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SOLID SUMMABILITY FIELDS

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ABSTRACT. In the paper we investigate Köthe-Toeplitz solidity of the summability field of an infinite matrix of operators in Banach spaces.

1. Introduction

Let X and Y be Banach spaces over the complex field $\mathbb{C}$, and denote by $B(X,Y)$ the space of bounded linear operators on X into Y. Following the notation of Maddox [3, pp. 4, 5] we denote by $s(X)$ the linear space of all X-valued sequences, and by $c(X)$ the subspace of all norm convergent X-valued sequences.

We shall be concerned with a generalized notion of solid sequence space in the vector-valued setting, and we shall determine the conditions for solidity of certain general summability fields.

2. Basic definitions

Generalizing the idea of solid (or normal) scalar sequence space due to Köthe and Toeplitz [2], we say that a subspace E of $s(X)$ is solid if $x = (x_n) \in E$ and $\|y_n\| \le \|x_n\|$ for all $n \ge 1$ imply $y \in E$. For example, $s(X)$ is solid but $c(X)$ is not.

If $A = (A_{nk})$, $n, k = 1, 2, \dots$, is an infinite matrix of operators $A_{nk} \in B(X,Y)$ and $x = (x_k) \in s(X)$, then we say that x is summable A to $z \in Y$ if and only if

$$A_n(x) = \sum_{k=1}^{\infty} A_{nk} x_k$$

converges in the norm of Y for each n and $A_n(x) \to z$ as $n \to \infty$. We define the summability field of A to be

$$c_A = \{x \in s(X) : (A_n(x)) \in c(Y)\} .$$

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In the present work we shall be concerned with the case in which $Y = C$, the complex field, and with A a diagonal matrix, that is $A_{nk} = 0$ for $n \neq k$. With these assumptions, writing $A_n = A_{nn}$ we see that $x \in c_A$ if and only if $A_n(x_n) \to z$ as $n \to \infty$, where each A_n is a continuous linear functional on X.

We shall determine necessary and sufficient conditions for the solidity of c_A for the two Banach spaces $X = C$ and $X = c_0$, where c_0 denotes the scalar null sequences.

3. The main results

First we obtain a necessary condition for the solidity of c_A for an arbitrary Banach space X.

THEOREM 3.1. *If c_A is solid, then $\{n : A_n = 0\}$ is an infinite set.*

Proof. Suppose if possible that $\{n : A_n = 0\}$ is finite. Then there exists p such that $A_n \neq 0$ for all $n > p$, whence there exists $z_n \in X$ with $A_n(z_n) \neq 0$. Now define $x_n = z_n/A_n(z_n)$ for $n > p$; $y_n = (-1)^n x_n$ for $n > p$ and $x_n = y_n = 0$ for $n \leq p$. Then $\|y_n\| = \|x_n\|$ for all $n \geq 1$ and $A_n(x_n) = 1$ for $n > p$, so that $x \in c_A$. Since c_A is solid we must have $y \in c_A$, contrary to the fact that $A_n(y_n) = (-1)^n$ for $n > p$. This proves the theorem.

Next we show that the condition $\{n : A_n = 0\}$ infinite is not generally sufficient for c_A to be solid.

PROPOSITION 3.2. *Let $X = c_0$, the space of all null scalar sequences with $\|x\| = \sup_k |s_k|$ for each $x = (s_k) \in c_0$. Define $A_n = 0$ for n odd and $A_n x = s_n$ for n even. Then c_A is not solid.*

Proof.

If we write $x_n = (s_{nk}) = (s_{n1}, s_{n2}, \ldots)$ and $y_n = (t_{nk}) = (t_{n1}, t_{n2}, \ldots)$, let us define $s_{n1} = 1$ and $s_{nn} = n^{-1}$, with $s_{nk} = 0$ otherwise, and $t_{nn} = 1$, with $t_{nk} = 0$ otherwise. Then x_n and y_n are in c_0 and $\|y_n\| = \|x_n\|$ for all $n \geq 1$.

Hence $A_n(x_n) = 0$ or $A_n(x_n) = n^{-1}$ as n is odd or even, whence $x = (x_n) \in c_A$. But $A_n(y_n) = 0$ or $A_n(y_n) = 1$ as n is odd or even, and so $y \notin c_A$. Thus c_A is not solid even though $\{n \colon A_n = 0\}$ is infinite. This completes the proof.

Now let us take $X = C$ and identify the A_n with complex numbers a_n, so that $A_n z = a_n z$ for each $z \in C$. In this case the condition of Theorem 3.1 is necessary and sufficient:

THEOREM 3.3. *In the case $X = C$ we have that c_A is solid if and only if $\{n : A_n = 0\}$ is infinite.*

P r o o f . In view of Theorem 3.1 we need only consider the sufficiency. Supposing that $a_n = 0$ for $n = n_1, n_2, \dots$ with $n_1 < n_2 < \dots$, let $(x_n) \in c_A$ and $|y_n| \le |x_n|$ for all $n \ge 1$. The $a_n x_n \to \ell$ implies $a_{n_i} x_{n_i} \to \ell$, and so $0 = \ell$. Hence for all $n \ge 1$, $|a_n y_n| \le |a_n||x_n|$, which implies $a_n y_n \to 0$, so $(y_n) \in c_A$, as required.

We next consider the case in which $X = c_0$ the space of scalar null sequences. Here we shall show that a stronger condition on (A_n) is required for solidity:

THEOREM 3.4. *In case $X = c_0$ we have that c_A is solid if and only if $A_n = 0$ eventually in n.*

P r o o f . The sufficiency is trivial, since if $A_n = 0$ eventually in n, then c_A is the space $s(c_0)$, which is certainly solid.

Conversely, let c_A be solid but assume that $A_n \ne 0$ for $n = n_1, n_2, \dots$ with $n_1 < n_2 < \dots$, and $A_n = 0$ for $n \ne n_i$.

Since A_n is a continuous linear functional in c_0 we may write

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} s_k$$

for each $x = (s_k) \in c_0$, with $\sum\limits_{k=1}^{\infty} |a_{nk}| < \infty$ for each $n \ge 1$. For this representation of A_n see B a n a c h [1] or M a d d o x [4].

Now for each n_i there exists k_i such that $a_{n_i k_i} \ne 0$. Take any n_i. Then from

$$\sum_{k=1}^{\infty} |a_{n_i k}| < \infty$$

it follows that there exists $r_i > k_i$ such that

$$|a_{n_i r_i}| < |a_{n_i k_i}| i^{-1} .$$

Now write $x_n = (s_{nk}) = (s_{n1}, s_{n2}, \dots)$ and $y_n = (t_{nk}) = (t_{n1}, t_{n2}, \dots)$. Define $x_n = y_n = 0$ for $n \ne n_i$, and for $n = n_i$ define

$$\begin{aligned} s_{nk} &= i^{-1} a_{nk}^{-1} && \text{when } k = k_i \\ s_{nk} &= a_{nk_i}^{-1} && \text{when } k = r_i \\ s_{nk} &= 0 && \text{(otherwise)} \\ t_{nk} &= a_{nk}^{-1} && \text{when } k = k_i \\ t_{nk} &= 0 && \text{(otherwise)}. \end{aligned}$$

Then it is clear that $\|x_n\| = \|y_n\|$ for all $n \geq 1$, and for $n = n_i$ we have

$$|A_n(x_n)| = |i^{-1} + a_{nr_i} a_{nk_i}^{-1}| < 2i^{-1}$$
$$A_n(y_n) = 1\,.$$

Since $A_n(x_n) = A_n(y_n) = 0$ for $n \neq n_i$ we see that $x \in c_A$ but $y \notin c_A$, contrary to the fact that c_A is solid. This proves the theorem.

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