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# WEAK SOLUTIONS OF A BOUNDARY VALUE PROBLEM FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATION OF SECOND ORDER IN BANACH SPACES

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ABSTRACT. Using the measure of weak noncompactness we give sufficient conditions for the existence of a weak solution of a boundary value problem for the equation $x'' = f(t,x,x')$ in Banach space.

Let $J = \langle 0, a\rangle$ be a compact interval in $\mathbb{R}$ and let $E$ be a weakly sequentially complete real Banach space. In this paper we give an existence theorem for weak solutions of the boundary value problem

$$x'' = f(t,x,x'), \qquad x(0) = x(a) = 0. \tag{1}$$

Our approach is to impose on $f$ weak compactness type conditions in terms of the measure of weak noncompactness introduced by De Blasi [4]. Let us recall that similar study relative to the strong topology has attracted much attention in recent years [2], [8], [11].

A function $x\colon J \to E$ is called a *weak solution of* (1) if $x$ has a weak second derivative $x''$ on $J$, $x(0) = x(a) = 0$ and

$$x''(t) = f\big(t, x(t), x'(t)\big) \qquad \text{for} \quad t \in J.$$

Throughout this paper we shall assume that $f$ is a weakly-weakly continuous function (cf. [3], [10]) from $J \times E \times E$ into $E$ and $\|f(t,x,y)\| \le M$ for $t \in J$, $x, y \in E$. It is well known that (1) is equivalent to the integral equation

$$x(t) = \int_0^a G(t,s) f\big(s, x(s), x'(s)\big)\, \mathrm{d}s, \tag{2}$$

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where $\int$ denotes the weak Riemann integral and

$$G(t,s) = \begin{cases} (t-a)\,s/a & \text{if } 0 \le s \le t \le a\,, \\ (s-a)\,t/a & \text{if } 0 \le t \le s \le a\,. \end{cases}$$

Moreover

$$\int_0^a |G(t,s)|\,\mathrm{d}s \le \frac{a^2}{8} \quad \text{and} \quad \int_0^a \Big|\frac{\partial G}{\partial t}(t,s)\Big|\,\mathrm{d}s \le \frac{a}{2} \qquad \text{for} \quad t \in J \tag{3}$$

(cf. [6, Ch. XII.4]).

As in [4], for a bounded subset $A$ of $E$ we denote by $\beta(A)$ the *measure of weak noncompactness of* $A$ defined by

$$\beta(A) = \inf\{\varepsilon > 0 : \text{ there exists a weakly compact subset } K \text{ such that } A \subset K + \varepsilon Q\}\,,$$

where $Q$ is the unit ball. Recall that $\beta$ has the following properties:

1° If $A \subset B$, then $\beta(A) \le \beta(B)$;
2° $\beta(A) = 0$ if and only if $A$ is relatively weakly compact in $E$;
3° $\beta(A \cup B) = \max(\beta(A), \beta(B))$;
4° $\beta(\overline{A}^w) = \beta(A)$, ( $\overline{A}^w$ denotes the weak closure of $A$);
5° $\beta(A + B) \le \beta(A) + \beta(B)$;
6° $\beta(\lambda A) = |\lambda|\,\beta(A)$;
7° $\beta(\operatorname{conv} A) = \beta(A)$;
8° $\beta\Big(\bigcup_{|\lambda| \le h} \lambda A\Big) = h\,\beta(A)$.

For any set $H$ of functions from $J$ into $E$ put

$$H(t) = \big\{u(t) :\ u \in H\big\}\,, \qquad H(J) = \big\{u(t) :\ u \in H,\ \ t \in J\big\}\,.$$

Arguing similarly as in the proof of Lemma 2.2 in [1], we can prove the following:

**Lemma 1.** *If $H$ is a strongly equicontinuous and uniformly bounded set of functions from $J$ into $E$, then*

$$\beta\big(H(J)\big) = \sup_{t \in J} \beta\big(H(t)\big)\,.$$

Denote by $C_w(J, E)$ the space of weakly continuous functions $J \to E$ endowed with the topology of weak uniform convergence.

In what follows we shall need the following Krasnosielskii-type:

**Lemma 2.** *Let $g$ be a weakly-weakly continuous function from $J \times E$ into $E$. Then for any $\varphi \in E^*$, $\varepsilon > 0$ and $u \in C_w(J,E)$ there exists a weak neighbourhood $U$ of $0$ in $E$ such that*

$$\left|\varphi\big(g(t,x(t)) - g(t,u(t))\big)\right| \le \varepsilon$$

*for all $t \in J$ and $x \in C_w(J,E)$ such that $x(s) - u(s) \in U$ for $s \in J$ (cf.* [12, Lemma 2]).

Our main result is the following:

**Theorem.** *If there exist positive numbers $p$, $q$ such that $p\frac{a^2}{8} + q\frac{a}{2} < 1$ and*

$$\beta\big(f(J \times X \times Y)\big) \le p\,\beta(X) + q\,\beta(Y) \tag{4}$$

*for every bounded subsets $X$, $Y$ of $E$, then the problem (1) has a weak solution.*

Proof of the Theorem. Let $C_{1w}(J,E)$ be the space of all weakly continuous functions $u\colon J \to E$ having weakly continuous weak derivative $u'$, endowed with the topology of weak uniform convergence. (More precisely, a net $(x_\alpha)$ converges to $x$ in $C_{1w}(J,E)$ if and only if $x_\alpha \to x$ and $x'_\alpha \to x'$ weakly uniformly). Denote by $B$ the set of all functions $x \in C_{1w}(J,E)$ which satisfy the inequalities

$$\max\Big(\sup_{t\in J}\|x(t)\|\,,\ \sup_{t\in J}\frac{a}{4}\|x'(t)\|\Big) \le M\frac{a^2}{8}\,,$$
$$\|x'(t) - x'(\tau)\| \le M\,|t-\tau|\,,\quad \|x(t) - x(\tau)\| \le Ma\,|t-\tau|/2\,,\qquad (t,\tau \in J)\,.$$

It is clear that $B$ is a convex closed subset of $C_{1w}(J,E)$.

We define an operator $F$ by

$$F(x)(t) = \int_0^a G(t,s) f\big(s, x(s), x'(s)\big)\,\mathrm{d}s \qquad (x \in B\,,\ t \in J)\,.$$

By (3) for each $x \in B$ the function $u = F(x)$ satisfies the inequalities

$$\|u''(t)\| = \|f\big(t,x(t),x'(t)\big)\| \le M\,,\qquad \|u'(t)\| \le M\frac{a}{2}\,,\qquad \|u(t)\| \le M\frac{a^2}{8}\,,$$

for $t \in J$, and consequently by the mean value theorem

$$\|u'(t) - u'(\tau)\| \le M\,|t-\tau|\,,\quad \|u(t) - u(\tau)\| \le Ma\,\frac{|t-\tau|}{2}\,,\qquad \text{for}\quad t,\tau \in J\,.$$

This proves that

$$F(B) \subset B\,. \tag{5}$$

Moreover, by Lemma 2, for given $\varphi \in E^*$, $y \in B$ and $\varepsilon > 0$ we can choose a weak neighbourhood $U$ of 0 in $E$ such that

$$\left|\varphi\big(f\big(t,x(t),x'(t)\big) - f\big(t,y(t),y'(t)\big)\big)\right| \le \varepsilon\,,$$

for $t \in J$ and $x \in B$ such that $x(s) - y(s) \in U$ and $x'(s) - y'(s) \in U$ for $s \in J$.

Hence, by (3),

$$\begin{aligned}&\left|\varphi\big(F(x)(t) - F(y)(t)\big)\right| \\ &\quad = \left|\int_0^a G(t,s)\,\varphi\big(f\big(s,x(s),x'(s)\big) - f\big(s,y(s),y'(s)\big)\big)\;\mathrm{d}s\right| \le a^2\varepsilon/8\,,\end{aligned}$$

$$\begin{aligned}&\left|\varphi\big(\big(F(x)\big)'(t) - \big(F(y)\big)'(t)\big)\right| \\ &\quad = \left|\int_0^a \frac{\partial G}{\partial t}(t,s)\,\varphi\big(f\big(s,x(s),x'(s)\big) - f\big(s,y(s),y'(s)\big)\big)\;\mathrm{d}s\right| \le a\,\varepsilon/2\,,\end{aligned}$$

for $t \in J$ and $x \in B$ such that $x(s) - y(s) \in U$ and $x'(s) - y'(s) \in U$ for $s \in J$.

From this we deduce that the operator $F$ is continuous.

Now we shall prove the following:

**LEMMA 3.** *If $V \subset B$ and*

$$V \subset \overline{\mathrm{conv}}\big(F(V) \cup \{0\}\big)\,, \tag{6}$$

*then $V$ is relatively compact in $C_{1w}(J,E)$.*

P r o o f. As $V \subset B$, the sets $V$ and $V' = \{x' :\ x \in V\}$ are uniformly bounded and strongly equicontinuous.

Since for convex subsets of $E$ the closure in the norm topology coincides with the weak closure (cf. [5, Th. II.1]), it is clear from (6) that

$$\begin{aligned}V(t) &\subset \overline{\mathrm{conv}}\left(\left\{\int_0^a G(t,s)f\big(s,x(s),x'(s)\big)\;\mathrm{d}s :\ x \in V\right\} \cup \{0\}\right),\\ V'(t) &\subset \overline{\mathrm{conv}}\left(\left\{\int_0^a \frac{\partial G}{\partial t}(t,s)f\big(s,x(s),x'(s)\big)\;\mathrm{d}s :\ x \in V\right\} \cup \{0\}\right), \qquad (t \in J)\,.\end{aligned} \tag{7}$$

Fix $t \in J$. We divide the interval $J$ into $m$ parts $0 = t_0 < t_1 < \cdots < t_m = a$ in such a way that $\Delta t_i = t_i - t_{i-1} = \frac{a}{m}$ $(i = 1, \ldots, m)$. Put $T_i = \langle t_{i-1}, t_i \rangle$ and $h_i = \sup\{|G(t,s)| : s \in T_i\} = |G(t,s_i)|$, where $s_i \in T_i$. Since for each $x \in V$

$$\int_0^a G(t,s) f\big(s, x(s), x'(s)\big)\, ds = \sum_{i=1}^m \int_{T_i} G(t,s) f\big(s, x(s), x'(s)\big)\, ds$$

$$\in \sum_{i=1}^m \Delta t_i \, \overline{\mathrm{conv}}\{G(t,s) f\big(s, x(s), x'(s)\big) : x \in V, \ s \in T_i\}$$

$$\subset \sum_{i=1}^m \Delta t_i \, \overline{\mathrm{conv}}\Big( \bigcup_{|\lambda| \le h_i} \lambda f\big(J \times V(J) \times V'(J)\big) \Big),$$

from (7), (4) and corresponding properties of $\beta$ it follows that

$$\beta\big(V(t)\big) \le \beta\Big(\Big\{ \int_0^a G(t,s) f\big(s, x(s), x'(s)\big)\, ds : x \in V \Big\}\Big)$$

$$\le \sum_{i=1}^m \Delta t_i h_i \beta\big(f\big(J \times V(J) \times V'(J)\big)\big)$$

$$\le \sum_{i=1}^m \Delta t_i |G(t,s_i)| \big(p\beta\big(V(J)\big) + q\beta\big(V'(J)\big)\big).$$

On the other hand, if $m \to \infty$, then

$$\sum_{i=1}^m \Delta t_i |G(t,s_i)| \to \int_0^a |G(t,s)|\, ds.$$

Thus

$$\beta\big(V(t)\big) \le \int_0^a |G(t,s)|\, ds \, \big(p\beta\big(V(J)\big) + q\beta\big(V'(J)\big)\big),$$

and by (3)

$$\beta\big(V(t)\big) \le \frac{a^2}{8} \big(p\beta\big(V(J)\big) + q\beta\big(V'(J)\big)\big).$$

Analogously, we can prove that

$$\beta\big(V'(t)\big) \le \frac{a}{2} \big(p\beta\big(V(J)\big) + q\beta\big(V'(J)\big)\big).$$

By Lemma 1 the above inequalities imply that

$$\max\big(\beta(V(J)), \tfrac{a}{4}\beta(V'(J))\big) \le \frac{a^2}{8}\big(p\beta(V(J)) + q\beta(V'(J))\big)$$
$$\le \Big(p\frac{a^2}{8} + q\frac{a}{2}\Big) \max\big(\beta(V(J)), \tfrac{a}{4}\beta(V'(J))\big).$$

As $p\frac{a^2}{8} + q\frac{a}{2} < 1$, this shows that $\beta(V(J)) = \beta(V'(J)) = 0$, i.e. the sets $V(J)$ and $V'(J)$ are relatively weakly compact in $E$.

By Ascoli's theorem, this proves that the sets $V$ and $V'$ are relatively compact in $C_w(J,E)$, so that $V$ is relatively compact in $C_{1w}(J,E)$. This ends the proof of Lemma 3.

Now we return to the proof of the Theorem. We define a sequence $(y_n)$ by $y_0 = 0$, $y_{n+1} = F(y_n)$ $(n \in \mathbb{N})$. Let $Y = \{y_n : n \in \mathbb{N}\}$. As $Y \subset B$ and $Y = F(Y) \cup \{0\}$, from Lemma 3 it follows that $Y$ is relatively compact in $C_{1w}(J,E)$. Denote by $Z$ the set of all limit points of $(y_n)$. We shall show that $Z = F(Z)$. If $y \in F(Z)$, then $y = F(x)$ for some $x \in Z$. Thus there exists a subnet $(x_\alpha)$ of $(y_n)$ such that $x_\alpha \to x$. From the continuity of $F$ it follows that $F(x_\alpha) \to F(x) = y$. As $(F(x_\alpha))$ is also a subnet of $(y_n)$, we see that $y \in Z$. Conversely, let $y \in Z$. Then there exists a subnet $(y_\alpha)$ of $(y_n)$ such that $y_\alpha \to y$ and $y_\alpha = F(x_\alpha)$, where $(x_\alpha)$ is also a subnet of $(y_n)$. Since the set $Y$ is relatively compact, $(x_\alpha)$ has a convergent subnet $(x_\gamma)$. Let $x = \lim x_\gamma$. Then $x \in Z$ and $y_\gamma = F(x_\gamma) \to F(x)$. On the other hand, $y_\gamma \to y$. Hence $y = F(x) \in F(Z)$.

Let us put $R(X) = \overline{\text{conv}} F(X)$ for $X \subset B$, and let $\Omega$ denote the family of all subsets $X$ of $B$ such that $Z \subset X$ and $R(X) \subset X$. From (5) it is clear that $B \in \Omega$. Denote by $V$ the intersection of all sets of the family $\Omega$. As $Z \subset V$, $V$ is nonempty and $Z = F(Z) \subset R(Z) \subset R(V)$. Since $R(V) \subset R(X) \subset X$ for all $X \in \Omega$, $R(V) \subset V$, and therefore $V \in \Omega$. Moreover, $R(R(V)) \subset R(V)$, and hence $R(V) \in \Omega$. Consequently, $V = R(V)$, i.e. $V = \overline{\text{conv}} F(V)$. In view of Lemma 3 this implies that $V$ is a compact subset of $B$. Applying now the Schauder-Tychonoff fixed point theorem to the mapping $F|_V$, we conclude that there exists $x \in V$ such that $x = F(x)$. It is clear that $x$ satisfies (2) and hence $x$ is a weak solution of (1).


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