ON THE SEQUENTIAL ORDER

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ABSTRACT. We show that the sequential order in some distinguished convergence groups and convergence rings is $\omega_1$.

A topological space is said to be sequential if each of its sequentially closed subsets is closed. In fact, every sequential space is the topological modification of a sequential convergence space, i.e., of a set $X$ equipped with a convergence of sequences and the associated sequential closure operator $\text{cl}$ assigning each subset $A$ of $X$ the set $\text{cl}A$ of all limits of all convergent sequences ranging in $A$. In order to guarantee the usual properties of a closure it suffices to assume that each constant sequence $(x)$ converges to $x$ and if a sequence converges to some point, then each of its subsequences converges to this point as well. For every ordinal number $\alpha$ and for every subset $A$ define $\alpha - \text{cl}A$ as follows:

\[
\begin{align*}
0 - \text{cl}A &= A, \\
\alpha - \text{cl}A &= \text{cl}(\beta - \text{cl}A) \text{ if } \alpha = \beta + 1, \\
\alpha - \text{cl}A &= \bigcup_{\beta < \alpha} \beta - \text{cl}A \text{ if } \alpha \text{ is a limit ordinal.}
\end{align*}
\]

Then each $\alpha - \text{cl}$ is a closure operator for $X$ and since for each subset $A$ of $X$ we have $\text{cl}(\omega_1 - \text{cl}A) = \omega_1 - \text{cl}A$, $\omega_1 - \text{cl}$ is idempotent and hence topological. The least ordinal number $\alpha$ ($\alpha \leq \omega_1$) such that $\alpha - \text{cl}A$ is sequentially closed for all subsets $A$ of $X$ is said to be the sequential order of $X$.

The sequential order in the realm of sequential convergence spaces has been investigated by J. Novák in [18], where an example of a countable space (with unique limits) having the sequential order $\omega_1$ can be found (cf. Example L8). The literature on the sequential order in the realm of topological spaces is much more extensive and the sequential order within the theory of ordinal invariants has been thoroughly investigated in [13]. An often cited example by A. V. Arhangel'skij and S. P. Franklin of a countable zero-dimensional Hausdorff homogeneous space, the sequential order of which is $\omega_1$, has been

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given in [1]. G. H. Greco in [12] proved that the space known as the rational plane $\mathbb{Q}^2$ with the cross topology has (besides other nice properties) the sequential order $\omega_1$, too.

The aim of the present paper is to prove that each of the sets $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, and $\mathbb{R}$ can be equipped with a distinguished sequential convergence with unique limits so that it is compatible with the group (ring) structure and the sequential order of the resulting space is $\omega_1$. In particular, the rational torus $\mathbb{Q}/\mathbb{Z}$ becomes a countable sequentially compact convergence group with unique limits and the sequential order $\omega_1$.

The proof by G. H. Greco and our proof are via transfinite induction and, as we point out, they follow the same schema based on the fact that for a certain class of diagonal sequences either all subsequences converge to the same point or no subsequence converges at all. This in fact provides a control on the closure of countable unions of certain inductively defined sets.

1. By a convergence group we understand a group equipped with a FLUSH-convergence of sequences (besides the axioms on constant sequences and subsequences we assume unique limit, the Urysohn axiom and the sequential continuity of the algebraic operations). We say that a convergence group is coarse if the underlying group admits no strictly coarser (i.e. larger) FLUSH convergence. Convergence rings and coarse convergence rings are defined analogously. Using the Zorn-Kuratowski lemma, it can be shown that each group (ring) convergence can be enlarged to a coarse one. For further information on coarse groups and rings the reader is referred to [10], [5] and [6].

Denote by $\mathbb{Q}_c$ the group of rational numbers equipped with a coarse group convergence coarser than the usual metric one. It is known (cf. [5]) that $\mathbb{Q}_c$ is complete (every Cauchy sequence converges) and, even though the sequential limits are unique, no two points of $\mathbb{Q}_c$ can be separated by disjoint neighbourhoods (neither by $(\omega_1-\text{cl})$-neighbourhoods nor by $\text{cl}$-neighbourhoods). In fact, for each $q \in \mathbb{Q}$ and for each interval $I = \{x \in \mathbb{Q}; a \leq x \leq b\}$, $a, b \in \mathbb{Q}$, $a < b$, there is a one-to-one sequence $\langle x_n \rangle$ in $I$ converging in $\mathbb{Q}_c$ to $q$. From the latter property it follows that the sequential order of $\mathbb{Q}_c$ is greater than one ([5]). We shall prove that the sequential order of $\mathbb{Q}_c$ is actually $\omega_1$ (mentioned in [5] as a remark added in the proof).

In the proof of the next theorem we shall use the following convention: by an interval we mean a set of the form $\{x \in \mathbb{Q}; a \leq x \leq b\}$, $a, b \in \mathbb{Q}$, $a < b$, and if $q \in \mathbb{Q}$ and $\langle I_n \rangle$ is a sequence of intervals, then by $\langle I_n \rangle \to q$ we mean that for each $a \in \mathbb{Q}$, $a > 0$, we have $I_n \cap \{x \in \mathbb{Q}; q-a \leq x \leq q+a\}$ for all but finitely many $n$, $n = 1, 2 \ldots$ (observe that if $q_n \in I_n$, then $\langle q_n \rangle$ converges in the usual metric to $q$).

**Theorem 1.1.** The sequential order of $\mathbb{Q}_c$ is $\omega_1$.  

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Proof. Clearly, it suffices to prove that for each ordinal number \( \alpha \), \( \alpha < \omega_1 \), the following proposition \( P(\alpha) \) holds true:

\[
P(\alpha) \quad \text{For each } p \in \mathbb{Q} \text{ and for each interval } I \text{ there is a set } A \subset I \text{ such that } \\
a - \text{cl} A \subset I \setminus \{p\} \text{ and } (\alpha + 1) - \text{cl} A = \{p\} \cup \alpha - \text{cl} A.
\]

Let \( p \in \mathbb{Q} \) and let \( I \) be an interval. We proceed by transfinite induction.

1. Let \( \alpha = 0 \). As mentioned before (cf. condition (c) on page 476 in [5]), in \( I \) there is a one-to-one sequence \( \langle x_n \rangle \) converging in \( \mathbb{Q}_c \) to \( p \). It suffices to put \( A = \{x_n; n = 1, 2, \ldots\} \setminus \{p\} \). Now, let \( \alpha > 0 \) and assume that \( P(\beta) \) holds for all ordinal numbers \( \beta \), \( \beta < \alpha \).

2. Let \( \alpha = \beta + 1 \). Choose \( q \in I \), \( q \neq p \), disjoint intervals \( I_n \subset I \), \( p \notin I_n \), \( n = 0, 1, \ldots \), such that \( \langle I_n \rangle \to q \) and a one-to-one sequence \( \langle p_n \rangle \) in \( I_0 \) converging in \( \mathbb{Q}_c \) to \( p \). By the inductive assumption, for each \( n \), \( n = 1, 2, \ldots \), choose a set \( A_n \subset I_n \) such that \( \beta - \text{cl} A_n \subset I_n \setminus \{p_n\} \) and \( (\beta + 1) - \text{cl} A_n = \{p_n\} \cup \beta - \text{cl} A_n \). It suffices to put \( A = \{q\} \cup \bigcup_{n=1}^{\infty} A_n \).

3. Let \( \alpha \) be a limit ordinal number. Let \( \langle \alpha_n \rangle \) be an increasing sequence of ordinal numbers converging to \( \alpha \). Choose \( q \), \( \langle I_n \rangle \) and \( \langle p_n \rangle \) as in the case of \( \alpha = \beta + 1 \). By the inductive assumption, for each \( n \), \( n = 1, 2, \ldots \), choose a set \( A_n \subset I_n \) such that \( \alpha_n - \text{cl} A_n \subset I_n \setminus \{p_n\} \) and \( (\alpha_n + 1) - \text{cl} A_n = \{p_n\} \cup \alpha_n - \text{cl} A_n \). It suffices to put \( A = \{q\} \cup \bigcup_{n=1}^{\infty} A_n \). This completes the proof.

Let \( \mathbb{Q}/\mathbb{Z} \) be the rational torus and let \( d \) be the usual metric for \( \mathbb{Q}/\mathbb{Z} \). Denote by \( (\mathbb{Q}/\mathbb{Z})_c \) the rational torus equipped with a coarse group convergence coarser than the usual metric one.

**Theorem 1.2.**

(i) \( (\mathbb{Q}/\mathbb{Z})_c \) is sequentially compact.

(ii) For each \( p, q \in \mathbb{Q}/\mathbb{Z} \) and for each positive real number \( \varepsilon \) there is in \( \mathbb{Q}/\mathbb{Z} \) a sequence \( \langle q_n \rangle \) such that \( \langle q_n \rangle \) converges in \( (\mathbb{Q}/\mathbb{Z})_c \) to \( p \) and \( d(q, q_n) < \varepsilon \), \( n = 1, 2, \ldots \).

(iii) No two points of \( (\mathbb{Q}/\mathbb{Z})_c \) can be separated by disjoint neighbourhoods.

(iv) The sequential order of \( (\mathbb{Q}/\mathbb{Z})_c \) is \( \omega_1 \).

**Proof.**

(i) Let \( \langle p_n \rangle \) be a sequence of points of \( \mathbb{Q}/\mathbb{Z} \). Then there is a subsequence \( \langle q_n \rangle \) of \( \langle p_n \rangle \) which is Cauchy in the metric \( d \). Since the convergence in \( (\mathbb{Q}/\mathbb{Z})_c \) is coarser than that induced by \( d \), \( \langle q_n \rangle \) is also a Cauchy sequence in \( (\mathbb{Q}/\mathbb{Z})_c \). By Proposition 3.1 in [5], \( (\mathbb{Q}/\mathbb{Z})_c \) is complete. Thus \( \langle q_n \rangle \) converges in \( (\mathbb{Q}/\mathbb{Z})_c \).

Proposition (ii) can be proved virtually in the same way as condition (c) on page 476 in [5].
Proposition (iii) follows immediately from (ii).

Using (ii), proposition (iv) can be proved exactly in the same way as Theorem 1.1. This completes the proof of the theorem.

Denote by \( \mathbb{Q}_r \) the ring of rational numbers equipped with a coarse ring convergence, coarser than the usual metric one. Basic properties of \( \mathbb{Q}_r \) have been described in [6]. E.g., no Cauchy sequence of rational numbers converging in the real line to an algebraic number does converge in \( \mathbb{Q}_r \) and hence the convergence in \( \mathbb{Q}_r \) fails to be a coarse group convergence for \( \mathbb{Q} \). Further, \( \mathbb{Q}_r \) does not have a ring completion. Even though the convergence in \( \mathbb{Q}_r \) is finer than that in \( \mathbb{Q}_c \), it follows from the proof of Proposition 6 in [6] that for each \( p, q \in \mathbb{Q} \) and for each positive real number \( \varepsilon \) there is a sequence \( \langle q_n \rangle \) in \( \mathbb{Q} \) such that \( \langle q_n \rangle \) converges in \( \mathbb{Q}_r \) to \( p \) and \( |q - q_n| < \varepsilon, \ n = 1, 2, \ldots \). Consequently, the proof of Theorem 1.1 yields the following result:

**Corollary 1.3.** The sequential order of \( \mathbb{Q}_r \) is \( \omega_1 \).

**Remark 1.4.** The nontrivial part (\( \alpha > 0 \)) of the proof of Theorem 1.1 can be summarized as follows.

Let \( X \) be a set equipped with a sequential convergence. Assume that for each point \( p \in X \), each “admissible” set \( I \subset X \) and each nondecreasing sequence \( \langle \alpha_n \rangle \) of isolated ordinal numbers \( \alpha_n < \omega_1 \) there are:

- (a₁) A one-to-one sequence \( \langle p_n \rangle \) of points of \( X, \ p_n \neq p, \ n = 1, 2, \ldots \);
- (a₂) A sequence \( \langle I_n \rangle \) of disjoint “admissible” sets \( I_n \subset I, \ p \notin I_n, \ n = 1, 2, \ldots \);
- (a₃) A sequence \( \langle A_n \rangle \) of sets \( A_n \subset I_n, \ n = 1, 2, \ldots \);
- (a₄) A point \( q \in X, \ q \neq p \);

such that

- (b₁) The sequence \( \langle p_n \rangle \) converges to \( p \);
- (b₂) \( \alpha_n - \text{cl} A_n \subset I_n \setminus \{p_n\}, \ (\alpha_n + 1) - \text{cl} A_n = \{p_n\} \cup \alpha_n - \text{cl} A_n, \ n = 1, 2, \ldots \);
- (b₃) Each sequence \( \langle q_n \rangle \) of points \( q_n \in I_n, \ n = 1, 2, \ldots \), converges in \( X \) to \( q \).

Then the sequential order of \( X \) is \( \omega_1 \). Indeed, the set \( A = \{q\} \cup \bigcup_{n=1}^{\infty} A_n \) satisfies condition \( P(\alpha) \) both for isolated \( \alpha \) (with \( \alpha_n + 1 = \alpha, \ n = 1, 2, \ldots \)) and limit \( \alpha \) (with \( \alpha = \lim \alpha_n \)).

It is easy to verify that if we leave out condition (a₄) and replace condition (b₃) by

- (b₃') No diagon. ′ sequence \( \langle q_n \rangle, \ q_n \in I_n, \ n = 1, 2, \ldots \), has a convergent subsequence;

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then the sequential order of $X$ is $\omega_1$, too. Now, the set $A = \bigcup_{n=1}^{\infty} A_n$ does the trick.

This modified scheme generalizes the proof of Theorem in [12], asserting that the sequential order of $\mathbb{Q}^2$ equipped with the cross topology is $\omega_1$. The role of the “admissible” sets $I_n$ is played by the discs $\Omega_n$ “tangent at $x_n$ ($= p_n$) to the line through $x$ ($= p$) and points $x_n$”. We shall use the latter scheme in the next section to prove that both the group categorical completion of $\mathbb{Q}$ and the ring categorical completion of $\mathbb{Q}$ have the sequential order $\omega_1$.

2. There are striking differences between the completion theories of topological groups (rings) and convergence groups (rings). Every abelian convergence group has a completion and it can have several nonhomeomorphic ones ([19], [7]), whereas there are convergence groups having no two-sided completion ([9]). As shown in [14] (see also [15]), if a convergence group has a completion, then it has the categorical one, yielding the epireflection into the subcategory of complete groups. The categorical completion, also called the Novák completion, has been explicitly constructed for abelian sequential convergence groups in [19] (see also [8]). The necessary and sufficient condition for the Novák completion of a Fréchet abelian group to be Fréchet has been given in [16] and a nontrivial example of a Fréchet abelian group the Novák completion of which is Fréchet has been given in [4]. As shown in [3], a sequential convergence commutative ring need not have a completion, but in special cases, e.g. for $\mathbb{Q}$ equipped with the usual metric convergence, the categorical ring completion has been constructed in [17]. We shall prove that both the group categorical completion $\nu \mathbb{Q}$ of $\mathbb{Q}$ and the ring categorical completion $\rho \mathbb{Q}$ of $\mathbb{Q}$ have the sequential order $\omega_1$.

The Novák completion $\nu \mathbb{Q}$ of $\mathbb{Q}$ can be briefly described as follows (cf. [19]). The underlying group of $\nu \mathbb{Q}$ is the group $\mathbb{R}$ of real numbers and the convergence in $\nu \mathbb{Q}$ is the Urysohn modification of the convergence in which a sequence $(z_n)$ converges to $z$ if and only if there is a Cauchy sequence $(q_n)$ of rational numbers such that $z_n = q_n - q + z$, where $q$ is the real number to which $(q_n)$ converges in the real line. Thus, the sequence $(\pi/n)$ does not converge in $\nu \mathbb{Q}$ to 0 (since $\pi/n - \pi/m$ is not a rational number for $n \neq m$) and, in fact, the following proposition holds ([19]).

**Proposition 2.1.** Let $(q_n)$ be a sequence of real numbers such that $z_n - z_m$ is not a rational number whenever $n \neq m$. Then no subsequence of $(z_n)$ converges in $\nu \mathbb{Q}$.

**Theorem 2.2.** The sequential order of $\nu \mathbb{Q}$ is $\omega_1$.

**Proof.** Let $H$ be a set of irrational numbers such that $\{1\} \cup H$ is a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. Given an infinite countable set $B \subset H$ and $r \in \mathbb{R}$, define
$I(B, r)$ to be the set of all real numbers of the form $r + \sum_{i=1}^{k} (b_i - q_i)$, where $k$ is a natural number ($k \neq 0$), $q_i \in \mathbb{Q}$, $b_i \in B$, $b_i \neq b_j$ for $i \neq j$, $i, j = 1, \ldots, k$; each set $I(B, r)$ will be called admissible.

Clearly, it suffices to prove that for each ordinal number $\alpha$, $\alpha < \omega_1$, the following proposition $R(\alpha)$ holds true:

$R(\alpha)$ For each $p \in \mathbb{R}$ and for each admissible set $I(B, p)$ there is a set $A \subseteq I(B, p)$ such that $\alpha - \text{cl} A \subseteq I(B, p) \setminus \{p\}$ and $(\alpha + 1) - \text{cl} A = \{p\} \cup \alpha - \text{cl} A$.

Let $p \in \mathbb{R}$ and let $I(B, p)$ be an admissible set. We proceed by transfinite induction.

1. Let $\alpha = 0$. Choose $b \in B$ and let $\langle q_n \rangle$ be a one-to-one sequence of rational numbers converging in $\nu \mathbb{Q}$ to $b$. It suffices to put $A = \{p + (b - q_n); n = 1, 2, \ldots\}$. Now, let $\alpha > 0$ and assume that $R(\beta)$ holds for all ordinal numbers $\beta$, $\beta < \alpha$.

2. Let $\alpha > 0$. Assume $\alpha = \beta + 1$. There is a finite subset $B_p \subset H$ such that $p$ is a $\mathbb{Q}$-linear combination of elements of $\{1\} \cup B_p$. Choose $b \in B \setminus B_p$. Let $\langle q_n \rangle$ be a one-to-one sequence of rational numbers such that $|b - q_n| < 1/100n$, $n = 1, 2, \ldots$. Put $p_n = p + 1/n + b - q_n$, $n = 1, 2, \ldots$. Let $B \setminus (\{b\} \cup B_p)$ be the union of disjoint infinite sets $B_n$, $n = 1, 2, \ldots$. Consider the admissible sets $I(B_n, p_n)$. Then $I(B_n, p_n) \cap I(B_m, p_m) = \emptyset$ whenever $n \neq m$, $I(B_n, p_n) \subset I(B, p)$ \setminus \{p\}$ and $p_n \notin I(B_n, p_n)$ for each $n$, $n = 1, 2, \ldots$, and no diagonal sequence $\langle x_n \rangle$, $x_n \in I(B_n, p_n)$, $n = 1, 2, \ldots$, has a convergent subsequence. By the inductive assumption, for each $n$, $n = 1, 2, \ldots$, choose a set $A'_n \subset I(B_n, p_n)$ such that $\beta - \text{cl} A'_n \subseteq I(B_n, p_n) \setminus \{p_n\} = I(B_n, p_n)$ and $(\beta + 1) - \text{cl} A'_n = \{p_n\} \cup \beta - \text{cl} A'_n$. It suffices to put $A_n = A'_n \cap (p + 1/n - 1/10n, p + 1/n + 1/10n)$ and $A = \bigcup_{n=1}^{\infty} A_n$ (cf. Remark 1.4).

For a limit ordinal number $\alpha$ the assertion $R(\alpha)$ can be proved analogously. We omit the details. This completes the proof.
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a sequence converges to a point in $\nu \mathbb{Q}$, then it converges to the same point in $\varrho \mathbb{Q}$ as well.

**Theorem 2.3.** The sequential order of $\mathbb{Q}$ is $\omega_1$.

**Proof.** The proof is analogous to that of Theorem 2.2. Indeed, the sequence converges to a point in $\varrho \mathbb{Q}$ whenever it does in $\nu \mathbb{Q}$ and no subsequence of any diagonal sequence $\langle x_n \rangle$, $x_n \in I(B_n, p_n)$, $n = 1, 2, \ldots$, does converge in $\varrho \mathbb{Q}$. Thus, for all ordinal numbers $\alpha$, $\alpha < \omega_1$, the assertion $R(\alpha)$ holds in $\varrho \mathbb{Q}$, too.

Remark 2.4. It is known that if $X$ is a group (ring, vector space, etc.) equipped with a compatible sequential convergence, then the first countable filter modification functor $\gamma$ yields a filter convergence compatible with the algebraic operations of $X$ (cf. [2]). The resulting filter convergence space has the same closure operator and hence all propositions concerning the sequential order proved in the present paper remain valid when applying the functor $\gamma$. For interesting applications of $\gamma$ see [2], [9], [6], [11], [17].

**REFERENCES**


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