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SOME DIOPHANTINE APPROXIMATION RESULTS CONCERNING LINEAR RECURRENCES

J. P. JONES*) — P. KISS**)1)

ABSTRACT. Let R_n and V_n ($n = 0, 1, 2, \dots$) be sequences of integers defined by $R_n = AR_{n-1} - BR_{n-2}$ and $V_n = AV_{n-1} - BV_{n-2}$, where A and B are fixed non-zero integers and $R_0 = 0$, $R_1 = 1$, $V_0 = 2$, $V_1 = A$. Furthermore let $D = A^2 - 4B$. We show that

$$\left| \sqrt{D} - \frac{V_n}{R_n} \right| < \frac{1}{c \cdot R_n^2}$$

holds for infinitely many n if and only if $|B| = 1$ and $c \leqq \sqrt{D}/2$. We also show that the "best" rational approximations of the irrational number $\sqrt{D}$ have the form $p/q = V_n/R_n$.

§1. Introduction

Let R_n and V_n, ($n = 0, 1, \dots$), be sequences of integers defined by a second order linear recurrence

$$R_n = A \cdot R_{n-1} - B \cdot R_{n-2} \qquad (n = 2, 3, \dots),$$
$$V_n = A \cdot V_{n-1} - B \cdot V_{n-2} \qquad (n = 2, 3, \dots),$$

where A and B are fixed non-zero integers and the initial terms of the sequences are $R_0 = 0$, $R_1 = 1$, $V_0 = 2$ and $V_1 = A$. Let α and β be the roots of the characteristic polynomial $x^2 - Ax + B$ and let D denote its discriminant. Then we have

$$\text{(i)} \quad D = A^2 - 4B\,, \qquad \text{(ii)} \quad A = \alpha + \beta\,, \qquad \text{(iii)} \quad B = \alpha\beta\,. \tag{1}$$

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Throughout the paper we suppose that $D > 0$, $D \neq \square$ (D is not a square) and also that $0 < A$, (see discussion of $0 < A$ below). Plainly $|\beta| = |\alpha|$ if and only if $D \leq 0$. Thus when $D > 0$ and $D \neq \square$, α and β are irrational real numbers and we can suppose that $|\beta| < |\alpha|$. Furthermore, since $\beta \neq \alpha$, the terms of the sequences R_n and V_n are given by

$$\text{(iv)} \quad R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \text{(v)} \quad V_n = \alpha^n + \beta^n. \tag{1}$$

For the derivation of (iv) and (v) see e.g. [1], [6] or [7]. From these equations it is not difficult to see that

$$\text{(i)} \quad \frac{R_{n+1}}{R_n} - \alpha = \frac{\sqrt{D}}{(\alpha/\beta)^n - 1}, \qquad \text{(ii)} \quad \frac{V_n}{R_n} - \sqrt{D} = \frac{2\sqrt{D}}{(\alpha/\beta)^n - 1}. \tag{2}$$

Since $|\beta| < |\alpha|$, it follows from (2) that

$$\text{(i)} \quad \lim_{n\to\infty} \frac{R_{n+1}}{R_n} = \alpha \qquad \text{and} \qquad \text{(ii)} \lim_{n\to\infty} \frac{V_n}{R_n} = \alpha - \beta = \sqrt{D}. \tag{3}$$

Thus R_{n+1}/R_n is an approximation to the irrational number α and V_n/R_n is an approximation to the irrational number $\sqrt{D}$. The quality of the approximation (3) (i) to α has been investigated in earlier papers. In [2] it was proved that the inequality

$$\left|\alpha - \frac{R_{n+1}}{R_n}\right| < \frac{1}{R_n^2}$$

holds for infinitely many n if and only if $|B| = 1$. In [2] it was also proved that when $|B| = 1$ and p/q is a rational number such that $(p, q) = 1$, then the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{D} \cdot q^2}$$

implies that $p/q = R_{n+1}/R_n$ for some $n \geq 1$. In some other special cases similar results follow from [3] and [8]. The quality of the approximation of α by the ratio R_{n+1}/R_n was studied in the papers [4] and [5], in the general setting when $|B| \neq 1$ and even for $D < 0$.

In this paper we consider the approximation of $\sqrt{D}$ by rationals of the form V_n/R_n. We shall see that the approximation by V_n/R_n is the best possibility when $|B| = 1$.

Throughout we will assume that $0 < A$. There is no loss of generality in making this assumption. To see this, let A be a positive integer and suppose V_n

and R_n are the sequences defined by A and B, with characteristic roots α and β, and V'_n and R'_n are the sequences defined by $-A$ and B, with characteristic roots α' and β'. Then $D' = D$ and the assumption $0 < A$ is equivalent to $\alpha - \beta = \sqrt{D}$. Hence from our assumption $0 < D$, i.e. that $|\beta| < |\alpha|$, we obtain

$$\alpha = \frac{A + \sqrt{D}}{2}, \qquad \beta = \frac{A - \sqrt{D}}{2},$$

and

$$\alpha' = \frac{-A - \sqrt{D}}{2} = -\alpha, \qquad \beta' = \frac{-A + \sqrt{D}}{2} = -\beta.$$

Therefore $\alpha'/\beta' = \alpha/\beta$. Hence we have from (1) (iv) and (v) that

$$\frac{V'_n}{R'_n} = -\frac{V_n}{R_n}.$$

Thus approximating $\sqrt{D}$ by rationals V_n/R_n, when $A < 0$, is equivalent to approximating $-\sqrt{D}$ by rationals V'_n/R'_n, when $0 < A$. So we shall suppose $0 < A$ together with our other assumptions, $0 < D$, $B \neq 0$ and $D \neq \Box$.

We shall prove the following theorems:

THEOREM 1. *Let c be a real number. Then the inequality*

$$\left|\sqrt{D} - \frac{V_n}{R_n}\right| < \frac{1}{c \cdot R_n^2}$$

holds for infinitely many n if and only if $|B| = 1$ and $c \leq \sqrt{D}/2$.

THEOREM 2. *Suppose $|B| = 1$ and $B + 5 \leq A$. All sufficiently large solutions p/q of*

$$\left|\sqrt{D} - \frac{p}{q}\right| < \frac{2}{\sqrt{D} \cdot q^2}, \tag{4}$$

have the form $p/q = V_n/R_n$ for some positive integer n.

THEOREM 3. *Suppose $|B| = 1$ and $B + 5 \leq A$. Then infinitely many rational numbers p and q satisfy the inequality*

$$\left|\sqrt{D} - \frac{p}{q}\right| < \frac{1}{c \cdot q^2} \tag{5}$$

if and only if $c \leq 2\sqrt{D}$. When $c = 2\sqrt{D}$, every sufficiently large rational solution p/q of (5) has the form $p/q = V_n/R_n$, for some positive integer n.

§2. Proof of the theorems

Proof of Theorem 1. From (1) (ii), (iii) we have $\alpha\beta = B$ and $\alpha - \beta = \sqrt{D}$ so that by (1) (iv), (v) we have

$$\begin{aligned}\frac{V_n}{R_n} - \sqrt{D} &= \sqrt{D}\left(\frac{\alpha^n + \beta^n}{\alpha^n - \beta^n} - 1\right) = \frac{2\sqrt{D}\beta^n}{\alpha^n - \beta^n} = \frac{2\beta^n(\alpha^n - \beta^n)}{\sqrt{D}\cdot R_n^2} \\ &= \frac{2B^n(1-(\beta/\alpha)^n)}{\sqrt{D}\cdot R_n^2}.\end{aligned} \tag{6}$$

Hence the inequality of Theorem 1 is equivalent to $|B|^n|1-(\beta/\alpha)^n| < \sqrt{D}/2c$. Since $|\beta| < |\alpha|$ we have $(\beta/\alpha)^n \to 0$ as $n \to \infty$. Theorem 1 follows.

In the proofs of Theorems 2 and 3 below we shall use the following lemma. A proof of it can be found in [9], (Chapter 7 in the 5th edition).

LEMMA 1. *Let γ be irrational. If there exist integers p and $q \geq 1$ such that*

$$\left|\gamma - \frac{p}{q}\right| < \frac{1}{2\cdot q^2},$$

then p/q is one of the convergents to the simple continued fraction expansion of γ, that is, $p/q = h_n/k_n$ holds for some n where h_n/k_n is the nth convergent to γ.

Proof of Theorem 2. We will consider four cases according as $B = \pm 1$ and A is odd or A is even. The assumption $B + 5 \leq A$ is equivalent to saying that when $B = -1$ and A is even, then $4 \leq A$; when $B = -1$ and A is odd, then $5 \leq A$; when $B = +1$ and A is even, then $6 \leq A$; and when $B = +1$ and A is odd, then $7 \leq A$. From these it follows that $2 < \sqrt{D}/2$ if $B = -1$ and $5/2 < \sqrt{D}/2$ if $B = +1$. We shall use these inequalities in the following when we apply Lemma 1.

First suppose that $B = -1$ and $A = 2a$, where a is an integer and $a \geq 2$. In this case $4 \leq A$ and we have $\sqrt{D} = \sqrt{4a^2+4}$. In this case it is easy to check that the simple periodic continued fraction expansion of $\sqrt{D}$ is

$$\sqrt{D} = \langle 2a, \overline{a, 4a}\,\rangle. \tag{7}$$

Let $\gamma = \sqrt{D}$. Since $D \neq \square$, γ is irrational. When $\gamma = \langle a_0, a_1, a_2, \ldots\rangle$ is the simple continued fraction expansion of an irrational number γ, then, as is well known, see [9], the nth convergent $r_n = \langle a_0, a_1, \ldots, a_n\rangle$ to γ is given by $r_n = h_n/k_n$, where h_n and k_n are sequences defined by

$$h_{-2} = 0, \qquad h_{-1} = 1, \qquad h_i = a_i h_{i-1} + h_{i-2}, \qquad (i = 0, 1, \ldots), \tag{8}$$

$$k_{-2} = 1, \qquad k_{-1} = 0, \qquad k_i = a_i k_{i-1} + k_{i-2}, \qquad (i = 0, 1, \ldots). \tag{9}$$

In our case, from (7) we have $a_0 = 2a$ and

$$a_{2i-1} = a \qquad \text{and} \qquad a_{2i} = 4a\,, \qquad (i = 0, 1, \ldots)\,. \tag{10}$$

Consequently by (8), $h_0 = 2a$, $h_1 = a \cdot 2a + 1 = 2a^2 + 1$ and $h_2 = 4a \cdot (2a^2 + 1) + 2a = 8a^3 + 6a$. On the other hand, from the definition of the sequence V_n, $V_0 = 2 = 2h_{-1}$, $V_1 = A = 2a = h_0$, $V_2 = 2a \cdot 2a + 2 = 2 \cdot h_1$ and $V_3 = 8a^3 + 6a = h_2$.

We now extend these equations by proving that

$$V_{2i} = 2 \cdot h_{2i-1}\,, \tag{11}$$

$$V_{2i+1} = h_{2i}\,, \tag{12}$$

for $i \geq 0$. Equations (11) and (12) will be proved by induction. The equations hold for $i = 0$ and $i = 1$. Suppose (11) and (12) hold for indices $0, 1, \ldots, i$. Then from (8) – (12) we have

$$V_{2(i+1)} = V_{2i+2} = 2a \cdot V_{2i+1} + V_{2i} = 2ah_{2i} + 2h_{2i-1} = 2(ah_{2i} + h_{2i-1}) = 2h_{2i+1}$$
$$= 2 \cdot h_{2(i+1)-1}.$$

Also

$$V_{2(i+1)+1} = 2aV_{2i+2} + V_{2i+1} = 4ah_{2i+1} + h_{2i} = h_{2(i+1)}.$$

Hence (11) and (12) are established for all $i \geq 0$.

Similarly as above, by (9) we have $R_0 = 0 = k_{-1}$, $R_1 = 1 = k_0$, $R_2 = 2a = 2k_1$, $R_3 = 4a^2 + 1 = k_2$ and we can show by induction that for any $i \geq 0$

$$R_{2i} = 2 \cdot k_{2i-1}\,, \tag{13}$$

and

$$R_{2i+1} = k_{2i}\,. \tag{14}$$

Now suppose (4) holds, i.e. $|\sqrt{D} - p/q| < 2/\sqrt{D}q^2$ for some p and q. Then, since $2 \leq \sqrt{D}/2$, Lemma 1 implies that $p/q = h_n/k_n$ for some n. Hence by (11), (12), (13) and (14), we have $p/q = V_n/R_n$, which implies the theorem.

Next suppose $B = -1$ and that $A = 2a + 1$ is odd. Since $5 \leq A$, we have $2 \leq a$. In this case $\sqrt{D} = \sqrt{4a^2 + 4a + 5}$. $\sqrt{D}$ is irrational and $2 < \sqrt{D}/2$. The periodic continued fraction of $\sqrt{D}$ is $\sqrt{D} = \langle 2a + 1, \overline{a, 1, 1, a, 4a + 2} \rangle = \langle a_0, a_1, a_2, \ldots \rangle$, where $a_0 = 2a + 1$ and

$$a_{5i+1} = a\,, \quad a_{5i+2} = 1\,, \quad a_{5i+3} = 1\,, \quad a_{5i+4} = a\,, \quad a_{5i+5} = 4a + 2\,,$$

for $i \geq 0$. By an argument similar to the above but longer, we can show that for $i \geq 0$

$$\begin{aligned} V_{3i} &= 2 \cdot h_{5i-1}, & R_{3i} &= 2 \cdot k_{5i-1}, \\ V_{3i+1} &= h_{5i}, & R_{3i+1} &= k_{5i}, \\ V_{3i+2} &= h_{5i+3}, & R_{3i+2} &= k_{5i+3}. \end{aligned} \tag{15}$$

Now suppose (4) holds for some rational p/q. Since $2 \leq \sqrt{D}/2$, Lemma 1 implies that $p/q = r_n = h_n/k_n$ for some n. Hence by (15) the theorem holds when $n = 5i - 1$, $n = 5i$ or $n = 5i + 3$. If n is of the form $n = 5i + 1$ or $n = 5i + 2$, then we still have to prove that

$$\frac{2}{\sqrt{D} \cdot k_n^2} < |\sqrt{D} - r_n|. \tag{16}$$

Suppose first that $n = 5i + 1$. By the elementary properties of the continued fraction expansion of an irrational number γ, we have

$$|\gamma - r_n| = \frac{1}{k_n(\theta_{n+1} k_n + k_{n-1})}, \tag{17}$$

where θ_j is defined by $\gamma = \langle a_0, a_1, a_2, \ldots, a_{j-1}, \theta_j \rangle$ and $\theta_j = \langle a_j, a_{j+1}, \ldots \rangle$. By (17), to prove (16), we have to show that

$$\theta_{n+1} + \frac{k_{n-1}}{k_n} < \frac{\sqrt{D}}{2}. \tag{18}$$

When $n = 5i + 1$ we have

$$\theta_{n+1} = \theta_{5i+2} = \langle \overline{1, 1, a, 4a + 2, a} \rangle$$

and one can check that

$$\theta_{5i+2} = \frac{2a - 1 + \sqrt{D}}{2a + 1}. \tag{19}$$

Furthermore, from (9), (15) and (3) we have

$$\begin{aligned} \frac{k_{n-1}}{k_n} &= \frac{k_{5i}}{k_{5i+1}} = \frac{k_{5i}}{ak_{5i} + k_{5i-1}} = \frac{R_{3i+1}}{aR_{3i+1} + R_{3i}/2} = \frac{1}{a + \dfrac{R_{3}i}{2 \cdot R_{3i+1}}} \\ &< \frac{1}{a + \dfrac{1}{2\alpha}} + \varepsilon \end{aligned} \tag{20}$$

for any $\varepsilon > 0$, if i is large enough. But

$$\alpha = \frac{A+\sqrt{D}}{2} = \frac{2a+1+\sqrt{D}}{2} \tag{21}$$

and from (19) and (21), after a short calculation, we have

$$\theta_{5i+2} + \frac{1}{a + \frac{1}{2\alpha}} = \frac{2\sqrt{D}}{2a+1} < \frac{\sqrt{D}}{2},$$

since $a \geq 2$. Together with (20), this proves inequality (18).

When $n = 5i+1$, we can prove inequality (16) by a similar argument.

We now consider the third case, $B = 1$ and A is even. Then $A = 2a$ and $B + 5 \leq A$ implies $3 \leq a$. In this case

$$\sqrt{D} = \sqrt{4a^2-4} = \langle 2a-1, \overline{1, a-2, 1, 4a-2} \rangle$$

and we have

$$\begin{aligned} V_{2i+1} &= h_{4i+1}, & R_{2i+1} &= k_{4i+1}, \\ V_{2i} &= 2 \cdot h_{4i-1}, & R_{2i} &= 2 \cdot k_{4i-1}, \end{aligned} \tag{22}$$

for $i \geq 0$. Suppose (4) holds for some rational p/q. Since $2 < \sqrt{D}/2$, Lemma 1 implies that p/q is a convergent to the continued fraction expansion of $\sqrt{D}$, i.e. that $p/q = r_n = h_n/k_n$. Hence from (22), $p/q = V_j/R_j$, if n is of the form $n = 4i+1$ or $n = 4i-1$. Similar to the above, for the other convergents we can prove that

$$\frac{2}{\sqrt{D} \cdot k_{4n+2}^2} < |\sqrt{D} - r_{4n+2}| \qquad \text{and} \qquad \frac{2}{\sqrt{D} \cdot k_{4n}^2} < |\sqrt{D} - r_{4n}|,$$

by using

$$\begin{aligned} \theta_{4n+3} &= \langle \overline{1, 4a-2, 1, a-2} \rangle = \frac{2a-4+\sqrt{D}}{4a-5} \\ \theta_{4n+1} &= \langle \overline{1, a-2, 1, 4a-2} \rangle = \frac{2a-1+\sqrt{D}}{4a-5}. \end{aligned}$$

This completes the proof of the theorem in this third case.

Finally assume $B = 1$ and A is odd. Then $A = 2a + 1$. $B + 5 \leq A$ implies $3 \leq a$. In this case $\sqrt{D} = \sqrt{4a^2 + 4a - 3} = \langle 2a, \overline{1, a-1, 2, a-1, 1, 4a} \rangle$, where a is an integer, and we can show that

$$\begin{aligned} V_{3i+1} &= h_{6i+1}, & R_{3i+1} &= k_{6i+1}, \\ V_{3i+2} &= h_{6i+3}, & R_{3i+2} &= k_{6i+3}, \\ V_{3i+3} &= 2 \cdot h_{6i+5}, & R_{3i+3} &= 2 \cdot k_{6i+5}, \end{aligned} \tag{23}$$

for all $i \geq 0$. Furthermore it can be shown that

$$\theta_{6n+1} = \langle \overline{1, a-1, 2, a-1, 1, 4a} \rangle = \frac{2a + \sqrt{D}}{4a - 3},$$

$$\theta_{6n+3} = \langle \overline{2, a-1, 1, 4a, 1, a-1} \rangle = \frac{2a - 1 + \sqrt{D}}{2a - 1},$$

$$\theta_{6n+5} = \langle \overline{1, 4a, 1, a-1, 2, a-1} \rangle = \frac{2a - 3 + \sqrt{D}}{4a - 3},$$

from which we obtain

$$\frac{2}{\sqrt{D} \cdot k_n^2} < |\sqrt{D} - r_n|$$

when $n = 6i$, $n = 6i + 2$ or $n = 6i + 4$, $(i = 0, 1, 2, \ldots)$, using $3 \leq a$. Hence the theorem is proved in all four cases.

P r o o f o f T h e o r e m 3. If a rational number p/q, with p and q sufficiently large, satisfies the inequality (5), with $c = 2\sqrt{D}$, then inequality (4) is also satisfied by p/q. Consequently by Theorem 2, there exists a positive integer n such that $p/q = V_n/R_n$.

If $c \geq 2$ and p/q is a solution of (5), then by Lemma 1 p/q is a convergent to the simple continued fraction expansion of $\sqrt{D}$ and so, by (11) - (15), (22) and (23), $p = V_n$ or $p = V_n/2$ and $q = R_n$ or $q = R_n/2$ for some n. From these by (6), with $V_n = 2p$ and $R_n = 2q$,

$$\left| \frac{p}{q} - \sqrt{D} \right| = \frac{1 - (\beta/\alpha)^n}{2\sqrt{D}q^2} \tag{24}$$

follows. From (5) and (24) we obtain the inequality $c \leq 2\sqrt{D}$. From (24) it also follows that (5) has infinitely many p, q integer solutions if $c \leq 2\sqrt{D}$. Thus we have proved every assertion of the theorem.

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