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OSCILLATION OF EVEN ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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ABSTRACT. Some new criteria for the oscillation of the differential equation $x^{(n)}(t) + q(t)F(x[g(t)]) = 0$, n is even, are established. The obtained results unify, extend and improve a well-known sufficient condition for the oscillation of the so-called Emden—Fowler equation $x^{(n)}(t) + q(t)|x[g(t)]|^{\gamma} \operatorname{sgn} x[g(t)] = 0$, n is even, where γ is any positive constant.

1. Introduction

Consider the differential equation

$$x^{(n)}(t) + q(t)F(x[g(t)]) = 0, \quad n \text{ is even}, \tag{1}$$

where $g, q\colon [t_0, \infty) \to \mathbf{R} = (-\infty, \infty)$, $F\colon \mathbf{R} \to \mathbf{R}$ are continuous, $q(t) \geq 0$ and not identically zero for all large t, $g(t) \to \infty$ as $t \to \infty$ and $xF(x) > 0$ for $x \neq 0$.

We assume that there exists a continuous function $\sigma\colon [t_0, \infty) \to [t_0, \infty)$ such that

$$\sigma(t) \leq \min\{t, g(t)\} \quad \text{and} \quad \sigma(t) \to \infty \quad \text{as} \quad t \to \infty.$$

Without further mention we will assume throughout that every solution $x(t)$ of equation (1) that is under consideration here is continuable to the right and is nontrivial, i.e., $x(t)$ is defined on some ray $[t_x, \infty)$ and $\sup\{|x(t)| \mid t > T\} > 0$ for every $T \geq T_x$. Such a solution will be called *oscillatory* if its set of zeros is unbounded and will be called *nonoscillatory* otherwise. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

The oscillation problem for equation (1) has been discussed by numerous authors by various techniques. As examples we refer the reader to the papers of Grace and Lalli [1—4], Kartsatos [5—6], Kiguradze [7], Kitamura

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and Kusano [8], Koplatadze and Chanturia [9], Kusano and Onose [10], Lovelady [11], Mahfoud [12], Philos [13] and Staikos [14].

A well-known and important *oscillation criterion for the Emden—Fowler equation*

$$x^{(n)}(t) + q(t)|x[g(t)]|^{\gamma} \operatorname{sgn} x[g(t)] = 0, \quad n \text{ is even}, \tag{E}$$

where $\gamma > 0$ and the functions g and q are as in equation (1), is given in the following theorem:

Theorem 0. *A sufficient condition for oscillation of equation* (E) *is that*

(i) *when* $\gamma > 1$, $\displaystyle\int^{\infty} \sigma^{n-1}(s)\, q(s)\, \mathrm{d}s = \infty$;

(ii) *when* $\gamma = 1$, $\displaystyle\int^{\infty} \sigma^{n-1-\varepsilon}(s)\, q(s)\, \mathrm{d}s = \infty$, *for some* $\varepsilon > 0$;

(iii) *when* $\gamma > 1$, $\displaystyle\int^{\infty} \sigma^{(n-1)\gamma}(s)\, q(s)\, \mathrm{d}s = \infty$,

where the function σ *is as defined above.*

Theorem 0 has been extended in the above mentioned papers to equation (1), where the function F is required to be either nondecreasing or locally of bounded variation (cf. [12]).

The main purpose of this paper is to establish some new oscillation criteria for equation (1) which extend and improve Theorem 0 and its generalizations in ([1]—[14]). The obtained results can be applied to cases in which Theorem 0 and the analogous results in ([1]—[14]) are not applicable.

2. Definitions and basic lemmas

The following notation will be used throughout this paper:

$$\mathbf{R}_{\alpha} = (-\infty, -\alpha] \cup [\alpha, \infty) \quad \text{if } \alpha > 0$$
$$= (-\infty, 0) \cup (0, \infty) \quad \text{if } \alpha = 0,$$

$C(\mathbf{R}) = \{F\colon \mathbf{R} \to \mathbf{R} \mid F$ is continuous and $xF(x) > 0$ if $x \neq 0\}$,
$C^1(\mathbf{R}_{\alpha}) = \{F \in C(\mathbf{R}) \mid F$ is continuously differentiable in $\mathbf{R}_{\alpha}\}$,
$C_p(\mathbf{R}_{\alpha}) = \{F \in C(\mathbf{R}) \mid F$ is of bounded variation on every interval $[a, b] \subset \mathbf{R}_{\alpha}\}$.

The following two lemmas will be needed in the proofs of our results. The first lemma can be found in [7] and [9] and the second appeared in [12].

Lemma 1. *Let* $x(t)$ *be a nonoscillatory solution of equation* (1) *and let*

$$x(t)\,x^{(n)}(t) \le 0 \quad \textit{for} \quad t \ge t_0.$$

Then there exist a $t_1 \ge t_0$ *and an integer* $\ell \in \{0, 1, \ldots, n-1\}$ *such that* $n + \ell$ *is odd and for all sufficiently large* $t \ge t_1$

$$x(t)\,x^{(k)}(t) > 0, \quad (k = 0, 1, \ldots, \ell - 1)$$

$$(-1)^{k+\ell}\,x(t)\,x^{(k)}(t) > 0, \quad (k = \ell, \ell + 1, \ldots, n - 1)$$

and

$$k\,|x^{(\ell-k)}(t)| \ge t\,|x^{(\ell-k+1)}(t)|, \quad (k = 1, 2, \ldots, \ell).$$

Lemma 2. *Suppose* $\alpha \ge 0$ *and* $F \in C(\mathbf{R})$. *Then* $F \in C_p(\mathbf{R}_\alpha)$ *if and only if* $F(x) = G(x)\,H(x)$ *for all* $x \in \mathbf{R}_\alpha$, *where* $G\colon \mathbf{R}_\alpha \to (0, \infty)$, *nondecreasing on* $(-\infty, -\alpha)$ *and nonincreasing on* (α, ∞) *and* $H\colon \mathbf{R}_\alpha \to \mathbf{R}$ *and nondecreasing in* $\mathbf{R}_\alpha$.

Definition. *We call G in Lemma 2 a positive component of F, H a nondecreasing component of F and the ordered pair* (G, H) *a pair of components of F.*

3. Main results

Theorem 1. *Suppose* $\alpha \ge 1$ *and* (G, H) *is a pair of components of F. Suppose moreover that*

$$-H(-xy) \ge H(xy) \ge KH(x)\,H(y), \quad xy > 0, \tag{2}$$

where K is a positive constant; and

$$\int^{\infty} \frac{\mathrm{d}u}{H(u)} < \infty \quad \textit{and} \quad \int^{-\alpha} \frac{\mathrm{d}u}{H(u)} < \infty. \tag{3}$$

If, for every integer $\ell \in \{1, 3, \ldots, n-1\}$

$$\int^{\infty} s^{n-\ell}\,q(s)\,G(k^* g^{n-1}(s))\,H\!\left(g^{\ell-1}(s)\,\frac{\sigma(s)}{s}\right)\mathrm{d}s = \infty, \tag{4}$$

for every $|k^*| \ge 1$, *then equation* (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Assume $x(t) > 0$ and $x[g(t)] > 0$ for $t \ge t_0 \ge \alpha$. By Lemma 1, there exists a $t_1 \ge t_0$ and an integer $\ell \in \{1, 3, \ldots, n-1\}$ such that for $t \ge t_1$

$$\begin{aligned} x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, \ell - 1) \\ (-1)^{k+\ell}\,x^{(k)}(t) > 0 \quad (k = \ell, \ell + 1, \ldots, n - 1). \end{aligned} \tag{5}$$

In particular, we have

$$x'(t) > 0 \quad \text{and} \quad x^{(n-1)}(t) > 0 \quad \text{for} \quad t \geq t_1. \tag{6}$$

Hence there exist a constant $k_1 > 0$ and $t_2 \geq t_1$ such that

$$x^{(n-1)}(t) \leq k_1 \quad \text{for all} \quad t \geq t_2. \tag{7}$$

Integrate (7) $(n-1)$-times from t_2 to t and choose $t_3 \geq t_2$ and $k_2 \geq 1$ so that

$$x(t) \leq k_2 t^{n-1} \quad \text{for} \quad t \geq t_3.$$

Since $g(t) \to \infty$ as $t \to \infty$, there exists a $t_4 \geq t_3$ so that

$$x[g(t)] \leq k_2 g^{n-1}(t) \quad \text{for} \quad t \geq t_4.$$

By Lemma 2,

$$\begin{aligned} F(x[g(t)]) &= G(x[g(t)])\,H(x[g(t)]) \\ &\geq G(k_2 g^{n-1}(t))\,H(x[g(t)]) \end{aligned}$$

and hence equation (1) yields the inequality

$$x^{(n)}(t) + q(t)\,G(kg^{n-1}(t))\,H(x[g(t)]) \leq 0 \quad \text{for} \quad t \geq t_4. \tag{8}$$

Observe that

$$x^{(\ell)}(t) = \sum_{j=\ell}^{n-1} (-1)^{j-\ell} \frac{(t-s)^{j-\ell}}{(j-\ell)!} x^{(j)}(s) + (-1)^{n-\ell} \int_t^s \frac{(u-t)^{n-\ell-1}}{(n-\ell-1)!} x^{(n)}(u)\,\mathrm{d}u$$

for any $t, s \geq t_4$. Using (5) and the fact that $n - \ell$ is odd, we have from the above

$$x^{(\ell)}(t) \geq -\int_t^s \frac{(u-1)^{n-\ell-1}}{(n-\ell-1)!} x^{(n)}(u)\,\mathrm{d}u \quad \text{for} \quad s \geq t \geq t_4,$$

which in view of (8) gives

$$x^{(\ell)}(t) \geq \int_t^s \frac{(u-t)^{n-\ell-1}}{(n-\ell-1)!} q(u)\,G(k_2 g^{n-1}(u))\,H(x[g(u)])\,\mathrm{d}u, \quad t \geq t_4. \tag{9}$$

Now, by Lemma 1, there exists a $t_5 \geq t_4$ so that

$$x[g(t)] \geq \frac{g^{\ell-1}(t)}{\ell!} x^{(\ell-1)}[g(t)] \quad \text{for} \quad t \geq t_5 \tag{10}$$

and

$$\frac{x^{(\ell-1)}(t)}{t} \quad \text{is nonincreasing for} \quad t \geq t_5. \tag{11}$$

Since $x^{(\ell-1)}(t)$ is nondecreasing for $t \geq t_5$ and $\sigma(t) \to \infty$ as $t \to \infty$, there exists a $t_6 \geq t_5$ such that

$$x^{(\ell-1)}[g(t)] \geq x^{(\ell-1)}[\sigma(t)] \quad \text{for} \quad t \geq t_6.$$

Thus

$$\frac{x^{(\ell-1)}[\sigma(t)]}{\sigma(t)} \geq \frac{x^{(\ell-1)}(t)}{t} \quad \text{for} \quad t \geq t_6,$$

and

$$x[g(t)] \geq \frac{g^{\ell-1}(t)}{\ell!} \frac{\sigma(t)}{t} x^{(\ell-1)}(t) \quad \text{for} \quad t \geq t_6. \tag{12}$$

Using (2) and (12) in (9), we have

$$\frac{x^{(\ell)}(t)}{H(x^{(\ell-1)}(t))} \geq$$

$$\geq K^2 H\left(\frac{1}{\ell!}\right) \int_t^s \frac{(u-t)^{n-\ell-1}}{(n-\ell-1)!} q(u) G(k_2 g^{n-1}(u)) H\left(g^{\ell-1}(u) \frac{\sigma(u)}{u}\right) du.$$

Integrating the last inequality from t_6 to $s \geq 2t_6$ we obtain

$$\int_{t_6}^s \frac{x^{(\ell)}(t)}{H(x^{(\ell-1)}(t))} dt \geq$$

$$\geq K^2 H\left(\frac{1}{\ell!}\right) \int_{t_6}^s \int_t^s \frac{(u-t)^{n-\ell-1}}{(n-\ell-1)!} q(u) G(k_2 g^{n-1}(u)) H\left(g^{\ell-1}(u) \frac{\sigma(u)}{u}\right) du\, dt \geq$$

$$\geq K^2 H\left(\frac{1}{\ell!}\right) \int_{2t_6}^s \frac{(u-t_6)^{n-\ell}}{(n-\ell)!} q(u) G(k_2 g^{n-1}(u)) H\left(g^{\ell-1}(u) \frac{\sigma(u)}{u}\right) du.$$

Letting $s \to \infty$ in the above inequality and using (3), we conclude that

$$K^2 H\left(\frac{1}{\ell!}\right) \int_{2t_6}^\infty \frac{u^{n-\ell}}{(n-\ell)!} q(u) G(k_2 g^{n-1}(u)) H\left(g^{\ell-1}(u) \frac{\sigma(u)}{u}\right) du \leq$$

$$\leq c \int_{x^{(\ell-1)}(t_6)}^\infty \frac{dv}{H(v)} < \infty,$$

which contradicts (4). A parallel argument holds if we assume that equation (1) has a negative solution. The proof is now complete.

Now, let us consider equation (E). Then Theorem 1 leads to the following corollary.

Corollary 1. *If $\gamma > 1$, and for every integer $\ell \in \{1, 3, \ldots, n-1\}$*

$$\int^\infty s^{n-\ell-\gamma}(g^{\ell-1}(s)\,\sigma(s))^\gamma q(s)\, ds = \infty, \tag{13}$$

then equation (1) is oscillatory.

Remark 1. If $n > 2$, then Corollary 1 improves Theorem 0(i). This case can be illustrated by the following example.

Example 1. Consider the differential equation

$$x^{(4)}(t) + t^{-7\,3}(x[t^{2\,5}])^{5\,3} = 0, \quad t \geq 1. \tag{14}$$

Here we take $g(t) = \sigma(t) = t^{2\,5}$.

All conditions of Corollary 1 are satisfied and hence equation (14) is oscillatory. Theorem 0(i) fails to apply to equation (14) since

$$\int_1^{\infty} \sigma^{n-1}(s)\, q(s)\, \mathrm{d}s = \int_1^{\infty} s^{-17\,15}\, \mathrm{d}s < \infty.$$

Remark 2. Since the function F in equation (1) is required to be a locally of bounded variation, our Theorem 1 extends and unifies the analogous results in ([1]—[14]). In particular, Theorem 1 can be applied to some cases in which Theorem 2 in [12] is not applicable. Such a case is described in Example 2 below.

Example 2. Consider the differential equation

$$x^{(4)}(t) + t^{-4\,3} \frac{(x[t^{2\,5}])^{5\,3}}{1 + |x[t^{2\,5}]|^{5\,6}} = 0, \quad t \geq 1. \tag{15}$$

Here we let $\sigma(t) = g(t) = t^{2\,5}$, $G(x) = \dfrac{1}{1 + |x|^{5\,6}}$ and $H(x) = x^{5\,3}$. Since $n = 4$, $\ell = 1$ or 3. If $\ell = 1$, then for all $|k| \geq 1$ we get

$$\int_1^{\infty} s^{n-\ell} q(s)\, G(kg^{n\ \ 1}(s))\, H\!\left(g^{\ell\ \ 1}(s)\, \frac{\sigma(s)}{s}\right) \mathrm{d}s = \int_1^{\infty} \frac{s^{2\,3}}{1 + |k|\, s}\, \mathrm{d}s = \infty.$$

When $\ell = 3$,

$$\int_1^{\infty} s^{n-\ell} q(s)\, G(kg^{n-1}(s))\, H\!\left(g^{\ell-1}(s)\, \frac{\sigma(s)}{s}\right) \mathrm{d}s = \int_1^{\infty} \frac{\mathrm{d}s}{1 + |k|\, s} = \infty,$$

for any $|k| \geq 1$. Thus, equation (15) is oscillatory by Theorem 1.

According to Theorem 2 in [12], equation (1) is oscillatory if there exists a nondecreasing differentiable function $\phi\colon [\alpha, \infty) \to (0, \infty)$ such that

$$\int_{\pm\alpha}^{\pm\infty} [\phi(|x|^{1\ n-1})\, H(x)]^{\ \ 1}\, \mathrm{d}x < \infty \quad \text{and} \tag{a}$$

$$\int_{\alpha}^{\infty} \sigma^{n-1}(s)\, q(s)\, (\phi[\sigma(s)])^{\ \ 1}\, G(kg^{n\ \ 1}(s))\, \mathrm{d}s = \infty, \quad \text{for every} \quad |k| \geq 1. \tag{b}$$

Theorem 2 in [12] cannot be applied to equation (15). Indeed, if we take $\phi(x) = 1$, condition (a) is satisfied, while condition (b) becomes

$$\int_1^\infty \sigma^{n-1}(s)\,q(s)\,(\phi[\sigma(s)])^{-1}\,G(kg^{n-1}(s))\,\mathrm{d}s =$$

$$= \int_1^\infty \frac{s^{-2/15}}{1+|k|\,s}\,\mathrm{d}s \le \frac{1}{|k|}\int_1^\infty s^{-17/15}\,\mathrm{d}s < \infty,$$

for every $|k| \ge 1$, which means that condition (iii) of Theorem 2 in [12] fails.

Theorem 2. *Assume $\alpha \ge 1$, $\sigma^{\cdot}(t) \ge 0$ for $t \ge t_0$ and (G, H) is a pair of components of F. Suppose that*

$$\frac{H(x)}{x} \ge c > 0 \quad \textit{for} \quad x \ne 0. \tag{16}$$

If for every $\ell \in \{1, 3, \ldots, n-1\}$

$$\limsup_{t\to\infty}\left[\int_{\sigma(t)}^{t} [u-\sigma(t)]^{n-\ell-1}\,q(u)\,\sigma^{\ell}(u)\,G(k^*g^{n-1}(u))\,\mathrm{d}u + \right.$$

$$\left. + \,\sigma(t)\int_t^\infty [u-\sigma(t)]^{n-\ell-1}\,q(u)\,\sigma^{\ell-1}(u)\,G(k^*g^{n-1}(u))\,\mathrm{d}u\right] > \tag{17}$$

$$> \frac{1}{c}\,(n-\ell-1)!\,\ell!,$$

for every $|k^*| \ge 1$, then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$ and $x[g(t)] > 0$ for $t \ge t_0 \ge \alpha$. As in the proof of Theorem 1, we obtain inequality (9). Now, using (16) in (9) and the fact that $x(t)$ is nondecreasing for $t \ge t_1$ we have

$$x^{(\ell)}(t) \ge c\int_t^\infty \frac{(u-t)^{n-\ell-1}}{(n-\ell-1)!}\,q(u)\,G(k_2 g^{n-1}(u))\,x[\sigma(u)]\,\mathrm{d}u \quad \text{for} \quad t \ge t_4.$$

By Lemma 1, there exists $t_5 \ge t_4$ such that

$$x[\sigma(t)] \ge \frac{\sigma^{\ell-1}(t)}{\ell!}\,x^{(\ell-1)}[\sigma(t)] \quad \text{for} \quad t \ge t_5.$$

Thus,

$$x^{(\ell)}[\sigma(t)] \ge c\left[\int_{\sigma(t)}^{t} \frac{[u-\sigma(t)]^{n-\ell-1}}{(n-\ell-1)!}\,q(u)\,\frac{\sigma^{\ell}(u)}{\ell!}\,G(k_2 g^{n-1}(u))\left(\frac{x^{(\ell-1)}[\sigma(u)]}{\sigma(u)}\right)\mathrm{d}u + \right.$$

$$+\int_t^\infty \frac{[u-\sigma(t)]^{n-\ell-1}}{(n-\ell-1)!} q(u)\frac{\sigma^{\ell-1}(u)}{\ell!} G(k_2 g^{n-1}(u)(x^{(\ell-1)}[\sigma(u)])\,\mathrm{d}u\Bigg].$$

Since $\dfrac{x^{(\ell-1)}}{t}$ is nonincreasing for $1 \le \ell \le n-1$ and $t \ge t_5$, $x^{(\ell-1)}(t)$ is nondecreasing for $t \ge t_1$ and $\sigma^{\cdot}(t) \ge 0$ for $t \ge t_0$, we get

$$\frac{(n-\ell-1)!\,\ell!}{c}\,\frac{\sigma(t)\,x^{\ell}[\sigma(t)]}{x^{(\ell-1)}[\sigma(t)]} \ge$$

$$\ge \int_{\sigma(t)}^t [u-\sigma(t)]^{n-\ell-1} q(u)\,\sigma^{\ell}(u)\,G(k_2 g^{n-1}(u))\,\mathrm{d}u +$$

$$+\sigma(t)\int_t^\infty [u-\sigma(t)]^{n-\ell-1} q(u)\,\sigma^{\ell-1}(u)\,G(k_2 g^{n-1}(u))\,\mathrm{d}u.$$

Next, we observe that

$$\limsup_{t\to\infty} \frac{\sigma(t)\,x^{(\ell)}[\sigma(t)]}{x^{(\ell-1)}[\sigma(t)]} \le 1.$$

Hence

$$\limsup_{t\to\infty}\Bigg[\int_{\sigma(t)}^t [u-\sigma(t)^{n-\ell-1} q(u)\,\sigma^{\ell}(u)\,G(k_2 g^{n-1}(u))\,\mathrm{d}u +$$

$$+\sigma(t)\int_t^\infty [u-\sigma(t)]^{n-\ell-1} q(u)\,\sigma^{\ell-1}(u)\,G(k_2 g^{n-1}(u))\,\mathrm{d}u\Bigg] \le$$

$$\le \frac{1}{c}(n-\ell-1)!\,\ell!,$$

which contradicts (17). A similar argument holds for $x(t)$ eventually negative, and this completes the proof.

Corollary 2. *Consider equation* (E) *with* $\gamma = 1$. *If for every* $\ell \in \{1, 3, \ldots, n-1\}$

$$\limsup_{t\to\infty}\Bigg[\int_{\sigma(t)}^t [u-\sigma(t)]^{n-\ell-1} q(u)\,\sigma^{\ell}(u)\,\mathrm{d}u + \sigma(t)\int_t^\infty [u-\sigma(t)]^{n-\ell-1} q(u)\,\sigma^{\ell-1}(u)\,\mathrm{d}u\Bigg] > (n-\ell-1)!\,\ell!, \tag{18}$$

then equation (E) *is oscillatory.*

The following examples are illustrative.

Example 3. Consider the differential equation

$$x^{(4)}(t) + ct^{-4}x\left[\frac{t}{2}\right] = 0, \quad t \geq 1, \tag{19}$$

where c is a positive constant. All conditions of Corollary 2 are satisfied if $c > \dfrac{48}{1 + \ln 2}$ and hence equation (19) is oscillatory. We may note that Theorem 0(ii) and Theorem 2 in [12] fail to apply to equation (19), while some of the results in ([1]—[4]) are applicable to equation (19) if $c > (48)\,(288)$. Thus, Corollary 2 improves Theorem 0(ii).

Example 4. Consider the differential equation

$$x^{(4)}(t) + ct^{-3}\frac{x\left[\frac{t}{2}\right]}{1 + \left|x\left[\frac{t}{2}\right]\right|} = 0, \quad t \geq 1, \tag{20}$$

where c is a positive constant. Here we take $G(x) = \dfrac{1}{1 + |x|}$, $H(x) = x$ and $\sigma(t) = g(t) = \dfrac{t}{2}$. The hypotheses of Theorem 2 are satisfied if $c > 24$ and hence equation (20) is oscillatory. It is easy to check that Theorem 4 in [4] can be applied to equation (20) if $c > (24)\,(1152)$, while Theorem 2 in [12] fails to apply to equation (20) for any function $\phi(x) = x^{\varepsilon}$, $\varepsilon > 0$ (see Example 2 above).

Therefore, our Theorem 2 improves Theorem 4 in [4] and Theorem 2 in [12]. Also, it extends and unifies some of the results in [1]—[4] and [6].

The following result is concerned with the oscillation of strongly sublinear equations of type (1).

Theorem 3. *Suppose $\alpha \geq 1$ and (G, H) is a pair of components of F. Assume that condition* (2) *holds, and*

$$\int_{+0} \frac{du}{H(u)} < \infty \quad \textit{and} \quad \int_{-0} \frac{du}{H(u)} < \infty. \tag{21}$$

If for every $\ell \in \{1, 3, \ldots, n-3\}$

$$\int^{\infty} H(s) \int_s^{\infty} (u - s)^{n-\ell-2} q(u)\, G(k^* g^{n-1}(u))\, H\left(g^{\ell-1}(u)\, \frac{\sigma(u)}{u}\right) du\, ds = \infty \tag{22}$$

and for $\ell = n - 1$

$$\int^{\infty} q(u)\, G(k^* g^{n-1}(u))\, H(g^{n-2}(u)\, \sigma(u))\, \mathrm{d}u = \infty, \tag{23}$$

for every $|k^*| \geq 1$, *then equation* (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Assume $x(t) > 0$ and $x[g(t)] > 0$ for $t \geq t_0 \geq \alpha$. By Lemma 1, there exist a $t_1 \geq t_0$ and an integer $\ell \in \{1, 3, \ldots, n-1\}$ such that (5) holds. Proceeding as in the proof of Theorem 1, we obtain inequalities (8) and (12). Now, we consider the following two cases:

Case 1. $\ell < n - 1$. By Taylor's formula with integral remainder

$$-x^{(\ell+1)}(t) = \sum_{i=\ell+1}^{n-1} (-1)^{i-\ell} \frac{(s-t)^{i-\ell-1}}{(i-\ell-1)!}\, x^{(i)}(s) + {} \\ + (-1)^{n-\ell} \int_t^s \frac{(u-t)^{n-\ell-2}}{(n-\ell-2)!}\, x^{(n)}(u)\, \mathrm{d}u.$$

Using (5) and (8) we obtain

$$-x^{(\ell+1)}(t) \geq \int_t^s \frac{(u-t)^{n-\ell-2}}{(n-\ell-2)!}\, q(u)\, G(k_2 g^{n-1}(u))\, H(x[g(u)])\, \mathrm{d}u, \quad t \geq t_4. \tag{24}$$

From (2), (12) and (24) we have

$$-x^{(\ell+1)}(t) \geq K \int_t^s \frac{(u-t)^{n-\ell-2}}{(n-\ell-2)!}\, q(u)\, G(k_2 g^{n-1}(u))\, H\!\left(\frac{g^{\ell-1}(u)}{\ell!}\, \frac{\sigma(u)}{u}\right) \times \\ \times H(x^{(\ell-1)}(u))\, \mathrm{d}u \geq K^2 H(x^{(\ell-1)}(t)) \int_t^s \frac{(u-t)^{n-\ell-2}}{(n-\ell-2)!} \times \\ \times q(u)\, G(k_2 g^{n-1}(u))\, H\!\left(\frac{1}{\ell!}\right) H\!\left(g^{\ell-1}(u)\, \frac{\sigma(u)}{u}\right) \mathrm{d}u. \tag{25}$$

Since

$$x^{(\ell-1)}(t) \geq t x^{(\ell)}(t) \quad \text{for} \quad t \geq t_6, \tag{26}$$

we get

$$\frac{-x^{(\ell+1)}(t)}{H(x^{(\ell)}(t))} \geq \frac{K^3 H\!\left(\frac{1}{\ell!}\right)}{(n-\ell-2)!}\, H(t) \times \\ \times \int_t^s (u-t)^{n-\ell-2} q(u)\, G(k_2 g^{n-1}(u))\, H\!\left(g^{\ell-1}(u)\, \frac{\sigma(u)}{u}\right) \mathrm{d}u.$$

Letting $s \to \infty$ in the above inequality and integrating from t_6 to ∞, we obtain

$$\int_{t_6}^{\infty} H(t) \int_t^{\infty} (u-t)^{n-\ell-2} q(u) G(k_2 g^{n-1}(u)) H\left(g^{\ell-1}(u) \frac{\sigma(u)}{u}\right) \mathrm{d}u \, \mathrm{d}t \leq$$

$$\leq \frac{(n-\ell-2)!}{K^3 H\left(\frac{1}{\ell!}\right)} \int_0^{x^{\ell}(t_6)} \frac{\mathrm{d}w}{H(w)} < \infty,$$

which contradicts (22).

Case 2. $\ell = n-1$. From inequalities (8) and (12), we have

$$-x^{(n)}(t) \geq q(t) G(k_2 g^{n-1}(t)) H\left(\frac{g^{n-2}(t)}{(n-1)!} \frac{\sigma(t)}{t} x^{(n-2)}(t)\right) \quad \text{for} \quad t \geq t_6. \tag{27}$$

Using (2) and (26), we get

$$-x^{(n)}(t) \geq K^2 H\left(\frac{1}{(n-1)!}\right) q(t) G(k_2 g^{n-1}(t)) H(g^{n-2}(t) \times$$
$$\times \sigma(t)) H(x^{(n-1)}(t)), \quad t \geq t_6.$$

Thus,

$$-\int_{t_6}^{t} \frac{x^{(n)}(s)}{H(x^{(n-1)}(s))} \mathrm{d}s \geq$$
$$\geq K^2 H\left(\frac{1}{(n-1)!}\right) \int_{t_6}^{t} q(s) G(k_2 g^{n-1}(s)) H(g^{n-2}(s) \sigma(s)) \mathrm{d}s.$$

Letting $t \to \infty$ in the above inequality, we conclude that

$$\int_{t_6}^{\infty} q(s) G(k_2 g^{n-1}(s)) H(g^{n-2}(s) \sigma(s)) \mathrm{d}s < \infty,$$

which contradicts (23). This completes the proof.

Next, we give an oscillation criterion for equation (E) when $0 < \gamma < 1$.

Corollary 3. *Consider equation* (E) *with* $0 < \gamma < 1$. *If for every* $\ell \in \{1, 3, \ldots, n-3\}$

$$\int^{\infty} s^{\gamma} \int_s^{\infty} (u-s)^{n-\ell-2} q(u) \left(g^{\ell-1}(u) \frac{\sigma(u)}{u}\right)^{\gamma} \mathrm{d}u \, \mathrm{d}s = \infty \tag{28}$$

and for $\ell = n-1$

$$\int^{\infty} q(s) (g^{n-2}(s) \sigma(s))^{\gamma} \mathrm{d}s = \infty, \tag{29}$$

then equation (E) *is oscillatory.*

The following examples are illustrative.

Example 5. The differential equation

$$x^{(4)}(t) + \frac{15}{16t^4}(x[t^3])^{1/3} = 0, \quad t \geq 1 \tag{30}$$

has the nonoscillatory solution $x(t) = \sqrt{t}$. All conditions of Corollary 3 are satisfied with $\sigma(t) = t$ except conditions (28) and (29).

Example 6. Consider the differential equation

$$x^{(4)}(t) + t^{-10/3}(x[t^3])^{1/3} = 0, \quad t \geq 1. \tag{31}$$

The hypotheses of Corollary 3 are satisfied and hence equation (31) is oscillatory.

It is easy to check that Theorem 0(iii) and the analogous results in [1]–[14] are not applicable to equation (31). Thus, Corollary 3 unifies Theorem 0(iii).

Example 7. Consider the differential equation

$$x^{(4)}(t) + t^{-7/3}\frac{(x[t^3])^{1/3}}{1 + |x[t^3]|^{1/9}} = 0, \quad t \geq 1. \tag{32}$$

Here we let $g(t) = t^3$, $\sigma(t) = t$,

$$G(x) = \frac{1}{1 + |x|^{1/3}} \quad \text{and} \quad H(x) = x^{1/3}.$$

Since $n = 4$, the integer ℓ is either 1 or 3. Now, for $\ell = 1$ and every $|k| \geq 1$, we have

$$\int_1^t H(s) \int_s^\infty (u - s)^{n-\ell-2} q(u)\, G(kg^{n-1}(u))\, H\left(g^{\ell-1}(u)\, \frac{\sigma(u)}{u}\right) du\, ds =$$

$$= \int_1^t s^{1/3} \left(\int_s^\infty \frac{(u - s)\, u^{-7/3}}{1 + |k|\, u}\, du \right) ds \geq$$

$$\geq \frac{1}{2|k|} \int_1^t \left(\int_s^\infty (u - s)\, u^{-4/3} du \right) ds \to \infty \quad \text{as} \quad t \to \infty.$$

Also, for $\ell = 3$ and every $|k| \geq 1$, we get

$$\int_1^t q(u)\, G(kg^{n-1}(u))\, H(g^{n-2}(u)\, \sigma(u))\, du = \int_1^t \frac{1}{1 + |k|\, u}\, du \to \infty \quad \text{as} \quad t \to \infty.$$

Thus, all conditions of Theorem 3 are satisfied and hence equation (32) is oscillatory.

On the other hand, Theorem 2 in [12] is not applicable to equation (32), since for $\phi(x) = x^{23+\varepsilon}$, $\varepsilon > 0$ (see Example 2), condition (iii) of that theorem takes the form

$$\int_1^t q(s)\,\sigma^{n-1}(s)\,(\phi[\sigma(s)])^{-1}\,G(kg^{n-1}(s))\,\mathrm{d}s = \int_1^t \frac{s^{-\varepsilon}}{1+|k|\,s}\,\mathrm{d}s < \infty \quad \text{as} \quad t \to \infty,$$

for every $\varepsilon > 0$ and all $|k| \geq 1$, which means that condition (iii) of Theorem 2 in [12] fails. Next, it is easy to check that Theorems 2 and 3 in [4] improve Theorem 2 in [12], however, they also fail to apply to equation (32).

Therefore, we conclude that Theorem 3 improves Theorems 2 and 3 in [4] and Theorem 2 in [12]. Also, it extends and unifies some of the results in [1]—[14].

In the following theorem we present an oscillation criterion for equation (1), via a comparison with the oscillatory behaviour of a set of second order ordinary differential equations. The obtained result extends Theorem 7 of Lovelady [11] in such a way that it can be applied in cases of nonlinear differential equations with deviating arguments.

Theorem 4. *Let $\alpha \geq 1$, condition* (2) *hold and* (G, H) *be a pair of components of F. If for every $|k^*| \geq 1$ either*

(i) *every solution of the equation*

$$y^{\cdot\cdot}(t) + K^2 H\left(\frac{1}{(n-1)!}\right) q(t)\,G(k^* g^{n-1}(t))\,H\left(g^{n-2}(t)\,\frac{\sigma(t)}{t}\right) H(y(t)) = 0 \tag{33}$$

is oscillatory; *or*

(ii) *for any $\ell \in \{1, 3, \ldots, n-3\}$, every solution of the equation*

$$y^{\cdot\cdot}(t) + \left(\frac{K^2 H\left(\frac{1}{\ell !}\right)}{(n-\ell-2)!}\right) \times$$

$$\times \left[\int_t^{\infty} (u-t)^{n-\ell-2}\,q(u)\,G(k^* g^{n-1}(u))\,H\left(g^{\ell-1}(u)\,\frac{\sigma(u)}{u}\right) \mathrm{d}u\right] H(y(t)) = 0 \tag{34}$$

is oscillatory, then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$ and $x[g(t)] > 0$ for $t \geq t_0 \geq \alpha$. As in the proof of Theorem 3, we consider the case when $\ell = n-1$ and obtain (27), also the case when $\ell \in \{1, 3, \ldots, n-1\}$ and get (25). Now, let $\ell = n-1$. From (27) we have

$$x^{(n)}(t) + \beta(t)\, H(x^{(n-2)}(t)) \leq 0 \quad \text{for} \quad t \geq t_6,$$

where

$$\beta(t) = K^2 H\left(\frac{1}{(n-1)!}\right) q(t)\, G(k^* g^{n-1}(t))\, H\left(g^{n-2}(t)\, \frac{\sigma(t)}{t}\right).$$

Thus, we have a positive solution of

$$w^{\cdot\cdot}(t) + \beta(t)\, H(w(t)) \leq 0.$$

By Lemma 2.1 in [5], equation (33) has a positive solution, a contradiction. Next, let $\ell \in \{1, 3, \ldots, n-3\}$. From (25) we get

$$x^{(\ell+1)}(t) + \gamma(t)\, H(x^{(\ell-1)}(t)) \leq 0 \quad \text{for} \quad t \geq t_6,$$

where

$$\gamma(t) = \frac{K^2 H\left(\frac{1}{\ell!}\right)}{(n-\ell-2)!} \int_t^\infty (u-t)^{n-\ell-2} q(u)\, G(k_2 g^{n-1}(u))\, H\left(g^{\ell-1}(u)\, \frac{\sigma(u)}{u}\right),$$

and hence the inequality

$$v^{\cdot\cdot}(t) + \gamma(t)\, H(v(t)) \leq 0$$

has a positive solution. Once again, by Lemma 2.1 in [5], equation (34) has a positive solution, a contradiction. This completes the proof.

Remarks.

1. The results of the present paper are presented in a form which is essentially new. We also mention that we do not stipulate that the function g in equation (1) be either retarded, advanced or a mixed type. Hence our theorems may hold for ordinary, retarded, advanced and a mixed type equations.

2. The results of this paper are applicable to many types of differential equations, e.g., linear or almost linear equations and strongly sublinear equations. In particular, our results extend and unify Theorem 0 and the analogous results in ([1]—[14]).

3. The results of this paper are extendable to more general equations of the form

$$x^{(n)}(t) + q(t)\, F(x[g_1(t)], x[g_2(t)], \ldots, x[g_m(t)]) = 0 \tag{35}$$

as well as the damped equations of the form

$$x^{(n)}(t) + p(t)\,|x^{(n-1)}(t)|^\beta + q(t)\, F(x[g_1(t)], x[g_2(t)], \ldots, x[g_m(t)]) = 0, \tag{36}$$

where $\beta \geq 0$, g_i, p, q: $[t_0, \infty) \to \mathbf{R}$, $i = 1, 2, \ldots, m$, F: $\mathbf{R}^m \to \mathbf{R}$ are continuous, $q(t) \geq 0$ and not identically zero for all large t, $p(t) \geq 0$ for $t \geq t_0$, $\lim_{t \to \infty} g_i(t) = \infty$, $i = 1, 2, \ldots, m$ and $y_1 F(y_1, y_2, \ldots, y_m) > 0$ for $y_1 \neq 0$. The function p is required to satisfy the following condition

$$\left(1 + \int_{t_0}^{t} p(s)\,\mathrm{d}s\right)^{-1/\beta} \notin L(t_0, \infty) \quad \text{if } \beta > 0,$$

$$\int^{\infty} \exp\left(\int_{t_0}^{s} -p(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s = \infty \quad \text{if } \beta = 0,$$

and $\sigma(t) \leq \min\{t, g_1(t), g_2(t), \ldots, g_m(t)\}$ and $\lim_{t \to \infty} \sigma(t) = \infty$.

Accordingly, the obtained results for equations (35) and (36) extend and improve some of the known results in ([1]—[14]).

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