Danuta Ozdarska; Stanisław Szufla
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WEAK SOLUTIONS OF A BOUNDARY VALUE PROBLEM FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATION OF SECOND ORDER IN BANACH SPACES

DANUTA OZDARSKA — STANISLAW SZUFLA

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ABSTRACT. Using the measure of weak noncompactness we give sufficient conditions for the existence of a weak solution of a boundary value problem for the equation $x'' = f(t, x, x')$ in Banach space.

Let $J = (0, a)$ be a compact interval in $\mathbb{R}$ and let $E$ be a weakly sequentially complete real Banach space. In this paper we give an existence theorem for weak solutions of the boundary value problem

$$x'' = f(t, x, x'), \quad x(0) = x(a) = 0. \quad (1)$$

Our approach is to impose on $f$ weak compactness type conditions in terms of the measure of weak noncompactness introduced by De Blasi [4]. Let us recall that similar study relative to the strong topology has attracted much attention in recent years [2], [8], [11].

A function $x: J \rightarrow E$ is called a weak solution of (1) if $x$ has a weak second derivative $x''$ on $J$, $x(0) = x(a) = 0$ and

$$x''(t) = f(t, x(t), x'(t)) \quad \text{for} \quad t \in J.$$

Throughout this paper we shall assume that $f$ is a weakly-weakly continuous function (cf. [3], [10]) from $J \times E \times E$ into $E$ and $||f(t, x, y)|| \leq M$ for $t \in J$, $x, y \in E$. It is well known that (1) is equivalent to the integral equation

$$x(t) = \int_0^a G(t, s)f(s, x(s), x'(s)) \, ds, \quad (2)$$

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where $\int$ denotes the weak Riemann integral and

$$G(t,s) = \begin{cases} (t-a) \frac{s}{a} & \text{if } 0 \leq s \leq t \leq a, \\ (s-a) \frac{t}{a} & \text{if } 0 \leq t \leq s \leq a. \end{cases}$$

Moreover

$$\int_0^a |G(t,s)| \, ds \leq \frac{a^2}{8} \quad \text{and} \quad \int_0^a \left| \frac{\partial G}{\partial t}(t,s) \right| \, ds \leq \frac{a}{2} \quad \text{for } t \in J \quad (3)$$

(cf. [6, Ch. XII.4]).

As in [4], for a bounded subset $A$ of $E$ we denote by $\beta(A)$ the measure of weak noncompactness of $A$ defined by

$$\beta(A) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact subset } K \text{ such that } A \subseteq K + \varepsilon Q\},$$

where $Q$ is the unit ball. Recall that $\beta$ has the following properties:

1° If $A \subseteq B$, then $\beta(A) \leq \beta(B)$;
2° $\beta(A) = 0$ if and only if $A$ is relatively weakly compact in $E$;
3° $\beta(A \cup B) = \max(\beta(A), \beta(B))$;
4° $\beta(A^w) = \beta(A)$, ($A^w$ denotes the weak closure of $A$);
5° $\beta(A + B) \leq \beta(A) + \beta(B)$;
6° $\beta(\lambda A) = |\lambda| \beta(A)$;
7° $\beta(\text{conv } A) = \beta(A)$;
8° $\beta\left( \bigcup_{|\lambda| \leq h} \lambda A \right) = h \beta(A)$.

For any set $H$ of functions from $J$ into $E$ put

$$H(t) = \{u(t) : u \in H\}, \quad H(J) = \{u(t) : u \in H, \ t \in J\}.$$  

Arguing similarly as in the proof of Lemma 2.2 in [1], we can prove the following:

**Lemma 1.** If $H$ is a strongly equicontinuous and uniformly bounded set of functions from $J$ into $E$, then

$$\beta(H(J)) = \sup_{t \in J} \beta(H(t)).$$

Denote by $C_w(J, E)$ the space of weakly continuous functions $J \to E$ endowed with the topology of weak uniform convergence.

In what follows we shall need the following Krasnosielskii-type:
LEMMA 2. Let g be a weakly-weakly continuous function from $J \times E$ into $E$. Then for any $\varphi \in E^*$, $\varepsilon > 0$ and $u \in C_w(J, E)$ there exists a weak neighbourhood $U$ of 0 in $E$ such that

$$|\varphi(g(t, x(t)) - g(t, u(t)))| \leq \varepsilon$$

for all $t \in J$ and $x \in C_w(J, E)$ such that $x(s) - u(s) \in U$ for $s \in J$ (cf. [12, Lemma 2]).

Our main result is the following:

THEOREM. If there exist positive numbers $p$, $q$ such that $p \frac{a^2}{8} + q \frac{a}{2} < 1$ and

$$\beta(f(J \times X \times Y)) \leq p \beta(X) + q \beta(Y)$$

(4)

for every bounded subsets $X$, $Y$ of $E$, then the problem (1) has a weak solution.

Proof of the Theorem. Let $C_w(J, E)$ be the space of all weakly continuous functions $u: J \rightarrow E$ having weakly continuous weak derivative $u'$, endowed with the topology of weak uniform convergence. (More precisely, a net $(x_\alpha)$ converges to $x$ in $C_w(J, E)$ if and only if $x_\alpha \rightarrow x$ and $x'_\alpha \rightarrow x'$ weakly uniformly). Denote by $B$ the set of all functions $x \in C_w(J, E)$ which satisfy the inequalities

$$\max\left(\sup_{t \in J} \|x(t)\|, \sup_{t \in J} \frac{a}{4} \|x'(t)\|\right) \leq M \frac{a^2}{8},$$

$$\|x'(t) - x'(\tau)\| \leq M |t - \tau|, \quad \|x(t) - x(\tau)\| \leq Ma |t - \tau|/2, \quad (t, \tau \in J).$$

It is clear that $B$ is a convex closed subset of $C_w(J, E)$.

We define an operator $F$ by

$$F(x)(t) = \int_0^a G(t, s)f(s, x(s), x'(s)) \, ds \quad (x \in B, \ t \in J).$$

By (3) for each $x \in B$ the function $u = F(x)$ satisfies the inequalities

$$\|u''(t)\| = \|f(t, x(t), x'(t))\| \leq M, \quad \|u'(t)\| \leq M \frac{a}{2}, \quad \|u(t)\| \leq M \frac{a^2}{8},$$

for $t \in J$, and consequently by the mean value theorem

$$\|u'(t) - u'(\tau)\| \leq M |t - \tau|, \quad \|u(t) - u(\tau)\| \leq Ma \frac{|t - \tau|}{2}, \quad \text{for} \ t, \tau \in J.$$
This proves that \( F(B) \subseteq B \). \hspace{1cm} (5)

Moreover, by Lemma 2, for given \( \varphi \in E^* \), \( y \in B \) and \( \varepsilon > 0 \) we can choose a weak neighbourhood \( U \) of 0 in \( E \) such that

\[
|\varphi(f(t,x(t),x'(t)) - f(t,y(t),y'(t)))| \leq \varepsilon,
\]

for \( t \in J \) and \( x \in B \) such that \( x(s) - y(s) \in U \) and \( x'(s) - y'(s) \in U \) for \( s \in J \).

Hence, by (3),

\[
|\varphi (F(x)(t) - F(y)(t))| = \left| \int_0^a G(t,s) \varphi (f(s,x(s),x'(s)) - f(s,y(s),y'(s))) \, ds \right| \leq a^2 \varepsilon / 8,
\]

\[
|\varphi ((F(x))'(t) - (F(y))'(t))| = \left| \int_0^a \frac{\partial G}{\partial t} (t,s) \varphi (f(s,x(s),x'(s)) - f(s,y(s),y'(s))) \, ds \right| \leq a \varepsilon / 2,
\]

for \( t \in J \) and \( x \in B \) such that \( x(s) - y(s) \in U \) and \( x'(s) - y'(s) \in U \) for \( s \in J \).

From this we deduce that the operator \( F \) is continuous.

Now we shall prove the following:

**Lemma 3.** If \( V \subseteq B \) and

\[
V \subseteq \overline{\text{conv}}(F(V) \cup \{0\}),
\]

then \( V \) is relatively compact in \( C_{1w}(J,E) \).

**Proof.** As \( V \subseteq B \), the sets \( V \) and \( V' = \{ x' : x \in V \} \) are uniformly bounded and strongly equicontinuous.

Since for convex subsets of \( E \) the closure in the norm topology coincides with the weak closure (cf. [5, Th. II.1]), it is clear from (6) that

\[
\begin{align*}
V(t) & \subseteq \overline{\text{conv}} \left( \left\{ \int_0^a G(t,s)f(s,x(s),x'(s)) \, ds : x \in V \right\} \cup \{0\} \right), \\
V'(t) & \subseteq \overline{\text{conv}} \left( \left\{ \int_0^a \frac{\partial G}{\partial t} (t,s)f(s,x(s),x'(s)) \, ds : x \in V \right\} \cup \{0\} \right), \quad (t \in J).
\end{align*}
\]
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Fix \( t \in J \). We divide the interval \( J \) into \( m \) parts \( 0 = t_0 < t_1 < \cdots < t_m = a \) in such a way that \( \Delta t_i = t_i - t_{i-1} = \frac{a}{m} \) \((i = 1, \ldots, m)\). Put \( T_i = (t_{i-1}, t_i) \) and
\[ h_i = \sup \{|G(t, s)| : s \in T_i\} = |G(t, s_i)|, \]where \( s_i \in T_i \). Since for each \( x \in V \)
\[
\int_0^a G(t, s)f(s, x(s), x'(s)) \, ds = \sum_{i=1}^m \int_{T_i} G(t, s)f(s, x(s), x'(s)) \, ds
\]

\[
\in \sum_{i=1}^m \Delta t_i \text{conv} \{G(t, s)f(s, x(s), x'(s)) : x \in V, \ s \in T_i\}
\]

\[
\subset \sum_{i=1}^m \Delta t_i \text{conv} \left( \bigcup_{|\lambda| \leq h_i} \lambda f(J \times V(J) \times V'(J)) \right),
\]

from (7), (4) and corresponding properties of \( \beta \) it follows that
\[
\beta(V(t)) \leq \beta \left( \left\{ \int_0^a G(t, s)f(s, x(s), x'(s)) \, ds : x \in V \right\} \right)
\]

\[
\leq \sum_{i=1}^m \Delta t_i h_i \beta(f(J \times V(J) \times V'(J)))
\]

\[
\leq \sum_{i=1}^m \Delta t_i |G(t, s_i)| (p \beta(V(J)) + q \beta(V'(J))).
\]

On the other hand, if \( m \to \infty \), then
\[
\sum_{i=1}^m \Delta t_i |G(t, s_i)| \to \int_0^a |G(t, s)| \, ds.
\]

Thus
\[
\beta(V(t)) \leq \int_0^a |G(t, s)| \, ds \ (p \beta(V(J)) + q \beta(V'(J))),
\]

and by (3)
\[
\beta(V(t)) \leq \frac{a^2}{8} (p \beta(V(J)) + q \beta(V'(J))).
\]

Analogously, we can prove that
\[
\beta(V'(t)) \leq \frac{a}{2} (p \beta(V(J)) + q \beta(V'(J))).
\]
By Lemma 1 the above inequalities imply that
\[
\max(\beta(V(J)), \frac{a}{4}\beta(V'(J))) \leq \frac{a^2}{8}(p\beta(V(J)) + q\beta(V'(J))) \\
\leq \left(p\frac{a^2}{8} + q\frac{a}{2}\right)\max(\beta(V(J)), \frac{a}{4}\beta(V'(J))).
\]
As \( p\frac{a^2}{8} + q\frac{a}{2} < 1 \), this shows that \( \beta(V(J)) = \beta(V'(J)) = 0 \), i.e. the sets \( V(J) \) and \( V'(J) \) are relatively weakly compact in \( E \).

By Ascoli’s theorem, this proves that the sets \( V \) and \( V' \) are relatively compact in \( C_w(J,E) \), so that \( V \) is relatively compact in \( C_{1w}(J,E) \). This ends the proof of Lemma 3.

Now we return to the proof of the Theorem. We define a sequence \( (y_n) \) by \( y_0 = 0 \), \( y_{n+1} = F(y_n) \ (n \in \mathbb{N}) \). Let \( Y = \{y_n : n \in \mathbb{N}\} \). As \( Y \subset B \) and \( Y = F(Y) \cup \{0\} \), from Lemma 3 it follows that \( Y \) is relatively compact in \( C_{1w}(J,E) \). Denote by \( Z \) the set of all limit points of \( (y_n) \). We shall show that \( Z = F(Z) \). If \( y \in F(Z) \), then \( y = F(x) \) for some \( x \in Z \). Thus there exists a subnet \( (x_\alpha) \) of \( (y_n) \) such that \( x_\alpha \rightarrow x \). From the continuity of \( F \) it follows that \( F(x_\alpha) \rightarrow F(x) = y \). As \( (F(x_\alpha)) \) is also a subnet of \( (y_n) \), we see that \( y \in Z \). Conversely, let \( y \in Z \). Then there exists a subnet \( (y_\alpha) \) of \( (y_n) \) such that \( y_\alpha \rightarrow y \) and \( y_\alpha = F(x_\alpha) \), where \( (x_\alpha) \) is also a subnet of \( (y_n) \). Since the set \( Y \) is relatively compact, \( (x_\alpha) \) has a convergent subnet \( (x_\gamma) \). Let \( x = \lim x_\gamma \). Then \( x \in Z \) and \( y_\gamma = F(x_\gamma) \rightarrow F(x) \). On the other hand, \( y_\gamma \rightarrow y \). Hence \( y = F(x) \in F(Z) \).

Let us put \( R(X) = \overline{\text{conv}}F(X) \) for \( X \subset B \), and let \( \Omega \) denote the family of all subsets \( X \) of \( B \) such that \( Z \subset X \) and \( R(X) \subset X \). From (5) it is clear that \( B \in \Omega \). Denote by \( V \) the intersection of all sets of the family \( \Omega \). As \( Z \subset V \), \( V \) is nonempty and \( Z = F(Z) \subset R(Z) \subset R(V) \). Since \( R(V) \subset R(X) \subset X \) for all \( X \in \Omega \), \( R(V) \subset V \), and therefore \( V \in \Omega \). Moreover, \( R(R(V)) \subset R(V) \), and hence \( R(V) \in \Omega \). Consequently, \( V = R(V) \), i.e. \( V = \overline{\text{conv}}F(V) \). In view of Lemma 3 this implies that \( V \) is a compact subset of \( B \). Applying now the Schauder-Tychonoff fixed point theorem to the mapping \( F|_V \), we conclude that there exists \( x \in V \) such that \( x = F(x) \). It is clear that \( x \) satisfies (2) and hence \( x \) is a weak solution of (1).

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Adam Mickiewicz University

Poznań

Poland