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δ-SETS, IRRESOLVABLE AND RESOLVABLE SPACES

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ABSTRACT. In a topological space (X, T), T^δ is the collection of all δ-sets $\{A\}$ where $\operatorname{int}\operatorname{cl} A \subset \operatorname{cl}\operatorname{int} A$; int and cl denote interior and closure with respect to the topology T. This paper considers the family of interiors of dense subsets of (X, T) and examines the relationship among this family, the collection T^δ and the concepts of hyperconnectedness, resolvability and irresolvability of the space. Properties of maximal resolvable spaces have also been dealt with.

Introduction

A topological space (X,T) is said to be *irresolvable* if any two dense sets in (X,T) intersect, otherwise (X,T) is said to be *resolvable*.

E. Hewitt has proved the following theorem.

THEOREM. [7] *Every space (X,T) can be represented uniquely as disjoint union of $X = F \cup G$, where F is closed and resolvable and G is open and hereditarily irresolvable.*

This representation will henceforth be called Hewitt representation of (X,T).

A space (X,T) is *submaximal* if every dense set is open. Clearly a submaximal space is hereditarily irresolvable. However, the converse is not true. See [5].

1. δ-sets, quasi-maximal spaces and irresolvable spaces

DEFINITION 1.1. *A topological space (X,T) is said to be a quasi-maximal space if for every dense set D in (X,T) with* $\operatorname{int} D \neq \emptyset$ *(the null set),* $\operatorname{int} D$ *is also dense in (X,T).*

We now prove the following theorem.

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THEOREM 1.1. *Let* $X = F \cup G$ *denote the Hewitt representation of a space* (X,T). *Then*

(i) *If* D *is dense in* (X,T), *then* $G \subset \operatorname{cl}\operatorname{int} D$.

(ii) *If* U *is open, then there exists a dense set* D *such that* $\operatorname{int} D = G \cup U$.

Proof.

(i) Suppose there exists $x \in G - \operatorname{cl}\operatorname{int} D$. Pick an open neighbourhood V of x with $V \subseteq G$ and $V \cap \operatorname{int} D = \emptyset$, i.e. $V \subseteq \operatorname{cl}(X - D)$. Then $V \cap D$ and $V \cap (X - D)$ are disjoint dense subsets of V, a contradiction, since $V \subseteq G$ is irresolvable.

(ii) (note that F, G, or U, are allowed to be empty.)
Let U be open and let $W = G \cup U$. Then $X - \operatorname{cl} W \subseteq F$. So choose $E_1, E_2 \subseteq X$ with $X - \operatorname{cl} W = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$ and $X - \operatorname{cl} W \subseteq \operatorname{cl} E_1$, $X - \operatorname{cl} W \subset \operatorname{cl} E_2$. Let $D = W \cup E_1$. Then $W = G \cup U \subseteq \operatorname{int} D$. If $x \in \operatorname{int} D$, then pick an open neighbourhood V of x such that $V \subseteq D = W \cup E_1$. Then $V \cap E_2 = \emptyset$, hence $V \cap \operatorname{cl} E_2 = \emptyset$ and so $V \cap E_1 = \emptyset$. Thus $V \subseteq W = G \cup U$. Therefore $\operatorname{int} D = G \cup U$. Hence (ii) is proved.

As a consequence of Theorem 1.1, we get the following characterization of quasi-maximal spaces.

THEOREM 1.2. *The following are equivalent for* (X,T):

(i) (X,T) *is quasi-maximal,*

(ii) (X,T) *is either resolvable and hyperconnected or* $\operatorname{cl} G = X$,

where $X = F \cup G$ *is the Hewitt representation of* (X,T).

From Theorem 1.1 also follows

THEOREM 1.3. *If* (X,T) *is submaximal, then it is irresolvable and quasi-maximal.*

However, the converse of Theorem 1.3 may not hold.

Example 1.1. Let X be an infinite set and $p \in X$. Let $T = \{X, \emptyset, \{p\}\}$. Then (X,T) is irresolvable and quasi-maximal but not submaximal.

DEFINITION 1.2. *A subset* A *of* (X,T) *is said to be an* α*-set* [9] *if* $A \subset \operatorname{int}\operatorname{cl}\operatorname{int} A$; *a* γ*-set* [1] *if* $A \subset \operatorname{int}\operatorname{cl} A$ *and a* δ*-set* [4] *if* $\operatorname{int}\operatorname{cl} A \subset \operatorname{cl}\operatorname{int} A$.

We denote by T^α, T^γ, T^δ, and $\mathcal{D}$, the collection of all α-sets, all γ-sets, all δ-sets and all dense sets, respectively, in (X,T). It is shown in [9] that T^α forms a topology on X.

(X,T) is said to be *open hereditarily irresolvable* if each open subset of X is irresolvable.

THEOREM 1.4. *For a topological space (X,T) the following are equivalent:*

(i) T^δ *is the discrete topology on* X
(ii) (X,T) *is open hereditarily irresolvable*
(iii) $\mathcal{D} \subset T^\delta$
(iv) (X,T) *is irresolvable and quasi-maximal*
(v) G *is dense in* (X,T), *where* $X = F \cup G$ *is the Hewitt representation of* (X,T)
(vi) (X, T^α) *is submaximal.*

Proof.

(i) $\Longrightarrow$ (ii): Let $O \in T$. Let D be any dense subset in O. Then $\operatorname{int}\operatorname{cl} D \supset O$ and $\operatorname{int}\operatorname{cl} D \subset \operatorname{cl}\operatorname{int} D$ (by (i)) implies that $\operatorname{int}_{T_O} D \neq \emptyset$, where $\operatorname{int}_{T_O} D$ denotes the interior of D in O relative to T. Hence (X,T) is open hereditarily irresolvable.

(ii) $\Longrightarrow$ (i): We first prove the following lemmas.

LEMMA 1.1. *Each $A \in T^\alpha$ can be expressed as $A = B \cup C$, where $B \in T^\alpha$ and C is nowhere dense in (X,T).*

Proof of Lemma 1.1. Let $A \in T^\delta$. Then

$$\operatorname{int}\operatorname{cl} A \subset \operatorname{cl}\operatorname{int} A. \tag{1.1}$$

Now $A = B \cup C$, where $B = \operatorname{int}\operatorname{cl} A \cap A$ and $C = A - \operatorname{int}\operatorname{cl} A$. Clearly C is nowhere dense in (X,T) and $\operatorname{int}\operatorname{cl}\operatorname{int} B = \operatorname{int}\operatorname{cl}\operatorname{int} A \supset \operatorname{int}\operatorname{cl} A \supset B$ (by (1.1)). Thus $B \in T^\alpha$. Hence the Lemma is proved.

LEMMA 1.2. *T^δ is the discrete topology on X if for every nonempty set $A \in T^\gamma$, $\operatorname{int} A \neq \emptyset$.*

Proof of Lemma 1.2. Let $A \subset X$. Suppose $\operatorname{int}\operatorname{cl} A \neq \emptyset$. Now $A = B \cup C$, where $B = A \cap \operatorname{int}\operatorname{cl} A$ and $C = A - \operatorname{int}\operatorname{cl} A$. Clearly C is nowhere dense in (X,T). By Lemma 1.1, it suffices to show that $B \in T^\alpha$. Suppose $B \neq T^\alpha$. Let $x \in B$ such that $x \neq \operatorname{int}\operatorname{cl}\operatorname{int} B$. Then for every open set O_x containing x, there exists $G \in T$ with $G \neq \emptyset$ and $G \subset O_x$, we have $G \cap \operatorname{int} B \neq \emptyset$. Therefore we get

$$G \cap \operatorname{int} A = \emptyset. \tag{1.2}$$

But as $x \in B$, there exists an open set O'_x containing x such that $O'_x \subset \operatorname{cl} A$. Consider the corresponding open subset G of O'_x satisfying (1.2). Now $G \subset O'_x \subset \operatorname{cl} A \Longrightarrow G \subset \operatorname{int}\operatorname{cl}(G \cap A)$. Let $G_1 = G \cap A$. Then $G_1 \subset G \subset \operatorname{int}\operatorname{cl} G_1$. Therefore $G_1 \in T^\gamma$. Since $G_1 \neq \emptyset$, by hypothesis we have $\operatorname{int} G_1 \neq \emptyset$. But $\operatorname{int} G_1 = \operatorname{int}(G \cap A) = G \cap \operatorname{int} A = \emptyset$ (by (1.2)), a contradiction. Thus $B \in T^\alpha$. Hence the Lemma is proved.

Now we prove (ii) $\Longrightarrow$ (i). Let $(\emptyset \neq)\ A \in T^{\gamma}$. Then $A \subset \operatorname{int}\operatorname{cl} A$ and since A is dense in $\operatorname{int}\operatorname{cl} A$ and (X,T) is open hereditarily irresolvable, we get $\operatorname{int} A \neq \emptyset$. From Lemma 1.2 it now follows that T^{δ} is the discrete topology on X.

(i) $\Longrightarrow$ (iii): Follows easily.

(iii) $\Longrightarrow$ (iv): Let $D \in \mathcal{D}$. Then $D \in T^{\delta} \Longrightarrow \operatorname{int} D \neq \emptyset$ and $\operatorname{int} D \in \mathcal{D}$. Thus (X,T) is irresolvable and quasi-maximal.

(iv) $\Longrightarrow$ (ii): Proved in Theorem 2 of [6].

(iii) $\Longrightarrow$ (v): Suppose $\operatorname{cl} G \neq X$. Then $\operatorname{int} F \neq \emptyset$. Let O be a nonempty open set such that $O \subset F$. Then O is resolvable since F is so. Let $D \subset O$ be dense in O such that $\operatorname{int}_{T_O} D = \emptyset$. Then $D \cup (X - O)$ is dense in (X,T) and by (iii), $D' = \operatorname{int}\big(D \cup (X - O)\big)$ is also dense in (X,T). But we claim that $O \cap \operatorname{int} D' = \emptyset$. For, if there exists a nonempty open set $O' \subset O \cap D'$, then $O' \subset D$; a contradiction to the fact that $\operatorname{int}_{T_O} D = \emptyset$. Therefore we get that $\operatorname{int} D'$ is not dense in (X,T); a contradiction to (iii).

(v) $\Longrightarrow$ (iii): It is given that $\operatorname{cl} G = X$. Let $(\emptyset \neq)\ O \in T$. Then $O \cap G \neq \emptyset$ and is open in (X,T). Since G is hereditarily irresolvable, $O \cap G$ is irresolvable. Let $D \in \mathcal{D}$. Then $D \cap O \cap G$ is dense in $O \cap G$ and therefore $\operatorname{int}_{T_{O\cap G}}(D \cap O \cap G) \neq \emptyset$, i.e. $\operatorname{int}_{T_{O\cap G}}(D \cap O) \neq \emptyset$, i.e. $\operatorname{int}(D \cap O) \neq \emptyset$. Thus $\operatorname{int} D \in \mathcal{D}$ and so $D \in T^{\delta}$.

(iv) $\Longrightarrow$ (vi): Follows from Theorem 2 and Theorem 4 of [6].

(vi) $\Longrightarrow$ (iv): Follows from Theorem 2 and Theorem 4 of [6].

The following Lemma will be used in proving Theorem 1.5.

LEMMA 1.3. *$T = T^{\delta}$ if and only if every open set is closed in (X,T).*

Proof of Lemma 1.3. Suppose every open set is closed in (X,T). Then $\operatorname{int}\operatorname{cl} A \neq \emptyset$ for each $(\emptyset \neq)\ A \subset X$. Therefore $T^{\delta} = T^{\alpha}$ (by Lemma 1.1 and Definition 1.2) and $T^{\alpha} = T$ [9]. Thus $T = T^{\delta}$. Conversely, let $T = T^{\delta}$. Suppose there exists an open set O which is not closed. Then $X - O$ is closed and not open and $X - O \in T^{\delta}$ (from Definition 1.2). Thus $T \neq T^{\delta}$, a contradiction. Hence the Lemma is proved.

THEOREM 1.5. *If $T = T^{\alpha}$, then (X,T) is either hereditarily irresolvable or resolvable or not a quasi-maximal space.*

Proof. If (X,T) is discrete, then clearly (X,T) is hereditarily irresolvable. Suppose (X,T) is not discrete. Then by Theorem 1.4, (X,T) is not open hereditarily irresolvable (since $T = T^{\delta}$). Now we claim that $F \neq \emptyset$, where $X = F \cup G$ is the Hewitt representation of (X,T). If not, then $G = X$ would be hereditarily irresolvable, a contradiction since (X,T) is not open hereditarily irresolvable. By Lemma 1.3, it follows that F is open (since F is closed). Therefore $\operatorname{cl} G \neq X$ and then by Theorem 1.2 we find that (X,T) is either resolvable or not a quasi-maximal space.

Let $\mathcal{N}$ denote the collection of all nowhere dense sets in (X,T). Then from Definition 1.2, $\mathcal{N} \subset T^\delta$. We denote by $T^\delta - \mathcal{N}$ the collection of all δ-sets which are non-nowhere dense in (X,T).

We now investigate the interrelationship between the family $T^\delta - \mathcal{N}$ and the concept of hyperconnectedness and irresolvability of the space.

THEOREM 1.6. *For a topological space (X,T) the following are equivalent:*

(i) *(X,T) is hyperconnected*
(ii) *$T^\delta - \mathcal{N}$ is a filter on X*
(iii) *$T^\delta - \mathcal{N} \subset \mathcal{D}$.*

Proof.

(i) $\Longrightarrow$ (ii): Let $A, B \in T^\delta - \mathcal{N}$. Then $\operatorname{int} A \cap \operatorname{int} B$ is nonempty and dense in (X,T). So $A \cap B \in T^\delta - \mathcal{N}$. Again let $A \subset B$ where $A \in T^\delta - \mathcal{N}$. Then $\operatorname{int} B$ is nonempty and dense in (X,T) implying that $B \in T^\delta - \mathcal{N}$. Hence $T^\delta - \mathcal{N}$ forms a filter on X.

(ii) $\Longrightarrow$ (iii): Let $A \in T^\delta - \mathcal{N}$. Then $\operatorname{int} A \neq \emptyset$. We claim that $\operatorname{int}(X - A) = \emptyset$. For, if not, then $\operatorname{int}(X - A) \in T^\delta - \mathcal{N}$. Since $A \cap \operatorname{int}(X - A) = \emptyset$, $\emptyset$ would belong to $T^\delta - \mathcal{N}$; a contradiction. Therefore $\operatorname{int}(X - A) = \emptyset$ and so $A \in \mathcal{D}$.

(iii) $\Longrightarrow$ (i): Follows immediately.

THEOREM 1.7. *For a topological space (X,T) the following are equivalent:*

(i) *(X,T) is irresolvable and hyperconnected*
(ii) *$\mathcal{D} = T^\delta - \mathcal{N}$*
(iii) *$T^\delta - \mathcal{N}$ is an ultrafilter on X.*

Proof.

(i) $\Longrightarrow$ (ii): Let $D \in \mathcal{D}$. Then $\operatorname{int} D \neq \emptyset$ and $\operatorname{int} D \in \mathcal{D}$ (using (i)). Thus $D \in T^\delta - \mathcal{N}$. Now, let $A \in T^\delta - \mathcal{N}$. Then $\operatorname{int} A \neq \emptyset$ and $\operatorname{int} A \in \mathcal{D}$ (by (i)) and so $A \in \mathcal{D}$. Thus $\mathcal{D} = T^\delta - \mathcal{N}$.

(iii) $\Longrightarrow$ (ii): Let $A \subset X$ be such that $A \notin \mathcal{D}$. Then $\operatorname{int}(X - A) \neq \emptyset$ and $\operatorname{int}(X - A) \in T^\delta - \mathcal{N}$. Therefore $\operatorname{cl} A \notin T^\delta - \mathcal{N}$ (by (iii)) and this implies that $A \notin T^\delta - \mathcal{N}$. Thus $T^\delta - \mathcal{N} \subset \mathcal{D}$. Now suppose $A \notin T^\delta - \mathcal{N}$. Then $X - A \in T^\delta - \mathcal{N}$ (by (iii)) and $\operatorname{int}(X - A) \neq \emptyset \Longrightarrow A \notin \mathcal{D}$. Thus $\mathcal{D} \subset T^\delta - \mathcal{N}$. Hence $T^\delta - \mathcal{N} = \mathcal{D}$.

(i) $\Longrightarrow$ (iii): In [5] it has been shown that if (X,T) is irresolvable and hyperconnected, then $\mathcal{D}$ is an ultrafilter on X. Thus it follows that $T^\delta - \mathcal{N}$ is an ultrafilter on X.

(ii) $\Longrightarrow$ (i): Since the interior of each dense set in (X,T) is nonempty, it follows that (X,T) is irresolvable. Also since every nonempty open subset of X is dense in (X,T), it follows that (X,T) is hyperconnected.

2. Maximal resolvable spaces

A space (X,T) with property R is maximal R if, whenever T' is stronger that T ($T' \supset T$), then (X, T') does not have property R.

Maximal R spaces have been extensively studied by E. Hewitt [7], N. Smythe and C.A. Wilkins [11], A.B. Raha [10], D.E. Cameron [2], [3] among others.

In this section we investigate necessary and sufficient conditions for a space to be maximal resolvable.

We first recall [8] that for a topological space (X,T) and for any subset A of X such that $A \notin T$, the simple extension of T by A is the topology $T(A) = \{U \cup (V \cap A)\colon U, V \in T\}$.

We now prove

LEMMA 2.1. *Let (X,T) be resolvable and $A \subset X$ be a resolvable subspace. Then $(X, T(A))$ is resolvable.*

Proof. Let D ($\subset A$) be dense in A such that $\operatorname{int}_{T_A} D = \emptyset$. Two cases arise.

Case I. $X - \operatorname{cl} A = \emptyset$.
Then D is dense in (X,T) and $\operatorname{int} D = \emptyset$. Consider the topology $T(A)$. Clearly D is dense in $(X, T(A))$ and having an empty interior with respect to $T(A)$. So $(X, T(A))$ is resolvable in this case.

Case II. $X - \operatorname{cl} A \neq \emptyset$.
Clearly $X - \operatorname{cl} A$ is resolvable. Choose $D^* \subset X - \operatorname{cl} A$ such that D^* is dense in $X - \operatorname{cl} A$ with $\operatorname{int}_{T_{X-\operatorname{cl} A}} D^* = \emptyset$. Then consider $D \cup D^*$, which is dense in (X,T) and having an empty interior in (X,T); for, if there exists a nonempty open set $O \subset D \cup D^*$, then $O \cap D^* \neq \emptyset \Longrightarrow O \cap (X - \operatorname{cl} A) = O'$ (say) $\neq \emptyset \Longrightarrow \operatorname{int}_{T_{X-\operatorname{cl} A}} D^* \neq \emptyset$ (since $O' \cap A = \emptyset$); a contradiction to the choice of D^*. Now consider the topology $T(A)$. Clearly $D \cup D^*$ is dense in $(X, T(A))$ and having an empty interior in $(X, T(A))$; for, if $\emptyset \neq U \cup (V \cap A) \subset D \cup D^*$ for some $U, V \in T$, then $U = \emptyset$ and $V \cap A \subset D \cup D^* \Longrightarrow V \cap A \subset D$; a contradiction, since $V \cap A$ is nonempty open in (A, T_A) and $\operatorname{int}_{T_A} D = \emptyset$. Hence $(X, T(A))$ is resolvable in this case also.

Now we come to the main theorem of this section.

THEOREM 2.1. *For a topological space (X,T), the following are equivalent:*

(i) *(X,T) is maximal resolvable.*

(ii) *The set of all open subsets of X = the set of all resolvable subsets of X.*

(iii) *Any continuous bijection f from a resolvable space (Y, V) onto (X,T) is a homeomorphism.*

Proof.

(i) $\Longrightarrow$ (ii): Clearly every open subset is resolvable. Now suppose A $(\subset X)$ is resolvable but not open. Then by Lemma 2.1 $\big(X, T(A)\big)$ is resolvable. So (X,T) cannot be maximal resolvable since $T(A) \supsetneqq T$.

(ii) $\Longrightarrow$ (i): Suppose (X,T) is not maximal resolvable. Then there exists a topology T' containing T properly such that (X, T') is resolvable. Let $U \in T'$ such that $U \notin T$. Then U is resolvable in (X, T') and so resolvable in (X,T). This contradicts (ii).

(i) $\Longrightarrow$ (iii): If $f\colon (Y, V) \to (X, T)$ is a continuous bijection, then for $T' = \{f(G)\colon G \in V\}$, $f\colon (Y, V) \to (X, T')$ is a homeomorphism and (X, T') is resolvable (since the property of being resolvable is a topological property). Since $T' \supset T$ and (X,T) is maximal resolvable, it follows that $T' = T$.

(iii) $\Longrightarrow$ (i): If (X,T) is not maximal resolvable, then there is a topology $T' \supsetneqq T$ such that (X, T') is resolvable. The identity map $I\colon (X, T') \to (X, T)$ is a continuous bijection which is not a homeomorphism.

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