$ onto $Y = (0, 1] - D$,
2) it is continuous,
3) $\varrho^{-1}| Y$ is also continuous.

This implies that the space $\mathscr{U} - \mathscr{U}_0$ with Baire's metric is homeomorphic to the space Y with the Euclidean metric. The proof of the next theorem is short now.

Theorem 16. *The set* $\varrho(\mathscr{U}_1)$ *is of the type* $F_{\sigma\delta}$ *of the first Baire category in* $(0, 1]$.

Proof. Put $Z = \mathscr{U} - \mathscr{U}_0$. Then

$$\varrho(\mathscr{U}_1) = \varrho(\mathscr{U}_1 \cap Z) \cup \varrho(\mathscr{U}_1 \cap \mathscr{U}_0).$$

Since ϱ maps homeomorphically the spacae Z onto Y, Theorem 15 implies that $\varrho(\mathcal{U}_1 \cap Z)$ is of the type $F_{\sigma\delta}$ in Y and consequently also in (0, 1] (D is a countable set). Along similar lines it can be proved that $\varrho(\mathcal{U}_1 \cap Z)$ is of the first category in Y and in (0, 1].

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ОБ ОДНОМ КЛАССЕ РАВНОМЕРНО РАСПРЕДЕЛЕННЫХ ПОСЛЕДОВАТЕЛЬНОСТЕЙ

Štefan Porubský—Tibor Šalát—Oto Strauch

Резюме

Пусть X бесконечная последовательность действительных чисел $x(n)$ $(n = 1, 2, \ldots)$ в единичном интервале $[0, 1]$, которая разложена на непустые сегменты так, что n-тый сегмент X_n состоит из a_n элементов. Предположим, что элементы сегмента X_n упорядочены в порядке возрастаня членов и что для любого интервала $I \subset [0, 1]$

$$\lim_{n \to \infty} \frac{A(I, X_n)}{a_n} = |I|,$$

где $A(I, X_n)$ — количество членов $x(k) \in X_n$, для которых $x(k) \in I$, и $|I|$ длина интервала I.

В главе 1 доказано, что последовательность X равномерно распределена тогда и только тогда, когда

$$\lim_{n \to \infty} \frac{a_n}{a_1 + a_2 + \ldots + a_n} = 0 \tag{1}$$

В главе 2 проводится детальное изучение последовательностей $A = \{a_1 < a_2 < \ldots\}$ целых чисел a_n для которых имеет место (1).

В главе 3 доказано, что все (в цмысле установленного там соответствия) возрастающие последовательности A положительных целых чисел обладают свойством (1).