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ON o–MODULAR AND o–DISTRIBUTIVE SEMILATTICES

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ABSTRACT. In a finite case, the notions of modular and distributive semilattices make it possible to study lattices only. In contrast to this, using the definitions of o-modular and o-distributive semilattices, we have obtained larger classes of semilattices. In this paper, the connections between these notions are found and there is shown their connection with the lattice of ideals (including the empty set).

0. Introduction

It is well known that the modularity and the distributivity of a $\vee$-semilattice A is equivalent to the condition that the set of ideals of A is a modular and distributive lattice, respectively. Unfortunately, both definitions of modular and distributive semilattices lead in the case of finite semilattices to lattices only.

In [5], the notions of modular and distributive ordered sets are introduced which can also be used for finite ordered sets. The semilattices which are simultaneously distributive ordered sets (o-semilattices) are characterized in [6], where it is also shown that the class of such semilattices is larger than that of the distributive semilattices.

In this paper, the connections between modular and o-modular semilattices are found, the connections between distributive and o-distributive semilattices are completed, and it is shown that the properties of the lattice of ideals (including the empty set) of a semilattice determine the properties of the semilattice.

1. o-modular semilattices

Let $(A, \vee)$ be a $\vee$-semilattice. Let $\leqq$ denote the *induced order on* A, i.e., for any $a, b \in A$, $a \leqq b$ if and only if $a \vee b = b$. We say that A is a *modular semilattice* if

$$\forall a, b, c \in A;\ a \leqq b,\ b \leqq a \vee c \implies \exists c_1 \in A;\ c_1 \leqq c,\ b = a \vee c_1 .$$

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(See [3, p. 148].)

If $(A, \leqq)$ is an ordered set, $B \subseteq A$, then $L(B)$ and $U(B)$ will denote the *lower* and *upper cone of* B in A, respectively, i.e.,

$$L(B) = \{x \in A;\ x \leqq a,\ \text{for all } a \in B\},$$
$$U(B) = \{y \in A;\ a \leqq y,\ \text{for all } a \in B\}.$$

For $B = \{a_1, \dots, a_n\}$ we shall also write $L(B) = L(a_1, \dots, a_n)$, $U(B) = U(a_1, \dots, a_n)$.

In [5], the modularity of ordered sets is defined by the following condition: An ordered set $(A, \leqq)$ is called a *modular ordered set* if

$$\forall a, b, c \in A;\ a \leqq b \implies L\Big(U\big(a, L(c, b)\big)\Big) = L\big(U(a, c), b\big).$$

This notion is self-dual. (See [5].)

If a semilattice (lattice) is modular as an ordered set, then it is called an *o-modular semilattice* (*lattice*). It is easy to verify that in the case of lattices the notions of modularity and o-modularity are the same. (See also [5].) But for semilattices, we get two different notions. Namely, among others, every modular $\vee$-semilattice is *down-directed*, while, e.g., a semilattice isomorphic to that illustrated in Figure 1 is an o-modular semilattice which is not down-directed. We will show some connections between the two types of modularity in semilattices.

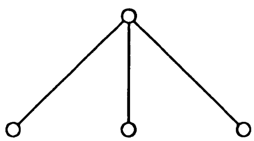


Figure 1

THEOREM 1.1. *Every modular* $\vee$*-semilattice is* o*-modular.*

Proof. Let A be a modular $\vee$-semilattice, $a, b, c \in A$, $a \leq b$. We have (in any $\vee$-semilattice)

$$L\Big(U\big(a, L(c, b)\big)\Big) \subseteq L(a \vee c, b).$$

Let $u \in L(a \vee c, b)$. Then $a \leqq a \vee u$, $a \vee u \leqq a \vee c$, hence from the modularity of the $\vee$-semilattice A we get that there exists an element $c_1 \in A$, $c_1 \leqq c$ such that $a \vee u = a \vee c_1$.

Let $x \in U(a, L(c,b))$. Then $x \geqq a$, $x \geqq c_1$, thus $x \geqq a \vee c_1 = a \vee c$, and so $u \leqq x$. Therefore

$$L(a \vee c, b) \subseteq L\Big(U\big(a, L(c,b)\big)\Big).$$

□

We say that an ordered set A satisfies the *restricted ascending chain condition* if every $L(a,b)$, where $a, b \in A$, satisfies the *ascending chain condition.*

THEOREM 1.2. *Every o-modular down-directed $\vee$-semilattice satisfying the restricted ascending chain condition is a modular semilattice.*

Proof. Let A satisfy the assumptions and let $a, b, c \in A$, $a \leqq b$, $b \leqq a \vee c$.

a) Let $a < b$. Then

$$L\Big(U\big(a, L(c,b)\big)\Big) = L(a \vee c, b) = L(b),$$

hence $b \leqq U\big(a, L(c,b)\big)$. Thus there must exist $u \in L(c,b)$ such that $a \ngeqq u$. We can suppose $u \| a$. (In the case $u > a$, we have $a \vee c = c$, and so $b = a \vee b$.)

If $a \vee u < b$, put $v_0 = a \vee u$. (See Figure 2a.) Then

$$L\Big(U\big(v_0, L(c,b)\big)\Big) = L(v_0 \vee c, b) = L(b),$$

hence $b \leqq U\big(v_0, L(c,b)\big)$. Thus there exists $u_1 \in L(c,b)$ such that $v_0 \ngeqq u_1$. Let us suppose $v_0 \| u_1$. (See Figure 2b.)

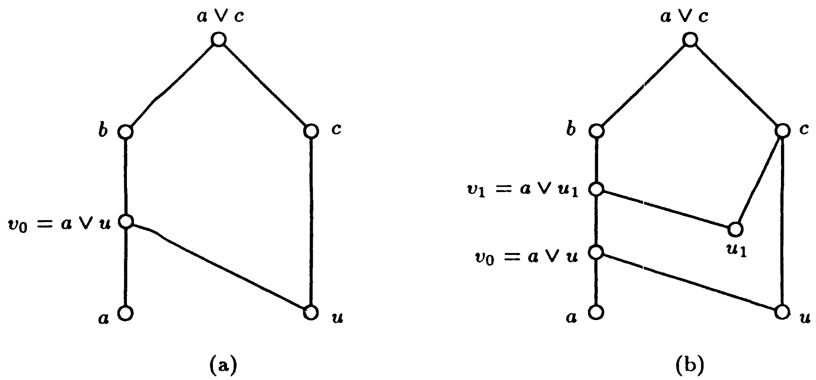


Figure 2

Since A satisfies the restricted ascending chain condition, there must exist $u_n \in A$, $u_n \leqq c$ with $b = a \vee u_n$.

b) Let $a = b$. By the assumption, A is down-directed, hence there exists $c_1 \in L(a, c)$. Then we have $a = a \vee c_1$. □

Recall that an *ideal of a* $\vee$*-semilattice* A is any nonvoid subset I of A such that

$$\forall a,\ b \in A; \quad a \vee b \in I \iff a,\ b \in I.$$

Let $I(A)$ denote the *set of all ideals of* A *ordered by set-inclusion.* Then $I_0(A)$ will mean the *ordinal sum* $\{\emptyset\} \oplus I(A)$. It is evident that $I_0(A)$ is a lattice. At the same time, the modular semilattices are characterized by the set of ideals as follows: A $\vee$-semilattice A is modular if and only if $I(A)$ is a modular lattice. Because the ordinal sum of two ordered sets A and B is a modular ordered set if and only if both of A and B are modular ordered sets (see [7]), for a modular $\vee$-semilattice A we have that $I_0(A)$ is a modular lattice. Conversely, we will consider the case that $I_0(A)$ is a modular lattice.

THEOREM 1.3. *If A is a $\vee$-semilattice such that $I_0(A)$ is a modular lattice, then A is an o-modular $\vee$-semilattice.*

Proof. Let $I_0(A)$ be a modular lattice. Let $a, b, c \in A$, $a \leqq b$. Let us suppose $x \in L(a \vee c, b)$, $y \in U\big(a, L(c, b)\big)$. Then

$$x \in L(a \vee c) \cap L(b) = \big(L(a) \vee L(c)\big) \cap L(b),$$

and according to the assumption $L(a) \subseteq L(b)$, we get

$$x \in L(a) \vee \big(L(c) \cap L(b)\big) = L(a) \vee L(c, b).$$

Further, $L(y) \supseteq L(a)$, $L(y) \supseteq L(c, b)$, hence

$$L(y) \supseteq L(a) \vee L(c, b).$$

Thus, $x \leqq y$, and this means

$$L(a \vee c, b) \subseteq L\Big(U\big(a, L(c, b)\big)\Big).$$

Since always

$$L\Big(U\big(a, L(c, b)\big)\Big) \subseteq L(a \vee c, b),$$

we obtain that A is an o-modular $\vee$-semilattice. □

2. o-distributive semilattices

Now, let us recall that a $\vee$-semilattice A is called *distributive* if

$$\forall a, b_0, b_1 \in A;\ a \leqq b_0 \vee b_1 \implies \exists\, a_0, a_1 \in A;\ a_0 \leqq b_0,\ a_1 \leqq b_1,\ a = a_0 \vee a_1 .$$

(See [3, p. 135].)

In [5], the distributivity of ordered sets is defined as follows: An ordered set $(A, \leqq)$ is called a *distributive ordered set* if

$$\forall a, b, c \in A;\ L\Big(U\big(L(a,c), L(b,c)\big)\Big) = L\big(U(a,b), c\big).$$

The distributivity of ordered sets is (similarly as the modularity) a self-dual notion. (See [5].) A semilattice (lattice) which is distributive as an ordered set will be also called an *o-distributive semilattice* (*lattice*). Analogously as for the modularity, a lattice is distributive if and only if it is o-distributive. (See e. g. [5].) However, for semilattices we also obtain two different notions. On the one hand, any distributive semilattice is o-distributive, but the converse assertion is not true. Moreover, any distributive $\vee$-semilattice is down-directed. Now, we will show a connection between the distributivity and the o-distributivity of *conditionally complete semilattices.* (By a *conditionally complete* $\vee$*-semilattice* we mean a $\vee$-semilattice in which any up-bounded non-void subset has the supremum.)

THEOREM 2.1. *If A is a conditionally complete $\vee$-semilattice, then the following conditions are equivalent:*

(i) *A is a distributive semilattice,*
(ii) *A is a down-directed o-distributive semilattice.*

Proof. (i) $\Longrightarrow$ (ii) is always true.
(ii) $\Longrightarrow$ (i): Let $a, b_0, b_1 \in A$, $a \leqq b_0 \vee b_1$. By the assumption, $L(b_0, a) \neq \emptyset$, $L(b_1, a) \neq \emptyset$. We have

$$L\Big(U\big(L(b_0, a), L(b_1, a)\big)\Big) = L(b_0 \vee b_1, a) = L(a),$$

hence $a \leqq U\big(L(b_0, a), L(b_1, a)\big)$, i.e., $a \leqq v$ for all $v \in A$ such that $v \geqq L(b_0, a)$, $v \geqq L(b_1, a)$. Moreover,

$$a \geqq L(b_0, a), \qquad a \geqq L(b_1, a),$$

and this means

$$a = \sup\big(L(b_0, a) \cup L(b_1, a)\big).$$

Let $a_0 = \sup L(b_0, a)$, $a_1 = \sup L(b_1, a)$. It is evident that $a_0 \vee a_1 \leqq a$. But also

$$a_0 \vee a_1 \geq L(b_0, a) \cup L(b_1, a),$$

hence $a \leqq a_0 \vee a_1$, and so $a = a_0 \vee a_1$. □

Let us recall that for distributive $\vee$-semilattices, the set of ideals has a similar sense as for modular ones. Namely (see [3]): If A is a $\vee$ semilattice, then A is distributive if and only if $I(A)$ is a distributive lattice. We can easily verify (see also [7]) that the ordinal sum of ordered sets A and B is a distributive ordered set if and only if both of A and B are distributive. Hence, for a distributive $\vee$-semilattice A, the ordered set $I_0(A)$ is a distributive lattice.

Conversely, let $I_0(A)$ be a distributive lattice. Hence we obtain the following theorem.

THEOREM 2.2. *Let A be a $\vee$-semilattice. If the lattice $I_0(A)$ is distributive, then A is an o-distributive semilattice.*

Proof. Let $I_0(A)$ be a distributive lattice. Let $a, b, c \in A$ and let $x \in L(a \vee b, c)$. Then

$$x \in L(a \vee b) \cap L(c) = \big(L(a) \vee L(b)\big) \cap L(c),$$

and thus, by the assumption,

$$x \in \big(L(a) \cap L(c)\big) \vee \big(L(b) \cap L(c)\big) = L(a, c) \vee L(b, c).$$

Consider $y \in U\big(L(a,c), L(b,c)\big)$. Then $L(y) \supseteq L(a,c)$, $L(y) \supseteq L(b,c)$, i. e. $L(y) \supseteq L(a,c) \vee L(b,c)$. Hence $x \leqq y$, and therefore

$$L(a \vee b, c) \subset L\Big(U\big(L(a,c), L(b,c)\big)\Big).$$

Because the converse implication is true in any ordered set, we have that A is an o-distributive $\vee$-semilattice. □

3. Concluding remarks

a) Distributive and modular semilattices have been intensively studied. Various characterizations of distributivity and modularity of semilattices have been found, for example, by W. H. Cornish [2].

In this paper, we are interested in some generalization of modularity and distributivity of semilattices. However, C. Jayaram [4] has introduced the notion of 0-modular semilattice which also generalizes a modular semilattice (in the dual form). Namely, a $\wedge$-semilattice A with the least element 0 is called *0-modular* if

$$\forall a, b, c \in A;\ (a \leqq c,\ b \wedge c = 0 \implies \exists\, d \in A;\ b \leqq d \quad \text{and} \quad a = c \wedge d).$$

Bounded 0-modular semilattices are characterized [4, Theorem 1] by means of the lattice of their filters as follows: A *bounded $\wedge$-semilattice* A is 0-modular if and only if the lattice $F(A)$ of all filters of A is 1-modular. (*1-modularity* is, for lattices, the dual notion of 0-modularity.)

As our Theorem 1.3 gives a characterization of o-modular semilattices in a similar way, we will compare, for bounded cases, the two types of generalized modularity of semilattices.

Let $A_1 = \{0, a, b, c, d, e, f, g, a_0, a_1, a_2, \dots\}$ be a $\wedge$-semilattice as shown in Figure 3, where the arrow indicates an infinite chain.

This semilattice is used in [4] as an example of a bounded 0-modular semilattice. On the other hand, we have

$$L\big(U(f,e), g\big) = L(g), \qquad L\big(U(f, e \wedge g)\big) = L(f),$$

and therefore A_1 is not an o-modular $\wedge$-semilattice.

Now, consider a bounded $\wedge$-semilattice $A_2 = \{0, a, b, c, c_1, c_2, \dots, d_0, d_1, \dots\}$ with diagram in Figure 4, where both arrows indicate infinite chains again. We have $c \wedge b = 0$, $a < c$, but there is no element $d \in A_2$ with $d \geqq b$ and $c \wedge d = a$. Hence A_2 is not 0-modular. At the same time it can be easily verified that A_2 is o-modular.

These two examples show that even for the case of bounded semilattices with 0, the concepts of 0-modular and o-modular semilattices are independent.

However, *Jayaram's theorem* makes it possible to show that the converse of Theorem 1.3 is not valid. Namely, for a bounded $\wedge$-semilattice A with 0, the modularity of the lattice $F(A)$ is equivalent to the modularity of the lattice $F_0(A) = \{\emptyset\} \oplus F(A)$, and any modular lattice is 1-modular as well. Hence even a bounded o-modular $\wedge$-semilattice A with 0 need not have the modular lattice $F_0(A)$.

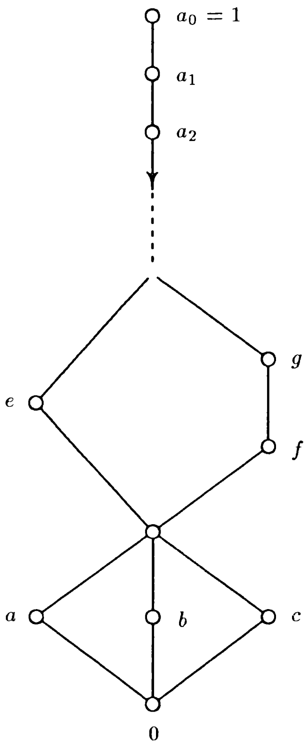


Figure 3

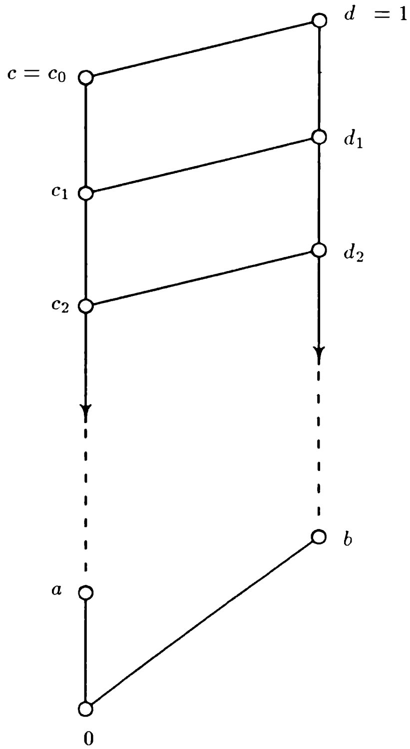


Figure 4

Analogously, we can compare the notions of o-distributive and 0-distributive semilattices (for the bounded case). Recall that a $\wedge$-semilattice A with the least element 0 is called 0-distributive if

$$\forall a, b, c \in A;\ (a \wedge b = 0 = a \wedge c \implies \exists\, d \in U(b,c);\ a \wedge d = 0).$$

Consider a $\wedge$-semilattice $A_3 = \{0, a, b, c, d, e, f, a_0, a_1, a_2, \dots\}$ in Figure 5, where the arrow indicates an infinite chain such that all its elements a_i are strictly greater than each of the elements a, b, c. It is evident that A_3 is 0-distributive. (Moreover, A_3 is modular, but it is not distributive.) However, we have

$$L\Big(U\big(L(a,b), L(a,c)\big)\Big) = L\big(U(a \wedge b, a \wedge c)\big) = L(f),$$
$$L\big(U(b,c), a\big) = L(a),$$

and so A_3 is not o-distributive.

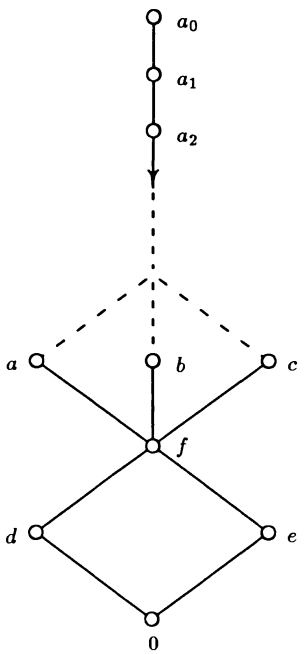


Figure 5

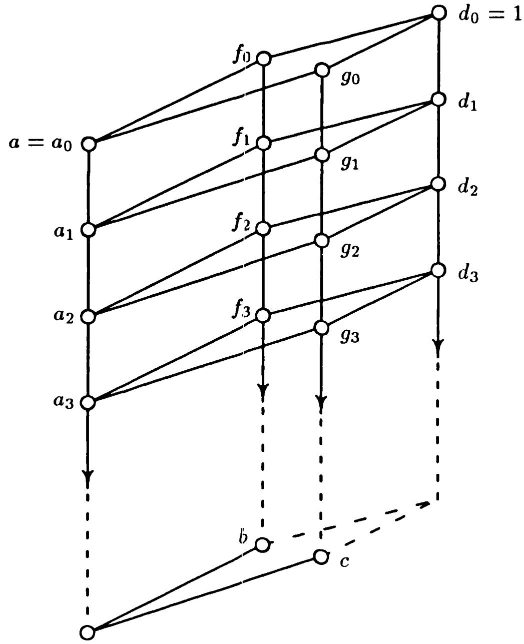


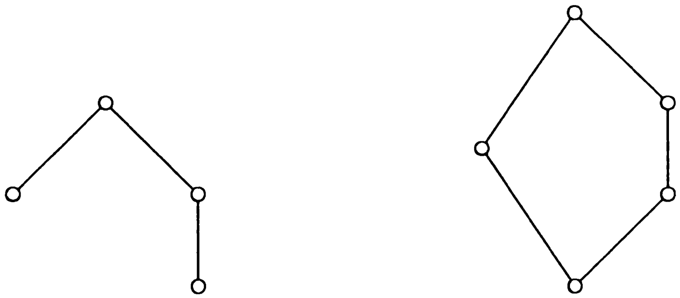
Figure 6

Conversely, let $A_4 = \{0, a, b, c, a_1, a_2, \ldots, f_0, f_1, f_2, \ldots, g_0, g_1, g_2, \ldots, d_0, d_1, d_2 \ldots\}$ be a $\wedge$-semilattice with diagram in Figure 6, where the elements $d_0, d_1, d_2, \ldots$ form $U(b,c)$, the elements $f_0, f_1, f_2, \ldots$ form $U(b) \setminus U(c)$, the elements $g_0, g_1, g_2, \ldots$ form $U(c) \setminus U(b)$, each of the four arrows indicates an infinite chain, and all elements of those four infinite sets have their connections indicated. We have $a \wedge b = 0 = a \wedge c$, but there is no element d_i in $U(b,c)$ with $a \wedge d_i = 0$, and so A_4 is not 0-distributive. It can be easily verified that A_4 is o-distributive. Therefore, even for bounded semilattices the concepts of 0-distributive and o-distributive (in the dual form) semilattices are different.

b) Let B be a subsemilattice of a $\vee$-semilattice A. We say that B is an LU *subsemilattice of* A if

$$\forall a, b \in B; \; L_B(a,b) = \emptyset \iff L(a,b) = \emptyset,$$

where $L_B(a,b) = L(a,b) \cap B$. (See [1].) Using [7, Theorem 2 and its proof] we can easily prove that the following proposition is true: If a $\vee$-semilattice A is not o-modular, then A contains an LU subsemilattice isomorphic to M_2 or to M_4 in Figure 7.



M_2 M_4

Figure 7

Further, a subsemilattice B of a $\vee$-semilattice A is called a *strong subsemilattice of* A if

$$\forall a, b \in B;\ U\big(L_B(a,b)\big) = U\big(L(a,b)\big).$$

(See [1].)

By [1, Theorem 4], we can get the following proposition: If a $\vee$-semilattice A contains an LU subsemilattice isomorphic to M_2 or if it contains a strong subsemilattice isomorphic to M_4, respectively, then A is not o-modular (and so, A is not modular).

REFERENCES

[1] CHAJDA, I.—RACHŮNEK, J.: *Forbidden configurations of modular and distributive ordered sets*, Order **5** (1989), 407–425.

[2] CORNISH, W. H.: *Characterizations of distributive and modular semilattices*, Math. Japon. **22** (1977), 159 174.

[3] GRÄTZER, G.: *General Lattice Theory*, Russian ed., Moscow, 1982.

[4] JAYARAM, C.: *0-modular semilattices*, Studia Sci. Math. Hungar. **22** (1987), 189 195.

[5] LARMEROVÁ, J.—RACHŮNEK, J.: *Translations of distributive and modular ordered sets*, Acta Univ. Palack. Olomouc. Fac. Rerum Natur. Math. **91** (1988), 13–23.

[6] RACHŮNEK, J.: *A characterization of o-distributive semilattices*, Acta Sci. Math. (Szeged) **54** (1990), 241–246.

[7] RACHŮNEK, J.: *Ordinal varieties of ordered sets*, (Preprint).

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