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WEAK RIESZ GROUPS

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ABSTRACT. In this paper a modification of the Riesz decomposition property is investigated on directed po-groups. Namely, a lattice characterization of the set of all directed convex subgroups of a directed po-group with this decomposition property is described.

Riesz groups are directly partially ordered groups (po-groups, briefly) which have the well-known interpolation property (or the decomposition property, equivalently – see [8] and [3]). Pedersen [6] proved that a weak variant of the Riesz decomposition property holds in C*-algebras. In this paper a similar modification of the Riesz decomposition property is investigated on directed po-groups. Namely, a lattice characterization of the set of all directed convex subgroups (a-ideals, respectively) of a directed po-group with this decomposition property is described.

1. Decomposition on C*-algebras

Pedersen in [6] shows that the following decomposition property is true in a C*-algebra $A$:

If $x, a, b \in A^+$, $0 \leq x \leq a + b$, then $u, v \in A$ exist such that $x = u^* \cdot u + v^* \cdot v$ and $u \cdot u^* \leq a$, $v \cdot v^* \leq b$.

In the case that $u, v$ are normal we obtain the Riesz decomposition property. All unexplained facts concerning C*-algebras can be found in Dixmier [1]. The set of all hermitian elements (positive elements) in a C*-algebra $A$ is denoted by $A_h$ ($A^+$). Let us denote $|a| = (a^* \cdot a)^{\frac{1}{2}}$ for $a \in A$ (see [2], preceding Th. 2.4).

PROPOSITION 1.1. If $A$ is a C*-algebra, then the following assertions are equivalent:

1. $A$ has the Riesz decomposition property.

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2. \( a \land b = 0 \) in \( A_h \) if and only if \( a \land b = 0 \) in \( A^+ \), for \( a, b \in A \).

3. \( A \) is commutative.

**Proof.**

1 \( \Rightarrow \) 2: If \( a \land b = 0 \) in \( A^+ \) and \( c \in A_h \) exists such that \( c \leq a, b, c \not\in 0 \), then \( c \parallel 0, c, 0 \leq a, b \) and the Riesz interpolation property implies an existence of \( z \in A \) with \( 0, c < z < a, b \) and the Riesz interpolation property implies an existence of \( z \in A \) with \( 0, c < z < a, b \), a contradiction.

2 \( \Rightarrow \) 3: If \( a, b \in A \) and \( |a| \land |b| = 0 \) in \( A_h \), then with regard to [10, 2.11] there holds \( a \cdot b = 0 \) and \( |a| \land |b^*| = 0 \) in \( A^+ \). We have \( a^* \cdot b = 0 \Rightarrow b^* \cdot a = 0 \Rightarrow |b^*| \land |a^*| = 0 \) in \( A^+ \Rightarrow |b^*| \land |a| = 0 \) in \( A_h \Rightarrow b^* \cdot a^* = 0 \Rightarrow a \cdot b = 0 \). Further, \( a \cdot b = 0 \Rightarrow (a^*)^* \cdot b = 0 \) and the previous consideration implies \( a^* \cdot b = 0 \). Finally, \( a^* \cdot b = 0 \iff a \cdot b = 0 \) holds, for each \( a, b \in A \) and according to [10, 2.13] \( A \) is commutative.

3 \( \Rightarrow \) 1 is clear.

**COROLLARY 1.2.** If \( A \) is non-commutative \( C^* \)-algebra, then \( A \) has the Pedersen decomposition property but not the Riesz decomposition property.

**PROPOSITION 1.3.** A \( C^* \)-algebra \( A \) has the following decomposition property:

If \( x, a, b \in A^+ \), \( 0 < x < a + b \), then \( k, l \in A \) exists such that \( k \leq l \leq x \leq k + l \) and \( k \in \langle a \rangle, l \in \langle b \rangle \) (\( \langle a \rangle \) is the closed ideal in \( A \) generated by \( a \)).

**Proof.** The Pedersen decomposition property implies an existence of \( u, v \in A \) such that \( x = |u|^2 + |v|^2, |u^*|^2 \leq a, |v^*|^2 \leq b \) hold. If \( k = |u|^2 \), \( l = |v|^2 \), then \( k, l \geq 0 \) and \( x = k + l \). We have \( |u|^4 = u^* \cdot u \cdot u^* \cdot u = u^* \cdot u \cdot u^* \cdot u \leq u^* \cdot u \cdot u \in \langle a \rangle \) and thus \( k = |u|^2 \in \langle a \rangle \) holds (see [2, 2.2]). Similarly \( l = |v|^2 \in \langle b \rangle \).

This decomposition property is a generalization of the Riesz decomposition property.

**PROPOSITION 1.4.** A \( C^* \)-algebra \( A \) has the following interpolation property:

If \( x_1, x_2, y_1, y_2 \in A, x_i \leq y_j \) for \( i, j \in \{1, 2\} \), then elements \( z, h, k \in A \) exists such that \( x_1 \leq z \leq h + y_1, x_2 - k \leq z \leq y_2, h, k \leq 0, h \in \langle x_1, y_1 \rangle, k \in \langle x_2, y_2 \rangle \), where \( \langle x_i, y_i \rangle \) is the ideal in \( A \) generated by \( x_i, y_i \) (\( i = 1, 2 \)).

**Proof.** We have \( y_j - x_i \geq 0 \) for \( i, j \in \{1, 2\} \) and \( y_2 - x_1 = (y_2 - x_2) + (x_2 - x_1) \leq (y_1 - x_1) + (y_2 - x_2) \). According to the Pedersen decomposition property there exist elements \( u, v \in A \) such that \( y_2 - x_1 = |u|^2 + |v|^2 \) and \( |u^*|^2 \leq y_1 - x_1, |v^*|^2 \leq y_2 - x_2 \). If we put \( z = |u|^2 + x_1 \), then \( z \geq x_1 \), \( y_1 \geq |u^*|^2 + x_1 = |u^*|^2 - |u|^2 + z \) and \( z \leq h + y_1 \) for \( h = |u|^2 - |u^*|^2 \). Further, \( y_2 = |u|^2 + |v|^2 + x_1 \geq |u|^2 + x_1 = z, x_2 \leq -|v^*|^2 + y_2 = -|v^*|^2 + |u|^2 + |v|^2 + x_1 = -|v^*|^2 + |v|^2 + z \) hold. Thus we have \( x_2 - k \leq z \) for \( k = |v|^2 + |v^*|^2 \). Finally,
2. Weak decomposition property on po-groups

Effros [2] in Theorem 2.8 describes a bijection between closed ideals of a C*-algebra $A$ and closed invariant order ideals in $A$. If $I$ is an ideal in $A$, then $I \cap A_h$ is an o-ideal (i.e., a directed convex normal subgroup) in a directed po-group $A_h$. These considerations give the following generalization.

**Definition 2.1.** Let $G$ be a directed po-group with the following property:

If $x, a, b \in G^+$, $0 \leq x \leq a + b$, then elements $k, l \in G$ exist such that $k, l \geq 0$, $x = k + l$, $k \in \langle a \rangle$ and $l \in \langle b \rangle$, where $\langle a \rangle$ ($\langle b \rangle$, resp.) is a directed convex subgroup in $G$ generated by $a$ ($b$, resp.). Then we say that $G$ is a weak Riesz group (or $G$ has the weak decomposition property).

Weak Riesz groups fulfil a theorem similar to the theorem of Størmer [9] for C*-algebras.

**Proposition 2.2.** Let $G$ be a directed po-group. Then $G$ is a weak Riesz group if and only if $I^+ + J^+ = (I + J)^+$ holds for arbitrary directed convex subgroups $I, J$ in $G$.

**Proof.**

$\Rightarrow$: Clearly $I^+ + J^+ \subseteq (I + J)^+$ and if $x \in (I + J)^+$, then $0 \leq x \leq a + b$ for suitable elements $a \in I^+$ and $b \in J^+$. Thus $k, l \in G$ exist such that $k, l \geq 0$, $x = k + l$, $k \in \langle a \rangle$, $l \in \langle b \rangle$ and it implies $x \in I^+ + J^+$.

$\Leftarrow$: If $x, a, b \in G^+$, $0 \leq x \leq a + b$ then there holds $x \in \langle a + b \rangle^+ = \langle a \rangle^+ + \langle b \rangle^+$. Finally, $k \in \langle a \rangle^+$ and $l \in \langle b \rangle^+$ exists such that $x = k + l$.

**Proposition 2.3.** If $G$ is a weak Riesz group, $0 \leq x \leq y_1 + y_2 + \cdots + y_n$ for $x, y_1, y_2, \ldots, y_n \in G^+$, then elements $x_1, x_2, \ldots, x_n \in G^+$ exist such that $x = x_1 + x_2 + \cdots + x_n$ and $x_i \in \langle y_i \rangle$ for $i = 1, 2, \ldots, n$.

**Proof.** can be done by induction.

**Proposition 2.4.** If $G$ is a weak Riesz group, then a sum of directed convex subgroups in $G$ is again a directed convex subgroup in $G$.

**Proof.** If $\{X_i: i \in I\}$ is a set of directed convex subgroups in $G$ and $X = \sum X_i$, then $X_i$ is generated by $X_i^+$ for $i \in I$ and thus $X$ is generated by a subset in $G^+$, i.e., $X$ is directed. If $0 \leq y \leq x$, $x \in X$, $y \in G$, then $x \leq \sum x_i$ for suitable $x_i \in X_i^+$ and $K \subseteq I$ finite. With regard to 2.3 we
have $y = \sum_{i \in K} y_i$ for $y_i \in (x_i)^+ \subseteq X_i$ $(i \in K)$. Finally, $X$ is a directed convex subgroup in $G$.

**Proposition 2.5.** Let $G$ be a weak Riesz group. Then there holds:

1. If $H$ is a directed convex subgroup in $G$, then $H$ is a weak Riesz group.

2. If $H$ is an o-ideal in $G$, then $G/H$ is a weak Riesz group.

**Proof.**

1. If $x, a, b \in H^+$, $0 \leq x \leq a + b$, then elements $k, l \in G^+$ exist such that $x = k + l$, $k \in \langle a \rangle$, $l \in \langle b \rangle$ and it implies that $k, l \in H$.

2. If $H \leq x + H \leq (a + H) + (b + H)$ for $x, a, b \in G^+$, then elements $c, d \in H^+$ exist such that $0 \leq x + c \leq a + b + d$. Further, there exist $k, l \in G^+$ such that $x + c = k + l$, $k \in \langle a \rangle$, $l \in \langle b + d \rangle$. Thus $x + H = (k + H) + (l + H)$, $k + H, l + H \in G/H^+$ and $k + H \in \langle a + H \rangle$, $l + H \in \langle b + H \rangle$ hold.

**Proposition 2.6.**

1. If $G$ is a weak Riesz group, then $G$ has the following interpolation property: If $x_1, x_2, y_1, y_2 \in G, x_i \leq y_j$ $(i, j \in \{1, 2\})$, then elements $z_1, z_2 \in G$ exist such that $x_1 \leq z_1 \leq y_2$, $x_2 \leq z_2 \leq y_1$ and $z_1, z_2 \in (\langle y_1 - x_1 \rangle + \{x_1 \}) \cap (\langle y_2 - x_2 \rangle + \{y_2 \})$.

2. Let $G$ be a commutative po-group. Then $G$ is a weak Riesz group if and only if $G$ has the following property:

If $x_1, x_2, y_1, y_2 \in G^+$, $x_1 + x_2 = y_1 + y_2$, then elements $z_{ij} \in G^+$ exist such that $x_i = z_{i1} + z_{i2}$, $y_j = z_{1j} + z_{2j}$ and $z_{ij} \in \langle y_j \rangle$ for $i, j \in \{1, 2\}$.

**Proof.**

1. We have $0 \leq y_2 - x_1 = (y_2 - x_2) + (x_2 - x_1) \leq (y_2 - x_2) + (y_1 - x_1)$ and thus there exist elements $k, l \in G$ such that $k, l \geq 0$, $y_2 - x_1 = k + l$, $k \in \langle y_2 - x_2 \rangle$, $l \in \langle y_1 - x_1 \rangle$. For $z_1 = l + x_1 = -k + y_2$ there holds $y_2 \geq z_1 \geq x_1$, $z_1 \in \langle y_1 - x_1 \rangle + \{x_1 \}$, $z_1 \in \langle y_2 - x_2 \rangle + \{y_2 \}$. Similarly we can prove existence of an element $z_2$ of required properties.

2. $\implies$: We have $0 \leq x_1 \leq y_1 + y_2$ and thus there exist $z_{11}, z_{12} \in G^+$ such that $x_1 = z_{11} + z_{12}$, $z_{11} \in \langle y_1 \rangle$, $z_{12} \in \langle y_2 \rangle$. For $z_{2j} = -z_{1j} + y_j$ there holds $y_j = z_{1j} + z_{2j}$ $(j = 1, 2)$ and $x_1 + x_2 = y_1 + y_2 = z_{11} + z_{21} + z_{12} + z_{22} = x_1 + z_{21} + z_{22}$. It implies $x_2 = z_{21} + z_{22}$, where $z_{21} \in \langle y_1 \rangle$, $z_{22} \in \langle y_2 \rangle$.

$\impliedby$: If $x, a, b \in G^+$, $0 \leq x \leq a + b$, then we have $a + b = (a + b - x) + x$ and thus there exist $z_{21}, z_{22} \in G^+$ such that $x = x_2 = z_{21} + z_{22}$, $z_{21} \in \langle a \rangle$, $z_{22} \in \langle b \rangle$. 260
**Weak Riesz Groups**

**Definition.** Let $G$ be a directed po-group with the following property:

If $x, a, b \in G^+$, $0 \leq x \leq a + b$, then elements $k, l \in G$ exist such that $k, l \geq 0$, $x = k + l$, $k \leq a$, $l \in \langle b \rangle$. Then we say that $G$ is a semiweak Riesz group (an sw-Riesz group).

**Proposition 2.7.**

1. An sw-Riesz group $G$ has the interpolation property from Proposition 2.6 and $z_1 \geq x_2$.

2. If $G$ is an sw-Riesz group, then a meet of two directed convex subgroups in $G$ is again a directed convex subgroup in $G$.

**Proof.**

1. If we repeat the proof of Prop. 2.6, 1., then we receive that $G$ has the interpolation property and $k \leq y_2 - x_2$, i.e., $z_1 = -k + y_2 \geq x_2$.

2. If $A, B$ are directed convex subgroups in $G$, then $A \cap B$ is a convex subgroup in $G$. If $x \in A \cap B$, then $p \in A$, $q \in B$ exist such that $0, x \leq p, q$. There exist elements $z_1, z_2 \in ((q - x) + \langle x \rangle) \cap \langle p \rangle$ such that $x \leq z_1 \leq p$, $0 \leq z_2 \leq q$, $z_1 \geq 0$. Finally, we have $z_1, z_2 \in A \cap B$, $z_1 + z_2 \geq 0$, $z_1 + z_2 \geq z_1 \geq x$, $z_1 + z_2 \in A \cap B$. $A \cap B$ is a directed subgroup in $G$.

**3. Lattice characterization**

The lattice of all convex l-subgroups of a lattice-ordered group $G$ was investigated by M. Jakubíková [4]. This lattice is a complete distributive lattice which is a complete sublattice of the lattice of all subgroups of $G$. This result was generalized by J. Rachunková [7] for the case of Riesz groups. Let us investigate a similar situation for sw-Riesz groups.

**Theorem 3.1.** If $G$ is an sw-Riesz group, then the set $C(G)$ of all directed convex subgroups in $G$ is a locale.

**Remark.** Let us recall that a locale is a complete lattice $L$ in which the infinite distributive law $a \land \bigvee S = \bigvee \{a \land s : s \in S\}$ holds for all $a \in L$ and $S \subseteq L$. The important examples of locales are lattices of all open sets of topological spaces. All unexplained facts concerning locales can be found in Johnstone [5].

**Proof of 3.1.** Let $A_i \in C(G)$ be for $i \in I$ and let $\left[ \bigcup_{i \in I} A_i \right]$ denote a subgroup generated by $\bigcup_{i \in I} A_i$. Then each element $x \in \left[ \bigcup_{i \in I} A_i \right]$ has the form $x = \sum_{i \in K} a_i$ for suitable elements $a_i \in A_i$ and a finite subset $K \subseteq I$. If $g \in G$, 261
0 ≤ g ≤ x then 2.3 implies an existence of elements \( g_i \in G^+ \) such that \( g = \sum_{i \in K} g_i \) and \( g_i \in \left[ \bigcup_{i \in I} A_i \right] \) for \( i \in K \). \( \left[ \bigcup_{i \in I} A_i \right] \) is convex and let us prove that it is also directed. If \( x = \sum_{i \in K} a_i \), \( y = \sum_{i \in L} b_i \) are two elements from \( \left[ \sum_{i \in I} A_i \right] \), then from the directness of \( A_i \) it implies that there exist \( z_i \in A_i \), \( z_i ≥ a_i \), 0 for all \( i \in K \) and \( z_i ≥ b_i \), 0 for all \( i \in L \). We have \( x, y ≤ \sum_{i \in K∪L} z_i \in \left[ \sum_{i \in I} A_i \right] \). Joins of \( A_i \in C(G) \) are also subgroups generated by \( \bigcup A_i \) and finite meets are meets of sets (see 2.7).

Now, let us verify the corresponding distributive law: If \( A, B_i \in C(G) \) for \( i \in I \), then \( A ∩ \bigvee_{i \in I} B_i ≥ \bigvee_{i \in I} (A ∩ B_i) \) clearly. If \( a \in A ∩ \bigvee_{i \in I} B_i \), then there exists an element \( \tilde{a} \in A ∩ \bigvee_{i \in I} B_i \) such that \( \tilde{a} ≥ a \), \( O ≥ −\tilde{a} \). We have \( \tilde{a} = \sum_{i \in K} \tilde{b}_i \) for suitable elements \( \tilde{b}_i \in B_i^+ \) and \( i \in K \), \( K \subseteq I \) finite (see 2.3) and thus \( \tilde{a} \in A ∩ B_i \) for \( i \in K \). Finally, \( \tilde{a} \in \bigvee_{i \in I} (A ∩ B_i) \) and from the convexity \( a ∈ \bigvee_{i \in I} (A ∩ B_i) \) holds.

**COROLLARY 3.2.** If \( G \) is an \( s w \)-Riesz group, then the set \( I(G) \) of all \( o \)-ideals in \( G \) is a locale with respect to arbitrary sums and finite meets.

**Proof** follows from 3.1 and 2.4.

Recall that a locale \( L \) is regular when \( l = \bigvee_{x \in L} (x^* \vee l = 1) \) holds for each \( l \in L \), where \( x^* = \bigvee_{y \in L} (y ∩ x = 0) \).

**PROPOSITION 3.3.** Let \( G \) be an \( s w \)-Riesz group. Then a locale \( I(G) \) is regular if and only if each principal \( o \)-ideal in \( G \) is a direct summand in \( G \).

**Proof.**

\[ \langle g \rangle = \sum_{i \in I} \langle X_i : X_i^* + \langle g \rangle = G \text{ for } i \in I \rangle \] holds for each \( g ∈ G^+ \). Since \( g = \sum_{i \in K} x_i \) for suitable \( x_i \in X_i \) and finite set \( K ⊆ I \) there holds \( \langle g \rangle = \sum_{i \in K} X_i \).

Distributivity of \( I(G) \) implies \( G = \bigcap_{i \in K} (X_i^* + \langle g \rangle) = \bigcap_{i \in K} X_i^* + \langle g \rangle \) and \( \bigcap_{i \in K} X_i = \left( \sum_{i \in K} X_i \right)^* = \langle g \rangle^* \).

\[ \iff \text{ Clearly, } A = \sum_{a \in A^+} \langle (a) : (a)^+ + \langle a \rangle = G \text{ holds for each } A ∈ I(G) \text{ and } \bigvee_{X ∈ I(G)} (X^* \vee A = G) ⊆ A \text{ because } X = G \land X = (X^* \vee A) \land X = (X^* \land X) \vee (A \land X) = A \land X \text{. Finally, } I(G) \text{ is regular.} \]
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REFERENCES


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