Mathematica Slovaca

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Mathematica Slovaca, Vol. 42 (1992), No. 3, 279--297

Persistent URL: `http://dml.cz/dmlcz/136557`

Math. Slovaca, **42** (1992), No. 3, 279–297

Mathematica Slovaca
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Slovak Academy of Sciences

CONTINUOUS SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR ODEs ON UNBOUNDED INTERVALS

MÁRIA KEČKEMÉTYOVÁ

ABSTRACT. The existence of a continuous solution defined on non-compact interval for a system of nonlinear differential equations with linear boundary conditions (BP) is proved.

Introduction

The aim of this paper is to prove the existence of a continuous solution for the system

$$\begin{aligned} \dot{x}(t) - A(t)x(t) &= f\big(t, x(t)\big) \\ Tx &= r \end{aligned} \tag{BP}$$

on non-compact interval $\langle a;\infty)$. The existence of a bounded solution of this system defined on the right open interval $\langle a;b)$ $(-\infty < a < b \leq +\infty)$, for the Banach space of all bounded continuous functions, has been studied by M. Cecchi, M. Marini, P. L. Zezza [1]. This method, that we shall use is to transform the system (BP) into the form of the equation

$$Lx = Nx\,, \tag{OE}$$

where L is a linear operator, N is generally non-linear. The existence of a bounded continuous solution for (BP) follows from the theorems of P. L. Zezza about equivalence between the set of solutions for (OE) and the set of fixed points of operator M defined by (1.9) and the continuation theorem [7].

The case that L is a Fredholm operator is studied by J. Mawhin. By this method this system is reduced to the operator equation (OE) which is solved by

AMS Subject Classification (1991): Primary 34B15.
Key words: Boundary value problem, Fixed point.

the local degree theory of Leray-Schauder. For the applications of this method see J. Mawhin – R. Gaines [5].

In this paper we shall prove the existence of a continuous solution bounded by a certain un-bounded function which is determined by the solutions of the associated linear system

$$\dot{y}(t) - A(t)y(t) = 0 .$$

If the fundamental matrix of this linear system is bounded on $\langle a;\infty)$, then that problem is reduced to the problem which is studied by M. Cecchi, M. Marini, P. L. Zezza on the interval $\langle a;\infty)$.

1. Let $C = C\big(\langle a,\infty),\mathbb{R}^n\big)$ be a vector space of continuous functions from $\langle a,\infty)$ into $\mathbb{R}^n$, $\psi \in C(\langle a,\infty),\mathbb{R})$ is a positive function on $\langle a,\infty)$. The space

$$C_\psi = \left\{ x(t) \in C\colon \ \sup_{t\in\langle a,\infty)} \frac{\|x(t)\|}{\psi(t)} < +\infty \right\},$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^n$, is a Banach space with respect to the norm

$$\|x\|_\psi = \sup_{t\in\langle a,\infty)} \frac{\|x(t)\|}{\psi(t)} \qquad \text{for each} \quad x \in C_\psi .$$

In this paper we shall investigate the existence of a solution for the system

$$\dot{x}(t) - A(t)x(t) = f\big(t,x(t)\big) , \tag{1.1}$$

which satisfies the boundary conditions:

$$Tx = r \qquad r \in \mathbb{R}^m \quad (1 \le m \le n) , \tag{1.2}$$

where $A(t)$ is a $n\times n$ matrix, continuous on $\langle a,\infty)$.

Let D be a space of all continuous solutions of the linear system

$$\dot{y}(t) - A(t)y(t) = 0 . \tag{1.3}$$

Let $\lambda(t)$ be the smallest eigenvalue and $\Lambda(t)$ the largest eigenvalue of the hermitian symmetric matrix

$$A^H(t) = \frac{1}{2}\left[A(t) + A^*(t)\right] ,$$

where if $A(t) = \big(a_{ij}(t)\big)_{i,j=1}^n$, then $A^*(t) = \big(\overline{a_{ji}(t)}\big)_{i,j=1}^n$ is the hermitian adjoint matrix of $A(t)$. It means that: $\lambda(t)$, $\Lambda(t)$ are solutions of the equation

$$\det\left[A^H(t) - \lambda E\right] = 0 .$$

These conditions assure that the Wazewski inequality

$$\|x(a)\| \exp\left(\int_a^t \lambda(s)\,\mathrm{d}s\right) \le \|x(t)\| \le \|x(a)\| \exp\left(\int_a^t \Lambda(s)\,\mathrm{d}s\right) \tag{1.4}$$

holds for all solutions $x(t)$ of the system (1.3), [3]. Let

$$\psi(t) = \exp\left(\int_a^t \Lambda(s)\,\mathrm{d}s\right), \tag{1.5}$$

then $\psi(t) > 0$ for each $t \in \langle a, \infty)$ and $\psi(t) \in C\big(\langle a, \infty), \mathbb{R}\big)$. Consequently, the space $\big(C_\psi; \|\cdot\|_\psi\big)$ with the weight function ψ defined by (1.5) is a Banach space.

R e m a r k 1.1. If $\psi(t)$ is bounded on $\langle a, \infty)$, then C_ψ need not be equal to the space of all bounded continuous functions. The equality of both spaces will be attained if $\psi(t)$ satisfies $0 < k \le \psi(t) \le K$ on $\langle a; \infty)$ with some positive constants $k < K$. This case was solved in [1].

Further, let $T\colon\ \operatorname{dom} T \subset C_\psi \to \mathbb{R}^m$, ($1 \le m \le n$) be a linear continuous operator, it means that:

$$\|Tx\| \le \|T\| \cdot \|x\|_\psi \qquad \text{for each} \quad x \in \operatorname{dom} T\,. \tag{1.6}$$

Let us assume that T satisfies the condition

$$D \subset \operatorname{dom} T\,, \quad T(D) = \mathbb{R}^m\,. \tag{1.7}$$

R e m a r k 1.2. These conditions assure that the linear problem associated to (1.1)–(1.2) for $f(t,x) \equiv 0$ has a solution for each $r \in \mathbb{R}^m$.

Let

$$L\colon\ \operatorname{dom} L \subset C_\psi \to C \times \mathbb{R}^m$$

be the linear operator defined by the relation:

$$x(\cdot) \mapsto \big(\dot{x}(\cdot) - A(\cdot)x(\cdot);\ Tx\big)\,,$$

where $\operatorname{dom} L = C^1\big(\langle a, \infty), \mathbb{R}^n\big) \cap \operatorname{dom} T$ and let $f:\ \langle a, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function,

$$N\colon\ \operatorname{dom} N = C_\psi \to C \times \mathbb{R}^m$$

be the operator which is determined by the relation:

$$x(\cdot) \mapsto \big(f(\cdot\,, x(\cdot)); r\big)\,.$$

Then the system (1.1) – (1.2) is equivalent to the equation of the form

$$Lx = Nx\,. \tag{1.8}$$

Now we introduce some theorems to be used later.

THEOREM 1.1. ([1]; p. 270). *Let X, Y be linear spaces. Let L be a linear operator,*

$$L\colon\ \operatorname{dom} L \subset X \to Y\,,$$

let N be an operator, possibly nonlinear,

$$N\colon\ \operatorname{dom} N \subset X \to Y\,.$$

Then the equation (1.8) is equivalent to

$$x = Mx \qquad x \in A\,, \tag{1.9}$$

where

$$A = \{x \in X\colon\ Nx \in \operatorname{Im} L\} = N^{-1}(\operatorname{Im} L) \neq \emptyset\,,$$
$$M\,:\ x \mapsto Px + K_P Nx\,,$$

$P\colon X \to \ker L$ is a projection onto $\ker L$, $X_{I-P} = \operatorname{Im}(I-P)$ and $K_P = \Big(L|_{\operatorname{dom} L\,\cap\, X_{I-P}}\Big)^{-1}$.

If $A = \emptyset$, then the problems (1.8) and (1.9) have no solution.

THEOREM 1.2. ([1]; p. 271). *Suppose that: X is a Banach space, $\dim(\ker L)$ is finite, the operator M is completely continuous. If Ω is an open, bounded neighbourhood of $0 \in X$, $\overline{\Omega} \subset \operatorname{dom} M$, such that*

$$x \in \partial\Omega\,, \qquad \lambda \in (0,1) \implies Lx \neq \lambda Nx$$

or (1.10)

$$x \in \partial\Omega\,, \qquad \lambda \in (0,1) \implies x \neq \lambda K_P Nx\,,$$

then the operator M has at least one fixed point in $\overline{\Omega}$.

The theorems 1.1, 1.2 imply that the equation (1.8) has at least one solution in $\overline{\Omega}$.

2. In this section we shall prove some existence theorems for the continuous solutions of the system (1.1)–(1.2) in C_ψ. First, we shall express the operator M.

Let $k = \dim(\ker L) = n - m$ ($k \neq 0$ if $m < n$). Let $\varphi_1;\dots;\varphi_k$ be a basis of $\ker L$. Let us extend it to obtain a basis of D:

$$\varphi_1;\dots;\varphi_k;\varphi_{k+1};\dots;\varphi_n \qquad \varphi_i \in C_\psi .$$

Letting $X(t) = \big(\varphi_1(t);\dots;\varphi_n(t)\big)$ we get a fundamental matrix for the equation (1.3). Since the inequality (1.4) holds for each solution of the system (1.3), there exists $H > 0$ such that

$$\sup_{t\in\langle a,\infty)} \frac{\|X(t)\|}{\psi(t)} \le H , \tag{2.1}$$

where $\|\cdot\|$ is a matrix norm which is compatible with a vector norm [2].

Under the hypotheses of section 1 there exists a topological projection $P\colon C_\psi \to \ker L \subset D$. Then it is possible to express the space C_ψ as a topological direct sum

$$C_\psi = \ker L \oplus (C_\psi)_{I-P} ,$$

where $I\colon C_\psi \to C_\psi$ is the identity mapping, $\ker L = \operatorname{Im} P = (C_\psi)_P$ and $(C_\psi)_{I-P} = \ker P$.

If we denote by J the immersion of $\mathbb{R}^m$ into $\mathbb{R}^n$

$$J(r_1;\dots;r_m) = (0;\dots;0;r_1;\dots;r_m), \qquad r = (r_1;\dots;r_m) \in \mathbb{R}^m$$

and

$$T_0 = (T\varphi_{k+1};\dots;T\varphi_n),$$

then the operator

$$K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap (C_\psi)_{I-P}, \qquad K_P = \Big(L|_{\operatorname{dom} L \cap (C_\psi)_{I-P}}\Big)^{-1}$$

is defined by the relation

$$K_P\colon \big(b(t),r\big) \mapsto X(t)JT_0^{-1}\left(r - T\left(\int_a^t X(t)X^{-1}(s)b(s)ds\right)\right) + \int_a^t X(t)X^{-1}(s)b(s)\,\mathrm{d}s \qquad \big(b(t),r\big) \in \operatorname{Im} L . \tag{2.2}$$

Remark 2.1. ([1]; p. 274) The operator K_P defined in (2.2) depends on P, because the choice of the fundamental matrix $X(t)$ is related to the form of P. If $m = n$, the matrix $TX(t)$ is invertible, hence:

$$K_P(b(t), r) = X(t)\big(TX(t)\big)^{-1}\left(r - T\left(\int_a^t X(t)X^{-1}(s)b(s)\,\mathrm{d}s\right)\right) + \int_a^t X(t)X^{-1}(s)b(s)\,\mathrm{d}s\,. \tag{2.3}$$

Let, in addition to the hypotheses of section 1, the following hold: there are two functions $p(t),\ q(t) \in C\big(\langle a,\infty),\mathbb{R}\big)$, non-negative integrable on $\langle a,\infty)$ such that

(i) $$\int_a^\infty p(t)\,\mathrm{d}t = \Gamma < +\infty\,, \qquad \int_a^\infty q(t)\,\mathrm{d}t = \Lambda < +\infty\,,$$

(ii) $$\psi(t)\|X^{-1}(t)f(t,u)\| \le p(t)\|u\| + q(t)\psi(t)$$

for each $t \in \langle a,\infty)$ and for each $u \in \mathbb{R}^n$.

Remark 2.2. ([1], p. 275) With respect to (2.2), the operator M is defined on the set:

$$A = \left\{\, g \in C_\psi\colon\ \int_a^t X(t)X^{-1}(s)f\big(s,g(s)\big)\,\mathrm{d}s \in \operatorname{dom} T \right\}.$$

LEMMA 2.2. *Under the hypotheses if* $\operatorname{dom} T = C_\psi$, *then the operator* M *is defined on* C_ψ *and is continuous.*

Proof. From definitions of the operators L and N, we have: if $g \in C_\psi$, then $Ng = \big(f(\cdot,g(\cdot)),r\big) \in \operatorname{Im} L$ if and only if there exists a solution $x \in \operatorname{dom}\,T$ of the system

$$\begin{aligned} &\text{(a)} \qquad \dot{x}(t) - A(t)x(t) = f\big(t,g(t)\big)\\ &\text{(b)} \qquad Tx = r\,. \end{aligned} \tag{2.4}$$

Let $g \in C_\psi$, we shall prove that there exists $x(t)$ satisfying (2.4). Let $x(t)$ be a solution of (2.4)(a),

$$x(t) = y(t) + \int_a^t X(t)X^{-1}(s)f\big(s,g(s)\big)\,\mathrm{d}s \qquad a \le t < +\infty\,,$$

where $y(t)$ is a solution of (1.3) such that $y(a) = x(a)$.
Since $y \in \operatorname{dom} T$, $x \in \operatorname{dom} T$ if and only if

$$\int_a^t X(t)X^{-1}(s)f\big(s, g(s)\big)\,\mathrm{d}s \in \operatorname{dom} T = C_\psi\,. \tag{2.5}$$

Using (i), (ii) we obtain:

$$\begin{aligned}\left\|\int_a^t X(t)X^{-1}(s)f(s,g(s))\,\mathrm{d}s\right\| &\le \|X(t)\| \int_a^t \big\|X^{-1}(s)f\big(s,g(s)\big)\big\|\,\mathrm{d}s \\ &\le \|X(t)\| \left(\int_a^t p(s)\frac{\|g(s)\|}{\psi(s)}\,\mathrm{d}s + \int_a^t q(s)\,\mathrm{d}s \right),\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\sup_{t\in\langle a,\infty)} \frac{1}{\psi(t)}\left\|\int_a^t X(t)X(s)^{-1}f\big(s,g(s)\big)\,\mathrm{d}s\right\| \\ &\le \sup_{t\in\langle a,\infty)} \frac{\|X(t)\|}{\psi(t)} \left(\int_a^t p(s)\frac{\|g(s)\|}{\psi(s)}\,\mathrm{d}s + \int_a^t q(s)\,\mathrm{d}s \right) \le H\left(\Gamma\|g\|_\psi + \Lambda\right).\end{aligned} \tag{2.7}$$

The last inequality implies (2.5). Let

$$T\left(\int_a^t X(t)X^{-1}(s)f\big(s,g(s)\big)\,\mathrm{d}s \right) = r_0\,,$$

then it is always possible to choose $y \in D$ such that $Ty = r - r_0$ and so $Tx = r$.
Therefore for each $g \in C_\psi$, $Ng \in \operatorname{Im} L$, $A = \operatorname{dom} M = C_\psi$.

Now we shall prove the continuity of $M = P + K_P N$. Since P is a continuous projection, it is sufficient to prove the continuity of $K_P N$. Let $\{x_j\}_{j=1}^\infty$ be a sequence of functions from C_ψ such that it is converging to x in C_ψ. Let us prove that $\{K_P N x_j\}_{j=1}^\infty$ converges to $K_P N x$ in C_ψ. According to (2.2) it suffices to show that

$$X(t)\int_a^t X^{-1}(s)\big[f\big(s,x_j(s)\big) - f\big(s,x(s)\big)\big]\,\mathrm{d}s \qquad j \in \mathbb{N} \tag{2.8}$$

converges to 0 in C_ψ. Since the function f is continuous on $\langle a,\infty) \times \mathbb{R}^n$, the sequence

$$X^{-1}(t)\big[f\big(t,x_j(t)\big) - f\big(t,x(t)\big])\big] \xrightarrow{\text{pointwise}} 0 \qquad \text{as} \quad j \to \infty\,, \tag{2.9}$$

$X(t)$ is bounded in C_ψ and there holds:

$$\begin{aligned}\big\|X^{-1}(t)\big[f\big(t,x_j(t)\big) - f\big(t,x(t)\big)\big]\big\| &\le \big\|X^{-1}(t)f\big(t,x_j(t)\big)\big\| + \big\|X^{-1}(t)f\big(t,x(t)\big)\big\| \\ &\le p(t)\left(\frac{\|x_j(t)\|}{\psi(t)} + \frac{\|x(t)\|}{\psi(t)}\right) + 2q(t) \le p(t)\left(2\frac{\|x(t)\|}{\psi(t)} + \varepsilon\right) + 2q(t) \\ &\le p(t)\big(2\|x\|_\psi + \varepsilon\big) + 2q(t) \qquad \text{for each} \quad j \ge j_\varepsilon\,. \end{aligned} \tag{2.10}$$

Hence the sequence (2.8) converges pointwise to 0 by the Lebesgue dominated convergence theorem.

Now let us prove the convergence of (2.8) in C_ψ. We shall use the following assertion, [4]:

Let the following conditions hold:

(a) *the sequence* $\{f_n(t)\}_{n\in\mathbb{N}}$ *converges pointwise to* 0 *on* $\langle a,\infty)$ *as* $n \to \infty$,

(b) *there exists* $\lim\limits_{t\to\infty} f_n(t) = f_n$ *for each* $n \in \mathbb{N}$,

(c) $\lim\limits_{n\to\infty} f_n = 0$,

(d) $\{f_n(t)\}_{n\in\mathbb{N}}$ *is equicontinuous on each compact interval of* $\langle a,\infty)$,

(e) $\forall\varepsilon > 0 \quad \exists K(\varepsilon) > 0$ *such that for* $\forall t \ge K(\varepsilon) \quad \forall n \in \mathbb{N}$:

$$\|f_n(t) - f_n\| < \varepsilon\,,$$

then $\{f_n(t)\}$ *uniformly converges to* 0 *on* $\langle a,\infty)$.

Since $\sup\limits_{t\in\langle a,\infty)} \dfrac{\|X(t)\|}{\psi(t)} \le H$, it is sufficient to verify (b), (c), (d), (e) for

$$\left\{\int_a^t X^{-1}(s)\big[f\big(s,x_j(s)\big) - f\big(s,x(s)\big)\big]\,\mathrm{d}s\right\}_{j=1}^{\infty}.$$

Let j be an arbitrary but fixed natural number, by (2.10) the integral

$$\int_a^\infty X^{-1}(s)\big[f\big(s,x_j(s)\big) - f\big(s,x(s)\big)\big]\,\mathrm{d}s \tag{2.11}$$

is absolutely convergent, therefore the condition (b) is satisfied.

Condition (c) follows from (2.9), (2.10) by the Lebesgue dominated convergence theorem.

To prove (d) let $t_1, t_2 \in \langle a, \infty)$; $t_1 < t_2$, then it holds:

$$\left\| \int_{t_1}^{t_2} X^{-1}(s)\big[f\big(s, x_j(s)\big) - f\big(s, x(s)\big)\big]\,\mathrm{d}s \right\|$$

$$\leq \int_{t_1}^{t_2} p(s)\left(\frac{\|x_j(s)\|}{\psi(s)} + \frac{\|x(s)\|}{\psi(s)}\right)\mathrm{d}s + \int_{t_1}^{t_2} 2q(s)\,\mathrm{d}s \leq 2\int_{t_1}^{t_2} \big(\alpha p(s) + q(s)\big)\,\mathrm{d}s$$

since $\{x_j(t)\}_{j=1}^{\infty}$ converges in C_ψ, it is uniformly bounded on $\langle a, \infty)$, i. e. $\exists \alpha > 0$ such that:

$$\forall t \in \langle a, \infty) \quad \forall j \in \mathbb{N}\colon \ \frac{\|x_j(t)\|}{\psi(t)} \leq \alpha\,.$$

Now let us verify (e):

$$\left\| \int_{t}^{\infty} X^{-1}(s)\big[f\big(s, x_j(s)\big) - f\big(s, x(s)\big)\big]\,\mathrm{d}s \right\| \leq 2\int_{t}^{\infty} \big(\alpha p(s) + q(s)\big)\,\mathrm{d}s\,.$$

By the preceding assertion (2.8) converges in C_ψ.

LEMMA 2.3. *Under the preceding hypotheses, the operator*

$$M\colon \ \operatorname{dom} M = C_\psi \to C_\psi$$

transforms bounded sets into sets which are bounded in C_ψ and equicontinuous on each compact interval of $\langle a, \infty)$.

Proof. Since P is a linear continuous operator and $\dim(\operatorname{Im} P) < +\infty$ (hence P is compact), it is sufficient to prove the statement for the operator $K_P N$.

Let Ω be a bounded set in C_ψ, i. e. there exists $\mu > 0$ such that:

$$\text{if} \quad x \in \Omega, \quad \text{then} \qquad \|x\|_\psi \leq \mu\,. \tag{2.12}$$

Let $\tau \in \langle a;\infty)$ be an arbitrary but fixed number. Then we have:

$$
\begin{aligned}
\|K_p N x(\tau)\| \leq & \left\| X(\tau) J T_0^{-1}\left(r - T\int_a^t X(t)X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s\right)\right\| \\
& + \left\| X(\tau)\int_a^\tau X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s\right\| \\
\leq & \|X(\tau)\| \|JT_0^{-1}\| \left[\|r\| + \left\| T\int_a^t X(t)X^{-1}(s) f\big(s,x(s)\big)\,\mathrm{d}s\right\|\right] \\
& + \|X(\tau)\| \int_a^\tau \left\| X^{-1}(s) f\big(s,x(s)\big)\right\| \mathrm{d}s\,.
\end{aligned}
\tag{2.13}
$$

On the basis of the last result we get:

$$
\begin{aligned}
\|K_P N x\|_\psi = & \sup_{\tau\in\langle a,\infty)} \frac{\|K_P N x(\tau)\|}{\psi(\tau)} \\
\leq & \sup_{\tau\in\langle a,\infty)} \frac{\|X(\tau)\|}{\psi(\tau)} \|JT_0^{-1}\| \left(\|r\| + \|T\| \left\|\int_a^t X(t)X^{-1}(s) f\big(s,x(s)\big)\,\mathrm{d}s\right\|_\psi\right) \\
& + \sup_{\tau\in\langle a,\infty)} \frac{\|X(\tau)\|}{\psi(\tau)} \int_a^\tau \left\|X^{-1}(s) f\big(s,x(s)\big)\right\| \mathrm{d}s \\
\leq & H\|JT_0^{-1}\| \left[\|r\| + \|T\| H\big(\Gamma\|x\|_\psi + \Lambda\big)\right] + H\big(\Gamma\|x\|_\psi + \Lambda\big) \\
\leq & H\|JT_0^{-1}\| \left[\|r\| + \|T\| H(\Gamma\mu + \Lambda)\right] + H(\Gamma\mu + \Lambda) = \nu
\end{aligned}
\tag{2.14}
$$

for each $x \in \Omega$. Therefore $M(\Omega)$ is bounded in C_ψ. It remains to prove the equicontinuity of $M(\Omega)$ in C_ψ.

Let $t_1, t_2 \in \langle a,\infty)$; $t_1 < t_2$. Putting

$$
\delta(t,x) = \int_a^t X^{-1}(s) f\big(s,x(s)\big)\,\mathrm{d}s\,, \qquad a \leq t < +\infty\,,
$$

$$
V = JT_0^{-1}\big(r - TX(t)\delta(t,x)\big)\,,
$$

using (1.6), (2.6), (2.7), (2.13), (2.14) we obtain:

$$\left\|\frac{K_P N x(t_2)}{\psi(t_2)} - \frac{K_P N x(t_1)}{\psi(t_1)}\right\|$$

$$= \left\|\frac{X(t_2)}{\psi(t_2)}V + \frac{X(t_2)}{\psi(t_2)}\delta(t_2, x) - \frac{X(t_1)}{\psi(t_1)}V - \frac{X(t_1)}{\psi(t_1)}\delta(t_1, x)\right\|$$

$$\leq \left\|\frac{X(t_2)}{\psi(t_2)} - \frac{X(t_1)}{\psi(t_1)}\right\| (\|V\| + \|\delta(t_2, x)\|) + \left\|\frac{X(t_1)}{\psi(t_1)}\right\| \cdot \left\|\int_{t_1}^{t_2} X^{-1}(s) f(s, x(s))\, \mathrm{d}s\right\|$$

$$\leq \left\|\frac{X(t_2)}{\psi(t_2)} - \frac{X(t_1)}{\psi(t_1)}\right\| \left[\|JT_0^{-1}\| \left(\|r\| + \left\|T \int_a^t X(t) X^{-1}(s) f(s, x(s))\, \mathrm{d}s\right\|\right)\right.$$

$$\left. + \left\|\int_a^{t_2} X^{-1}(s) f(s, x(s))\, \mathrm{d}s\right\|\right] + \left\|\frac{X(t_1)}{\psi(t_1)}\right\| \int_{t_1}^{t_2} \|X^{-1}(s) f(s, x(s))\|\, \mathrm{d}s$$

$$\leq \left\|\frac{X(t_2)}{\psi(t_2)} - \frac{X(t_1)}{\psi(t_1)}\right\| \left\{\|JT_0^{-1}\| \left[\|r\| + \|T\| H(\Gamma \|x\|_\psi + \Lambda)\right] + (\Gamma \|x\|_\psi + \Lambda)\right\}$$

$$+ H\left(\|x\|_\psi \int_{t_1}^{t_2} p(t)\, \mathrm{d}t + \int_{t_1}^{t_2} q(t)\, \mathrm{d}t\right)$$

$$\leq \left\|\frac{X(t_2)}{\psi(t_2)} - \frac{X(t_1)}{\psi(t_1)}\right\| \left\{\|JT_0^{-1}\| \left[\|r\| + \|T\|\, H(\Gamma\mu + \Lambda)\right] + (\Gamma\mu + \Lambda)\right\}$$

$$+ H\left(\mu \int_{t_1}^{t_2} p(t)\, \mathrm{d}t + \int_{t_1}^{t_2} q(t)\, \mathrm{d}t\right).$$

The preceding inequality finishes the proof.

First, we are going to state some existence theorems for (1.1)-(1.2) in a special case. Let

$$C_{\psi,l} = \left\{x \in C_\psi\colon \lim_{t\to\infty} \frac{x(t)}{\psi(t)} = l_x \quad \|l_x\| < +\infty\right\},$$

$C_{\psi,l} \subset C_\psi$. We shall use the following lemma:

LEMMA 2.4. *Suppose that, for the system* (1.1)-(1.2) *the following hypotheses hold:*

(2.15) $A(t)$ *is a real valued* $n \times n$ *matrix, defined and continuous on* $\langle a, \infty)$,

$X(t)$ is a fundamental matrix of (1.3) and $H > 0$ such that:

$$\sup_{t\in\langle a,\infty)} \frac{\|X(t)\|}{\psi(t)} \le H\,,$$

(2.16) $$\lim_{t\to\infty} \frac{X(t)}{\psi(t)} = W\,, \qquad \textit{i.e.} \quad D \subset C_{\psi,l}\,,$$

(2.17) *$f \in C\big(\langle a,\infty) \times \mathbb{R}^n, \mathbb{R}^n\big)$ such that*

$$\psi(t)\,\big\|X^{-1}(t)f(t,u)\big\| \le p(t)\|u\| + q(t)$$

for each $t \in \langle a,\infty)$, $u \in \mathbb{R}^n$, where $p(t)$, $q(t) \in C\big(\langle a,\infty), \mathbb{R}\big)$ are non-negative, integrable functions such that:

$$\int_a^\infty p(t)\,\mathrm{d}t = \Gamma < +\infty\,; \qquad \int_a^\infty q(t)\,\mathrm{d}t = \Lambda < +\infty\,,$$

(2.18) *T is a bounded linear operator, $T\colon \operatorname{dom} T = C_{\psi,l} \to \mathbb{R}^m$ and the matrix $TX(t)$ has rank m.*

Then the operator M is defined on C_ψ, its range is contained in $C_{\psi,l}$ and it is completely continuous.

P r o o f. From the proof of Lemma 2.1 we have:

$$\int_a^t X(t)X^{-1}(s)f\big(s,x(s)\big)\,\mathrm{d}s \in C_\psi \qquad \forall x \in C_\psi$$

and from (2.16) and (2.17) this integral is absolutely convergent on $\langle a,\infty)$, it means that

$$\int_a^t X(t)X^{-1}(s)f(s,x(s))\,\mathrm{d}s \in C_{\psi,l} = \operatorname{dom} T\,; \qquad A = \operatorname{dom} M = C_\psi\,.$$

Since $\operatorname{Im} P = \ker L \subset D \subset C_{\psi,l}$, $\operatorname{Im} M \subset C_{\psi,l}$. Projection P is completely continuous, from Lemma 2.2 the continuity of M follows, therefore it suffices to prove that the operator $K_P N$ transforms bounded sets into relatively compact sets. Recall that $\Omega \subset C_{\psi,l}$ is relatively compact if and only if it is:

(1) bounded
(2) equicontinuous
(3) uniformly convergent, in the following sense:

$$\forall \varepsilon > 0 \quad \exists K > 0 \quad \text{such that} \quad \forall t > K \quad \forall g \in \Omega\colon \left\| \frac{g(t)}{\psi(t)} - l_g \right\| < \varepsilon\,.$$

The equicontinuity and the boundedness of $K_P N(\Omega)$ in C_ψ have been already proved in Lemma 2.3. Now let us prove the uniform convergence.

Let $\Omega \subset \operatorname{dom} M$ be bounded, i.e., if $x \in \Omega$ then $\|x\|_\psi \le \mu$.

(2.13), (2.14), (2.16) imply:

$$\begin{aligned}
&\left\| \frac{K_P N x(t)}{\psi(t)} - \lim_{t\to\infty} \frac{K_P N x(t)}{\psi(t)} \right\| \\
&\le \left\| \frac{X(t)}{\psi(t)} J T_0^{-1} \left(r - T \int_a^t X(t) X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s \right) \right. \\
&\qquad \left. - \lim_{t\to\infty} \frac{X(t)}{\psi(t)} J T_0^{-1} \left(r - T \int_a^t X(t) X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s \right) \right\| \\
&\qquad + \left\| \frac{X(t)}{\psi(t)} \int_a^t X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s - \lim_{t\to\infty} \frac{X(t)}{\psi(t)} \int_a^t X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s \right\| \\
&\le \left\| \frac{X(t)}{\psi(t)} - W \right\| \left\{ \|J T_0^{-1}\| \big[\|r\| + H \|T\| (\mu\Gamma + \Lambda) \big] \right\} \\
&\qquad + \left\| \frac{X(t)}{\psi(t)} \int_a^t X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s - W \int_a^\infty X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s \right\| .
\end{aligned}$$

But there holds:

$$\begin{aligned}
&\left\| \frac{X(t)}{\psi(t)} \int_a^t X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s - W \int_a^\infty X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s \right\| \\
&\le \left\| \frac{X(t)}{\psi(t)} - W \right\| \int_a^\infty \big\| X^{-1}(s) f\big(s, x(s)\big) \big\|\,\mathrm{d}s + \left\| \frac{X(t)}{\psi(t)} \right\| \int_t^\infty \big\| X^{-1}(s) f\big(s, x(s)\big) \big\|\,\mathrm{d}s \\
&\le \left\| \frac{X(t)}{\psi(t)} - W \right\| (\Gamma\mu + \Lambda) + H \left(\mu \int_t^\infty p(s)\,\mathrm{d}s + \int_t^\infty q(s)\,\mathrm{d}s \right), \qquad (2.19)
\end{aligned}$$

hence

$$\left\| \frac{K_P N x(t)}{\psi(t)} - \lim_{t\to\infty} \frac{K_P N x(t)}{\psi(t)} \right\| \le \left\| \frac{X(t)}{\psi(t)} - W \right\| \left\{ \|JT_0^{-1}\| \left[\|r\| + H\|T\|(\mu\Gamma + \Lambda) \right] + \Gamma\mu + \Lambda \right\} + H\left(\mu \int_t^\infty p(s)\,\mathrm{d}s + \int_t^\infty q(s)\,\mathrm{d}s \right),$$

from which the validity of (3) follows. $M:\ \operatorname{dom} M = C_\psi \to C_{\psi,l}$ is completely continuous.

THEOREM 2.1. *If the system* (1.1)-(1.2) *satisfies conditions* (2.15), (2.16), (2.17), (2.18) *and*

$$H^2\|JT_0^{-1}\| \cdot \|T\|\Gamma \exp(H\Gamma) < 1\,, \tag{2.20}$$

then the operator M *has at least one fixed point in* $C_{\psi,l}$.

P r o o f. The complete continuity of the operator $M\colon C_{\psi,l} \to C_{\psi,l}$ follows from Lemma 2.4. According to Theorem 1.2 it is sufficient to show that there exists an open, bounded neighbourhood $\Omega \subset C_{\psi,l}$ of 0 such that

$$x \ne \lambda K_P N x \qquad \forall x \in \partial\Omega\,, \quad \lambda \in (0,1)\,. \tag{2.21}$$

Let $\Omega = \{x \in C_{\psi,l}\colon\ \|x\|_\psi < \varrho\}$, ϱ will be specified later. Let $x_\lambda = \lambda K_P N x_\lambda$ for any $\lambda \in (0,1)$, then for each $t \in \langle a,\infty)$ we have:

$$\|x_\lambda(t)\| < \frac{1}{\lambda}\|x_\lambda(t)\| = \|K_P N x_\lambda(t)\| \le \|X(t)\| \cdot \|JT_0^{-1}\| \left(\|r\| + \left\| T \int_a^t X(t)X^{-1}(s) f\big(s, x_\lambda(s)\big)\,\mathrm{d}s \right\| \right) + \left\| \int_a^t X(t)X^{-1}(s) f\big(s, x_\lambda(s)\big)\,\mathrm{d}s \right\|.$$

From (2.6), (2.7), (2.14) there follows:

$$\begin{aligned}\frac{\|x_\lambda(t)\|}{\psi(t)} &< \frac{\|K_P N x_\lambda(t)\|}{\psi(t)} \\ &\leq \frac{\|X(t)\|}{\psi(t)} \|JT_0^{-1}\| \left[\|r\| + \|T\| H(\Gamma \|x_\lambda\|_\psi + \Lambda)\right] \\ &\quad + \frac{\|X(t)\|}{\psi(t)} \left(\int_a^t p(s) \frac{\|x_\lambda(s)\|}{\psi(s)} \, \mathrm{d}s + \int_a^t q(s) \, \mathrm{d}s \right) \\ &\leq H \left[\|JT_0^{-1}\| (\|r\| + \|T\| H \Lambda) + \Lambda \right] \\ &\quad + H^2 \|JT_0^{-1}\| \cdot \|T\| \Gamma \cdot \|x_\lambda\|_\psi + H \int_a^t p(s) \frac{\|x_\lambda(s)\|}{\psi(s)} \, \mathrm{d}s\end{aligned}$$

and applying Gronwall's lemma

$$\frac{\|x_\lambda(t)\|}{\psi(t)} \leq \left\{ H \|JT_0^{-1}\| \left[\|r\| + \|T\| H(\Gamma \|x_\lambda\|_\psi + \Lambda)\right] + + H\Lambda \right\} \exp(H\Gamma),$$

$$\begin{aligned}\|x_\lambda\|_\psi \leq H^2 \|JT_0^{-1}\| \cdot \|T\| \Gamma \cdot \|x_\lambda\|_\psi \exp(H\Gamma) \\ + H \left[\|JT_0^{-1}\| (\|r\| + \|T\| H \Lambda) + \Lambda \right] \exp(H\Gamma),\end{aligned}$$

$$\begin{aligned}\left[1 - H^2 \|JT_0^{-1}\| \cdot \|T\| \Gamma \exp(H\Gamma)\right] \|x_\lambda\|_\psi \\ \leq H \left[\|JT_0^{-1}\| (\|r\| + \|T\| H \Lambda) + \Lambda \right] \exp(H\Gamma).\end{aligned}$$

By (2.20) $\left[1 - H^2 \|JT_0^{-1}\| \cdot \|T\| \Gamma \exp(H\Gamma)\right] > 0$. If we choose ϱ sufficiently large,

$$\varrho > \frac{H \left[\|JT_0^{-1}\| (\|r\| + \|T\| H \Lambda) + \Lambda \right] \exp(H\Gamma)}{\left[1 - H^2 \|JT_0^{-1}\| \|T\| \Gamma \exp(H\Gamma)\right]}, \tag{2.22}$$

then (2.21) is satisfied and from Theorem 1.2 there exists at least one fixed point x of the operator M in $\overline{\Omega}$, i.e. $x = Mx$ and $\|x\|_\psi \leq \varrho$.

THEOREM 2.2. *If the conditions* (2.15), (2.16), (2.17), (2.18) *are valid and if*

$$H^2 \|JT_0^{-1}\| \|T\| \Gamma + H\Gamma < 1, \tag{2.23}$$

then the operator M has at least one fixed point in $C_{\psi,l}$.

Proof. Similarly as in the proof of Theorem 2.1, let

$$\Omega = \{x \in C_{\psi,l} \ : \ \|x\|_\psi < \varrho\},$$

let $x_\lambda = \lambda K_P N x_\lambda$ for any $\lambda \in (0,1)$, then using (2.14) we obtain:

$$\|x_\lambda\|_\psi = \|\lambda K_P N x_\lambda\|_\psi < \|K_P N x_\lambda\|_\psi$$
$$\le H\|JT_0^{-1}\|\big[\|r\| + H\|T\|(\Gamma\|x_\lambda\|_\psi + \Lambda)\big] + H(\Gamma\|x_\lambda\|_\psi + \Lambda),$$
$$\big[1 - (H^2\|JT_0^{-1}\| \cdot \|T\|\Gamma + H\Gamma)\big] \cdot \|x_\lambda\|_\psi \le H\big[\|JT_0^{-1}\|(\|r\| + H\|T\|\Lambda) + \Lambda\big]$$

and by (2.23):

$$\|x_\lambda\|_\psi \le \frac{H\big[\|JT_0^{-1}\|(\|r\| + H\|T\|\Lambda) + \Lambda\big]}{\big[1 - (H^2\|JT_0^{-1}\|\|T\|\Gamma + H\Gamma)\big]}.$$

If we choose ϱ sufficiently large, then (2.21) is satisfied and the theorem is proved.

We can now consider a more general case: the existence of solutions for the system (1.1) - (1.2) in C_ψ (omitting the hypothesis (2.16)). Let us suppose that $\operatorname{dom} T = C_\psi$ and that $\psi(t) \ge k$ on $\langle a;\infty)$ with a constant $k > 0$. If the function (1.5) does not fulfil this hypothesis, then we consider $\psi_1(t) = \max\limits_{t\in\langle a;\infty)} (\psi(t), k)$, k is some real number and we again write $\psi(t)$ instead of $\psi_1(t)$.

By Lemma 2.2 M maps C_ψ into C_ψ.

The existence of a fixed point for the operator M shall be proved using Theorem 2.1, or Theorem 2.2 and a diagonal process.

Let $\{a_i\}_{i\in N}$ be an increasing sequence of real numbers such that $a_1 = a$, $\lim\limits_{i\to\infty} a_i = \infty$. Let $I_i = \langle a, a_i\rangle$ and

$$C_\psi(I_i,\mathbb{R}^n) = \left\{g(t) \in C(I_i,\mathbb{R}^n)\colon \ \sup_{t\in I_i} \frac{\|g(t)\|}{\psi(t)} < \infty\right\}.$$

($C_\psi(I_i,\mathbb{R}^n)$ is isomorphic to $C(I_i,\mathbb{R}^n)$). Let $g(t) \in C_\psi(I_i,\mathbb{R}^n)$, $\overline{g}(t)$ is the following extension of $g(t)$:

$$\overline{g}(t) = \begin{cases} g(t) & \text{for } t \in I_i \\ g(a_i) & \text{for } t \in \langle a_i, \infty). \end{cases}$$

Let us denote by E_i the set of all such $\overline{g}(t)$.

E_i is a Banach space with respect to the norm

$$\|\overline{g}\| = \sup_{t\in\langle a,\infty)} \frac{\|\overline{g}(t)\|}{\psi(t)},$$

moreover E_i is isomorphic to $C_\psi(I_i,\mathbb{R}^n)$. The following lemma holds:

LEMMA 2.5. *Let the system* (1.1)-(1.2) *satisfy the conditions* (2.15), (2.17) *and*

(2.24) T *is a bounded, linear operator from* $\operatorname{dom} T = C_\psi$ *onto* $\mathbb{R}^m$ *and the matrix* $TX(t)$ *has rank* m.

If, moreover, the condition (2.20) *is satisfied, then the operator*

$$M_i\colon \ \operatorname{dom} M_i \subseteq E_i \to E_i$$

defined by

$$M_i\colon \ \overline{g}(t) \to \overline{x}(t),$$

where $x(t) = (M\overline{g})(t)$, $t \in I_i$ $g \in C_\psi(I_i, \mathbb{R}^n)$, $x \in C_\psi(\langle a;\infty);\mathbb{R}^n)$, *has at least one fixed point in* E_i.

Proof. The complete continuity of the operator M_i can be shown in a similar way as it was done for M in Lemma 2.4. If we consider the bounded neighbourhood $\Omega_i \subset E_i$ of 0, $\Omega_i = E_i \cap \Omega$, then from Theorem 2.1. there exists at least one fixed point of the operator M_i in $\overline{\Omega}_i$.

We can show that a solution of the system (1.1)-(1.2) exists in C_ψ.

THEOREM 2.3. *If the conditions* (2.15), (2.17), (2.20), (2.24) *are satisfied, then the system* (1.1)-(1.2) *has at least one solution in* C_ψ.

Proof. Using Lemma 2.5 we obtain a sequence $\{x_i\}_{i\in N}$, $\overline{x}_i \in E_i$ such that $\overline{x}_i = M_i\overline{x}_i$. From the definition of M_i it follows:

$$x_i(t) = (M_i\overline{x}_i)(t) = M\overline{x}_i(t) \qquad t \in I_i. \tag{2.25}$$

The sequence $\{x_i\}_{i\in\mathrm{N}}$ is uniformly bounded and locally equicontinuous in $C_\psi(I_1, \mathbb{R}^n)$. The uniformly boundedness follows from the proof of Theorem 2.1 and local equicontinuity in the same way as in Lemma 2.3. Hence, according to the Ascoli-Arzela theorem, there exists a subsequence $\{x_i^1(t)\}_{i\in\mathrm{N}}$ that converges uniformly to $z_1(t) \in C_\psi(I_1, \mathbb{R}^n)$, i.e.

$$\lim_{i\to\infty} \left\| \frac{x_i^1(t)}{\psi(t)} - \frac{z_1(t)}{\psi(t)} \right\| = 0 \qquad \text{uniformly in } \ I_1.$$

Analogously, there exists a subsequence $\{x_i^2(t)\}_{i\in\mathrm{N}}$ of $\{x_i^1(t)\}_{i\in\mathrm{N}}$ that converges to $z_2(t)$ in $C_\psi(I_2, \mathbb{R}^n)$ such that $z_2(t) = z_1(t)$ $\forall t \in I_1$. We can repeat this reasoning for each $i \in \mathbb{N}$. In this way we obtain a family of subsequences of $\{x_i\}_{i\in\mathbb{N}}$.

Let $\{x_i^i(t)\}_{i\in\mathbb{N}}$ be the subsequence of $\{x_i(t)\}_{i\in\mathbb{N}}$ obtained by the diagonal process. Since the sequence $\left\{\frac{\overline{x}_i^i(t)}{\psi(t)}\right\}_{i\in\mathbb{N}}$ converges uniformly on each compact interval of $\langle a,\infty)$, there exists $z(t) \in C\Big(\langle a,\infty), \mathbb{R}^n\Big)$ such that:

$$\lim_{i\to\infty}\left\|\frac{\overline{x}_i^i(t)}{\psi(t)} - \frac{z(t)}{\psi(t)}\right\| = 0 \tag{2.26}$$

uniformly on each compact interval of $\langle a,\infty)$.
Moreover, $z(t) \in C_\psi(\langle a,\infty), \mathbb{R}^n)$ because $\{\overline{x}_i^i(t)\}_{i\in\mathbb{N}}$ is uniformly bounded , $\|\overline{x}_i^i\| \le \varrho \quad \forall i \in \mathbb{N}$ where ϱ satisfies (2.22). It remains to prove that $z(t)$ is a solution of our problem. Let

$$y(t) = Mz(t) = Pz(t) + K_P N z(t)\,.$$

For fixed $c \in \langle a,\infty)$, for each $t \in \langle a,c\rangle$ and for i sufficiently large from (2.14), (2.26) we obtain:

$$\begin{aligned}\left\|\frac{\overline{x}_i^i(t)}{\psi(t)} - \frac{y(t)}{\psi(t)}\right\| &= \left\|\frac{M\overline{x}_i^i(t)}{\psi(t)} - \frac{y(t)}{\psi(t)}\right\| \\ &\le \|P\|\cdot\left\|\frac{\overline{x}_i^i(t)}{\psi(t)} - \frac{z(t)}{\psi(t)}\right\| + \left\|\frac{K_PN\overline{x}_i^i(t)}{\psi(t)} - \frac{K_PNz(t)}{\psi(t)}\right\| \\ &\le \|P\|\cdot\left\|\frac{\overline{x}_i^i(t)}{\psi(t)} - \frac{z(t)}{\psi(t)}\right\| + H\big(H\|JT_0^{-1}\|\cdot\|T\| + 1\big)\cdot \\ &\quad\cdot \int_a^\infty \left\|X^{-1}(s)\big[f\big(s,\overline{x}_i^i(s)\big) - f\big(s,z(s)\big)\big]\right\| \,\mathrm{d}s\,.\end{aligned}$$

From (2.26) and from the proof of Lemma 2.2 we can infer:

$$\lim_{i\to\infty}\left\|\frac{\overline{x}_i^i(t)}{\psi(t)} - \frac{y(t)}{\psi(t)}\right\| = 0 \qquad t \in \langle a,c\rangle\,. \tag{2.27}$$

Comparing (2.26) and (2.27) we can conclude

$$y(t) = z(t) = Mz(t) \qquad t \in \langle a,c\rangle\,.$$

Since c is arbitrary,

$$z(t) = Mz(t) \qquad t \in \langle a,\infty)\,.$$

The theorem is proved.

We can state a theorem similar to Theorem 2.3.

THEOREM 2.4. *If the conditions* (2.15), (2.17), (2.23) *and* (2.24) *are satisfied, then the system* (1.1)-(1.2) *has at least one solution in* C_ψ.

The proof is similar to the proof of the preceding theorem.

Acknowledgements

I wish to express my gratitude to Professor Valter Šeda for his constructive criticism in the preparation of this paper.

REFERENCES

[1] CECCHI, M.—MARINI, M.—ZŁZZA, P. L.: *Linear boundary value problems for systems of ordinary differential equations on non compact intervals*, Ann. Mat. Pura Appl. (4) **123** (1980), 267–285.

[2] COLLATZ, L.: *Funktionalanalysis und Numerische Mathematik*, Springer-Verlag, Berlin, 1964.

[3] DEMIDOVIČ, B. P.: *Lectures of mathematical stability theory.* (Russian), Izd. Nauka, Moscow, 1967, pp. 150.

[4] DUNFORD, N.—SCHWARTZ, J. T.: *Linear Operators, part I*, Interscience Publishers, New York, 1957.

[5] GAINES, R. E.—MAWHIN, J.: *Coincidence Degree and Nonlinear Differential Equations. Lectures Notes in Math. 568*, Springer, Berlin, 1977.

[6] RUDIN, W.: *Principles of Mathematical Analysis*, McGraw-Hill Book Company, New York, 1964.

[7] ZEZZA, P. L.: *An equivalence theorem for nonlinear operator equations and an extension of Leray-Schauder continuation theorem*, Boll. Un. Mat. Ital. A (5) **15** (1978), 545-551.

Received June 29, 1990

Revised September 24, 1991

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