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LOCALLY–CYCLIC GRAPHS COVERING COMPLETE TRIPARTITE GRAPHS

ROMAN NEDELA

ABSTRACT. A new construction of the so-called locally-\(C_n\) graphs, for \(n\) even, based on the technique of voltage graphs is presented.

Let \(G\) be a graph and \(u\) a vertex. Denote by \(G(u)\) the subgraph of \(G\) induced by the set of vertices adjacent to \(u\). The graph \(G\) is called locally \(H\) if \(G(u) \cong H\) for each vertex \(u\) of \(G\). Further we shall be interested only in the case \(H \cong C_n\), where \(n \geq 3\) is fixed and \(C_n\) is a cycle of length \(n\). The existence of finite locally-\(C_n\) graphs for each \(n \geq 3\) was established in [1] and also in [2]. Later Ronan in [7] showed that there are infinitely many such graphs for each \(n \geq 6\). A characterization of locally-\(C_n\) graphs is geometrical terms given by Vince [9] shows how to obtain locally-\(C_n\) graphs from groups. This was done in [8]. The relationship between locally-\(C_n\) graphs and 3-valent polygonal graphs is studied in [6]. In this note we present a way of constructing locally-\(C_{2n}\) graphs using voltage graphs.

An important and interesting property of locally \(C_n\) graphs is that each of them gives rise to a uniquely determined triangulation of a closed surface. In fact, denote for a given graph \(G\) by \(K(G)\) the simplicial complex the simplices of which are the cliques of \(G\) and the incidence relation is given by subgraph inclusion. Then we have

**Theorem 1.** ([5]) A graph \(G\) is locally \(C_n\) if and only if \(K(G)\) is an \(n\)-valent triangulation of a closed surface in which each cycle of length 3 forms a face-boundary.

We obtain a class of locally-\(C_{2n}\) graphs as covering triangulations of the well-known triangular embedding of complete tripartite graphs \(K_{n,n,n}\), \(n \geq 2\) even, described in [10].

Further it is assumed that the reader is familiar with the terminology and the basic concepts of the topological graph theory, namely with the theory of

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2-cell embeddings of graphs into closed surfaces and with the theory of voltage graphs (see [3, 11]).

First we recall some definitions. For a given graph $G$ choose for each edge of a graph $G$ one of the two possible orientations. Then to each edge $e$ of $G$ we associate two arcs $e, e^{-1}$ with the chosen and the opposite orientation, respectively. Denote by $D(G)$ the set of all arcs of $G$. Clearly, $|D(G)| = 2|E(G)|$.

A voltage graph is a triple $(G, \varphi, \Gamma)$, where $G$ is a graph and $\varphi$ is a mapping (sometimes called a voltage assignment) from $D(G)$ to a group $\Gamma$ with a unique restriction $\varphi(e)^{-1} = \varphi(e^{-1})$. For the given voltage graph $(G, \varphi, \Gamma)$ the derived covering graph $G \times^\varphi \Gamma$ is defined as follows: its vertex set is $V(G) \times \Gamma$ and each edge $e = uv$ of $G$ generates the edges $(e, g) = (u, g)(v, g\varphi(e))$ of $G \times^\varphi \Gamma$, where $g$ ranges over all the elements of the group $\Gamma$. It is easy to see that the natural projection, mapping an edge $(e, g)$ of $G \times^\varphi \Gamma$ to $e$ of $G$, is a covering mapping. If the original graph $G$ is embedded into some surface, then this embedding may be lifted in a natural way into the derived embedding of the graph $G \times^\varphi \Gamma$. The set of cycles forming face-boundaries of faces of the derived embedding consists of the cycles of $G \times^\varphi \Gamma$ covering the boundaries of faces of the embedding of $G$ in the natural projection sending an edge $(e, g)$ of $G \times^\varphi \Gamma$ to $e$ in $G$. It is not difficult to see that this new embedding forms a (branched) covering embedding of the original embedding. It is also known that the derived embedding is unbranched if and only if the product of voltages on a boundary of each face of the original embedding is the unit element of $\Gamma$. In the latter case a covering over a triangulation is again a triangulation.

**Construction.** We start with a triangular embedding $j$ of $K_{n,n,n}$ into an orientable surface $S$ described in [10]. Let $V(K_{n,n,n}) = A \cup B \cup C$ be the tri-partition of $K_{n,n,n}$. Since $j$ is the triangulation, then its restriction $r = j|_{A \cup B}$ is an embedding of an induced subgraph $(A \cup B) \cong K_{n,n}$ of $K_{n,n,n}$ into $S$. Clearly, the boundary of each face of $r$ forms a Hamiltonian cycle in $K_{n,n,n}$. We claim that if $n$ is even, then the edges of $K_{n,n}$ can be oriented in such a way that arcs lying on the boundary of each face of $r$ create a directed cycle. This follows from the fact that the dual embedding $r^*$ of $r$ is an embedding of a bipartite graph $H \hookrightarrow S$. In fact, the embedding $r$ can be obtained as the derived embedding of the embedding $k$ of the $n$-fold $K_2$ into the sphere (see [3, p. 210]). Since $k^*$ is an embedding of $C_n$ into the sphere and $r^*$ covers $k^*$, then $r^*$ must be bipartite if $n$ is even. Now define a voltage assignment mapping $\psi: D(K_{n,n,n}) \to \mathbb{Z}_{2n}$ as follows. Set $\psi(e) = 1$ if an arc $e$ of $K_{n,n} \cong (A \cup B)$ has the chosen orientation and set $\psi(e^{-1}) = -1$ for the arc $e^{-1}$. Let $\varrho_u$ be the local rotation of arcs emanating from a vertex $u$ of $C$ determined by the embedding $j: K_{n,n,n} \hookrightarrow S$. Let $(e_0, e_1, \ldots, e_{2n-1})$ be one of the rotations $\varrho_u, \varrho_u^{-1}$ which is consistent with the orientation on the boundary of a face of $r$ containing the
vertex \( u \). Then put \( \psi(e_i) = i \) and \( \psi(e_i^{-1}) = -i \) for all \( i = 0, \ldots, 2n - 1 \).

**Theorem 2.** Let \( \psi \) be the voltage assignments on the graph \( K_{n,n,n} \), \( n \geq 2 \) even, with values in the cyclic group \( \mathbb{Z}_{2n} \) defined above. Then \( G = K_{n,n,n} \times \mathbb{Z}_{2n} \) is a locally-\( C_{2n} \)-graph.

**Proof.** Since the sum of assignments of arcs on each triangle-face of \( j \) is 0, then the derived embedding \( i \) is unbranched, and consequently, it must be a \( 2n \)-valent triangulation. To complete the proof it is sufficient to show that each cycle \( ((u, x), (v, y)(w, z)) \) of length 3 in \( G \) forms a face-boundary. By the definition of \( G \) we have that \( (uvw) \) is a cycle of length 3 in \( K_{n,n,n} \) and \( \psi(uv) + \psi(vw) + \psi(wu) = 0 \). We may suppose that \( u \in C, v \in A, w \in B \). By the definition of \( \psi \) we have \( \psi(vw) = 1 \) or \( \psi(vw) = -1 \), and consequently, \( \psi(uv) \) and \( \psi(uw) \) differ by 1. Then either \( \varrho_u(v) = w \) or \( \varrho_u(w) = v \). In both cases we see that \( (uvw) \) forms the boundary of a triangle face in \( j \), hence \( ((u, x)(v, y)(w, z)) \) forms a face boundary. The assertion follows from Theorem 1. \( \square \)

**Concluding remark.** A triangulation \( T \) is called a clean triangulation if every cycle of length 3 in \( T \) forms a face-boundary. N. Hartshfield and G. Ringel [4] investigated the problem of determining of the minimum number \( T(S_p) \) of triangles of a clean triangulation of surface of genus \( p \). They proved \( \lim_{p \to \infty} \frac{T(S_p)}{p} = 4 \). Let \( T_k \) be the triangulation obtained using our construction for \( n = 2k \), denote by \( T(T_k) \) the number of triangles of \( T_k \) and by \( p_k \) the genus of the underlying surface. Then it is easy to compute \( T(T_k) = 32k^3 \), \( p_k = 8k^3 - 12k^2 + 1 \), and hence, \( \lim_{k \to \infty} \frac{T(T_k)}{p_k} = 4 \). Thus the sequence \( \{T_k\} \) is extremal in sense of Ringel and Hartshfield [4].

**References**


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