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ON THE EMBEDDING $H^w \subset V_p$

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ABSTRACT. In this paper a necessary and sufficient condition for the embedding $H^w \subset V_p$ are given.

Let p be a given constant for which $1 \le p < \infty$ holds. Then the function f defined on the interval $[a;b]$ is said to have finite p-variation if

$$V_p(f;a,b) = \sup_G \left\{ \sum_{k=1}^{N} |f(s_k) - f(s_{k-1})|^p \right\}^{1/p} < \infty ,$$

where the supremum is taken over all decompositions $G = \{s_k\}$ of the interval $[a;b]$ with $a \le s_0 < s_1 < \cdots < s_N \le b$. The set of all functions with finite p-variation on $[a;b]$ will be denoted by V_p.

The functions from V_p are important in the theory of Fourier series as we can see in the following proposition.

PROPOSITION. *Let $p > 1$ and let f be a continuous function with finite p-variation on $[0;2\pi]$. Then the trigonometrical Fourier series of f is uniformly convergent.* (See [2, p. 283].)

Embeddings of Lebesgue spaces into V_p are studied in papers of G. H. Hardy and J. E. Littlewood [3], A. P. Terjokhin [6], P. L. Uljanov [7] and others.

Remark. Just for simplicity we note, that the whole investigation will be proceed on the interval $[0;1]$.

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DEFINITION 1. *Any function* $w(t)$ *defined, continuous and nondecreasing on* $[0;\infty[$ *is called a modulus of continuity if* $w(0) = 0$ *and* $w(t_1 + t_2) \leq w(t_1) + w(t_2)$ *for any nonnegative* t_1 *and* t_2.

DEFINITION 2. *For any function* f *continuous on* $[0;1]$ *we define the modulus of continuity of* f *as follows*

$$w(t,f) = \sup_{\substack{0<h\leq t \\ 0\leq x\leq 1-h}} |f(x+h) - f(x)|\,. \tag{1}$$

DEFINITION 3. *Let* w *be a modulus of continuity. By* H^w *we denote the class of all functions* f *continuous on* $[0;1]$ *for which the moduli of continuity* (1) *satisfy the following condition*

$$w(t,f) \leq c \cdot w(t)\,,$$

where $w(t)$ *is a given modulus of continuity and* c *is some positive constant.*

DEFINITION 4. *Let* $0 < r \leq 1$. *For* $w(t) = a \cdot t^r$ *denote by* H^r *the set* H^{t^r}, *where* a *is some positive constant. It is a Hölder class of functions.*

LEMMA 1. *The inequality*

$$w(c \cdot t) \leq (c+1) \cdot w(t)$$

holds for any positive c *and for any modulus of continuity* $w(t)$.
(See [4, p. 177].)

LEMMA 2. *Let* $t \in [0;1]$. *Then for arbitrary modulus of continuity* $w(t) \not\equiv 0$ *there exists a concave modulus of continuity* $w^*(t)$ *such that* $w(t) \leq w^*(t) \leq 2 \cdot w(t)$. (See [4, pp. 182–183].)

Remark 1. On the symbols O and o see e.g. in [1].

Remark 2. From Lemma 2 we get that both of the functions $w(t)$ and $w^*(t)$ define the same set H^w. Therefore we can consider the modulus of continuity $w(t)$ to be concave.

Remark 3. According to the expression (1) and Remark we are interested in $t \in [0;1]$.

DEFINITION 5. *We shall say that the function* $g\colon \mathbb{R} \to \mathbb{R}$ *preserves the convergence of the series* $\sum a_n$ *if from the convergence of this series there follows the convergence of the series* $\sum g(a_n)$.

LEMMA 3. *The function $g\colon \mathbb{R} \to \mathbb{R}$ preserves the convergence of the series $\sum a_n$ if and only if there exists a real constant b such that $g(x) = b \cdot x$ holds for any x from some neighbourhood of zero.* (See [5, pp. 84–85].)

THEOREM. *Let p be a given constant from $[1;\infty[$ and $w(t)$ be a given modulus of continuity.*

a) *If $w(t) = O\{t^{1/p}\}$ for $t \to 0+$, then the embedding $H^w \subset V_p$ takes place.*
b) *If $w(t) \neq O\{t^{1/p}\}$ and, moreover, $t^{1/p} = o\{w(t)\}$ for $t \to 0+$, then there exists a function $f \in H^w$ such that $f \notin V_p$.*

Proof. Let $f \in H^w$ and $w(t) = O\{t^{1/p}\}$. Then there exists some positive constant d such that

$$w(t) \le d \cdot t^{1/p}$$

holds for any $t \in [0;1]$. For every decomposition G of $[0;1]$ we have

$$\sum_{k=1}^{N} |f(x_k) - f(x_{k-1})|^p \le \sum_{k=1}^{N} w^p(x_k - x_{k-1}, f) \le \sum_{k=1}^{N} c^p \cdot w^p(x_k - x_{k-1})$$
$$\le (c \cdot d)^p \sum_{k=1}^{N} |x_k - x_{k-1}|^{p/p} = (c \cdot d)^p .$$

Therefore $f \in V_p$.

Now let we have $t^{1/p} = o\{w(t)\}$. According to Lemma 2 and Remark 2 we can put $w(t) = w^*(t)$. From the continuity and concavity of $w(t)$ there follows an existence of some positive t_0 such that $w(t)$ will be increasing on $[0;t_0]$. If $t_0 > 1$, then according to Remark 3 we can put $t_0 = 1$. Denote $T_0 = w(t_0)$. Then for the function $w(t)$ there exists an inverse function $w_{-1}(t)$, which is defined on $[0;T_0]$. From the assumption $t^{1/p} = o\{w(t)\}$ we get

$$w_{-1}(t) = o\{t^p\} \qquad \text{for} \quad t \to 0+, \ \ t \in [0;T_0] .$$

According to Lemma 3 there exists a sequence $\{t_n\}$, $t_1 \le T_0$, $t_n \to 0+$, such that

$$\sum_{n=1}^{\infty} (t_n)^p = \infty \tag{2}$$

and

$$\sum_{n=1}^{\infty} w_{-1}(t_n) = S ,$$

where S is some positive constant.

We construct the decomposition G of $[0;1]$ using the following points

$$x_{2i} = (1/S)\sum_{k=1}^{i} w_{-1}(t_k) \quad \text{and} \quad x_{2i+1} = x_{2i} + (1/2S)\cdot w_{-1}(t_{i+1})$$

for $i = 0, 1, \dots$, putting $\sum_{k=1}^{0} w_{-1}(t_k) = 0$.

Define the function f as follows:

$$f(x) = \begin{cases} w\big(2S[x - x_{2i}]\big) & \text{if } x_{2i} \le x \le x_{2i+1}\,, \\ w\big(2S[x_{2i+2} - x]\big) & \text{if } x_{2i+1} \le x \le x_{2i+2}\,, \\ 0 & \text{if } x = 1\,, \end{cases}$$

for $i = 0, 1, \dots$.

We can see that the function f is continuous on $[0;1]$. We will now prove

$$w(t, f) \le c \cdot w(t)\,, \tag{3}$$

where $c = 2S + 1$. According to (1) we shall investigate the following absolute value

$$|f(x+h) - f(x)|\,, \qquad h \in [0;1]\,. \tag{4}$$

The supremum of (4) will be attained in some interval of monotonicity of function f. It is equivalent to the investigation on the corresponding interval of increasity of this function f. Then there exists a natural number i such that

$$x_{2i} \le x \le x_{2i+1} - h\,.$$

Using (1) we obtain

$$\sup_{0\le x\le 1-h} |f(x+h) - f(x)| = \sup_{0\le x\le 1-h} \big|w\big(2S[x+h-x_{2i+1}]\big) - w\big(2S[x - x_{2i+1}]\big)\big|\,.$$

From the Definition 1 we get

$$w(x+h) \le w(x) + w(h) \qquad \text{for any} \quad x \ge 0 \quad \text{and} \quad h \ge 0$$

and

$$w(x+h) - w(x) \le w(h)\,.$$

Therefore we obtain the supremum if $x - x_{2i+1} = 0$, i.e.

$$\sup_{0 \le x \le 1-h} |f(x+h) - f(x)| = w(2Sh).$$

According to (1) and using Lemma 1 we have

$$w(t, f) = \sup_{0 \le h \le t} w(2Sh) = w(2St) \le (2S+1) \cdot w(t).$$

It proves (3). Therefore $f \in H^w$.

Finally we shall prove $f \notin V_p$. For a given constant $p \ge 1$ we have

$$\sum_{k=1}^{\infty} |f(x_k) - f(x_{k-1})|^p$$
$$= \sum_{k=1}^{\infty} \{|f(x_{2k+1}) - f(x_{2k})|^p + |f(x_{2(k+1)}) - f(x_{2k+1})|^p\} = \sum_{k=1}^{\infty} 2t_k^p.$$

Using (2) we obtain

$$V_p(f; 0, 1) \ge 2 \sum_{k=1}^{\infty} (t_k^p)^{1/p} = \infty.$$

Therefore $f \notin V_p$. Proof of the Theorem is complete.

COROLLARY. *Let p be a given constant from $[1; \infty[$. Then the condition $r \ge 1/p$ will be the necessary and sufficient condition for embedding $H^r \subset V_p$.*

Proof.

Sufficiency we obtain from the proof of Theorem with respect to Definition 4.

Necessity. Let $r < 1/p$. Then there exists some positive $c < 1$ such that $r = c/p$. Hence according to Definition 4 there follows $w(t) = a \cdot t^{c/p}$, i.e. $w_{-1}(t) = a^{-p/c} \cdot t^{p/c}$. Choosing $x_k = k^{-1/p}$ from the proof of Theorem we get

$$\sum_{k=1}^{\infty} x_k^p = \sum_{k=1}^{\infty} k^{-1} = \infty$$

and

$$\sum_{k=1}^{\infty} w_{-1}(x_k) = \sum_{k=1}^{\infty} a^{-p/c} \cdot k^{(-1/p)(p/c)} = a^{-p/c} \cdot \sum_{k=1}^{\infty} k^{-1/c} < \infty.$$

The proof of the Corollary is complete.

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