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NOTE TO THE LAGRANGE STABILITY OF EXCITED PENDULUM TYPE EQUATIONS

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ABSTRACT. Higher-order pendulum-type equations under forcing are studied with respect to the Lagrange stability. The convergence result for the third-order equation is obtained as a partial answer for the so-called problem of Barbashin.

Introduction

This note was stimulated by two recent papers [27] and [15] dealing with the periodically excited mathematical and physical pendulum, respectively. In the first, there is positively solved the problem of J. Moser [14, p. 11] concerning the Lagrange stability on the cylinder, i.e. the boundedness of trajectories with respect to the angular velocity components. In the second one, the conjecture of J. Mawhin [12] on the existence of harmonics to the damped equation with the zero mean value of the applied torque is negatively answered by a counterexample. This means that in such a case all solutions are unbounded, according to the well-known Massera transformation theorem, and consequently a natural question arises whether the Lagrange stability (in general) can be proved for sufficiently large positive values, say a, of a viscous damping constant.

Since we already know (see [1]) that this can be affirmatively answered for $a > \alpha$, a suitable positive constant approaching $\sqrt{b/\pi}$ in the best case, where b is a positive multiple coefficient at the restoring torque, the above problem can be still specified as follows: "what is the sharpest lower estimate of α?"

In [16], R. Ortega has recently studied the stability of periodic solutions when only $a > 2\sqrt{b}$.

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In [1], we have already refined the related result of G. Seifert [22], provided only

$$\limsup_{t\to\infty} |p(t)| < \infty \quad \text{and} \quad \limsup_{t\to\infty} \left| \int_0^t p(s)\, \mathrm{d}s \right| < \infty ,$$

where $p(t)$ is a (not necessarily periodic) forcing term.

Although we are able to show in this paper that α can approach $\sqrt{b/(2+\pi)}$ in the best case, the sharpest estimate remains yet unknown.

In [25], F. Tricomi has shown that if the forcing term is constant (and thus considerable as a part of the restoring term), namely $p(t) \equiv p$, $0 < p < 1$, then a critical value, say $\hat{\alpha} = \hat{\alpha}(p)$, exists such that $(0 <)\ a < \hat{\alpha}$, $(b = 1)$, implies the existence of a 2π-periodic (unbounded) solution of the second kind. This partially may explain the role of the interval $(0, \alpha\rangle$ $(\supset (0, \hat{\alpha}))$.

We believe that the above problem is also of a big practical importance, because e.g. in electrical engineering it has the analogy with respect to the Josephson functions and especially the phase-locked loops; for the corresponding literature see e.g. [8], where also the detailed simulating circuit diagram for the phase-locked loop can be found. The higher-order analogies of damped pendulum equations can also be applied in modelling for the automatic control in TV systems realized by means of the RC-filters (see [23]). Therefore, we would like to enlarge here our results to the nth-order (especially for $n = 3$) equations, too.

Let us note that the clarification of the phase space to the third-order pendulum-type equations is usually called the problem of Barbashin (see [5, p. 286], [11], [3]).

There are not so many mathematical contributions (cf. [13]) on an excited nonlinear pendulum except those where the existence of harmonics is studied (for the survey of results see again [13]) or those where various aspects of chaos are treated as period doubling (cf. e.g. [8]) or the structure of the fractal basin boundaries (see e.g. [9], [18] and the references therein).

For further interesting information see [6], [10, pp. 201–204], [17] and [20, pp. 195–197], where mainly the Melnikov function technique is applied in the frame of the perturbation theory.

Boundedness of derivatives

Consider the equation

$$x'' + g(x)x' + h(x) = p(t) , \tag{1}$$

where $g(x)$, $h(x) \in C(\mathbb{R}^1)$ and $p(t) \in C(\langle 0, \infty))$.

Lemma 1. *Let nonnegative constants H, P and a positive constant a exist such that*

$$|p(t)| \le P \qquad \text{for} \quad t \ge 0, \tag{2}$$

$$|h(x)| \le H \qquad \text{for all} \quad x, \tag{3}$$

$$a \le g(x) \qquad \text{for all} \quad x. \tag{4}$$

Then

$$\limsup_{t\to\infty} |x'(t)| \le D' := (H+P)/a \tag{5}$$

is satisfied for all solutions $x(t)$ of (1).

P r o o f. Let $x(t)$ be a solution of (1). Because of

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[x'^2(t)\big] &= 2x'(t)x''(t) = 2x'(t)\big[p(t) - g\big(x(t)\big)x'(t) - h\big(x(t)\big)\big]\\ &\le 2|x'(t)|\big[-a|x'(t)| + H + P\big],\end{aligned}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x'^2(t)\big] \le -2\varepsilon(H+P+a\varepsilon) < 0 \qquad \text{for} \quad |x'(t)| \ge (H+P)/a + \varepsilon,$$

where ε is an arbitrary positive constant, and consequently (5) must be satisfied.

R e m a r k 1. The assertion of Lemma 1 remains obviously valid for the equation

$$x'' + g(t,x,x')x' + h(t,x,x') = p(t,x,x'),$$

where $g, h, p \in C\big(\langle 0,\infty) \times \mathbb{R}^2\big)$, $a \le g(t,x,y)$, $|h(t,x,y)| \le H$ and $|p(t,x,y)| \le P$ for $t \ge 0$ and all x, y.

R e m a r k 2. Although the sufficient conditions are known (see [21]) for the uniform ultimate boundedness of the derivatives $x^{(j)}(t)$, $j = 1,\dots,n-1$, of all solutions $x(t)$ to the equation

$$x^{(n)} + \sum_{k=1}^{n-1} g_k\big(x^{(n-k-1)}\big)x^{(n-k)} + h(x) = p(t), \tag{6}$$

where $h(x) \in C(\mathbb{R}^1)$ satisfies (3), $g_k(y) \in C(\mathbb{R}^1)$, $k = 1,\dots,n-1$, and $p(t) \in C\big(\langle 0,\infty)\big)$ satisfies (2), it is rather cumbersome to find the estimating constants explicitly; for $n = 3$ see [26].

Remark 3. In the special case of (6), namely

$$x^{(n)} + \sum_{k=1}^{n-1} a_k x^{(n-k)} + h(x) = p(t)\,, \tag{7}$$

where a_k, $k = 1, \dots, n-1$, are positive constants with the Hurwitz structure, i.e. all roots of the "characteristic" polynomial

$$\lambda^{n-1} + \sum_{k=1}^{n-1} a_k \lambda^{n-k-1} \tag{8}$$

have negative real parts, we are able (see [2]) to estimate ultimately the jth derivatives by means of the following constants:

$$D^{(j)} := (H+P) \sum_{k=0}^{n-2} 2^k \|A\|^k / (-M)^{k+1}\,, \qquad j = 1, \dots, n-1\,, \tag{9}$$

where $\|A\| = \max\Bigl(1, \sum\limits_{k=1}^{n-1} a_k\Bigr)$ and M is the maximum of the real parts of the roots of (8).

Remark 4. If all roots of the polynomials

$$\lambda^{n-i} + \sum_{k=1}^{n-i} a_k \lambda^{n-k-i}$$

are negative single for $i = 1, \dots, \max(1, n-4)$, then the estimating constants (9) can be still specified as follows (see [4]):

$$D^{(j)} := j(H+P)/a_{n-j} \qquad \text{for } j = 1, \dots, n-1\,. \tag{10}$$

For $n \leq 5$ it is even sufficient to assume (see [4]) that the coefficients a_k in (8) satisfy the Routh-Hurwitz conditions in order to obtain (10).

Remark 5. One can readily check that for $n = 2$ both (9) and (10) reduce to (5). For more detail and further information concerning such estimates see [2] and [4]. For example, applying the approach of [2], one can appropriately modify estimates (9) for (6).

Boundedness of solutions

Consider equation (6) again and assume that the jth derivatives, $j = 1, \dots, n-1$, of all solutions $x(t)$ of (6) are ultimately bounded by means of the constants $D^{(j)}$, $j = 1, \dots, n-1$, i.e.

$$\limsup_{t\to\infty} |x^{(j)}(t)| \le D^{(j)} \qquad \text{for} \quad j = 1, \dots, n-1\,; \tag{11}$$

for the special cases of (6) see Remark 2, Remark 3 and Remark 4.

Furthermore, let two sequences of intervals $\{\langle X_i^-, Y_i^-\rangle\}$ and $\{\langle X_j^+, Y_j^+\rangle\}$ exist such that

$$\lim_{i\to\infty} Y_i^- = -\infty\,, \qquad \lim_{j\to\infty} X_j^+ = \infty\,,$$

and $h(x)\operatorname{sgn} x > 0$ for $x \in (X_i^-, Y_i^-)$, $i \in \mathbb{N}$, as well as for $x \in (X_j^+, Y_j^+)$, $j \in \mathbb{N}$, where $h(X_i^-) = h(Y_i^-) = h(X_j^+) = h(Y_j^+) = 0$, $i, j \in \mathbb{N}$.

The following lemma is essential in order to precise the criteria for the Lagrange stability of (6).

Lemma 2. *Let ε be a positive constant and let $x(t)$ be a solution of* (6) *such that*

$$\limsup_{t\to\infty} x(t) = \infty\,. \tag{12}$$

Then

$$\int_{T_1}^{T_2} h\big(x(t)\big)\,\mathrm{d}t \ge \frac{1}{D' + \varepsilon} \int_X^Y h(s)\,\mathrm{d}s\,, \tag{13}$$

where T_1, T_2 $(T_1 < T_2)$ are suitable (in order that $h\big(x(t)\big) > 0$ for $t \in (T_1, T_2)$) sufficiently large positive reals with

$$h\big(x(T_1)\big) = h\big(x(T_2)\big) = 0\,, \qquad X = x(T_1) < x(T_2) = Y\,.$$

Proof. Let ε be a positive constant. Then a positive number T_0 exists such that $|x'(t)| \le D' + \varepsilon$ for $t \ge T_0$ and $T_1, T_2 \in \langle T_0, \infty)$, $T_1 < T_2$, exist with $x(T_1) < x(T_2)$, $h\big(x(T_1)\big) = h\big(x(T_2)\big) = 0$, $h\big(x(t)\big) > 0$ for $t \in (T_1, T_2)$.

Let m be an arbitrary natural number. Then a system of intervals $\big\{\langle u_k, v_k\rangle\big\}_{k=1}^m$ exists on $\langle T_1, T_2\rangle$ such that the intersection of each pair consists at most of the boundary points with the property

$$x(u_k) = X + \frac{Y - X}{m}(k-1)\,, \quad x(v_k) = X + \frac{Y-X}{m}k\,, \qquad k = 1, \dots, m\,,$$

and

$$x(u_k) \le x(t) \le x(v_k) \qquad \text{for} \quad t \in \langle u_k, v_k\rangle .$$

Thus, we have

$$\int_{T_1}^{T_2} h\big(x(t)\big)\, dt \ge \sum_{k=1}^{m} \int_{u_k}^{v_k} h\big(x(t)\big)\, dt = \sum_{k=1}^{m} h(\xi_k)(v_k - u_k)\,,$$

where $\xi_k \in \big(x(u_k), x(v_k)\big) = \big(X + \frac{Y-X}{m}(k-1),\, X + \frac{Y-X}{m}k\big)$.

Because of

$$\frac{Y-X}{m} = x(v_k) - x(u_k) = x'(\eta_k)(v_k - u_k) \leqq (D' + \varepsilon)(v_k - u_k)\,,$$

where $\eta_k \in (u_k, v_k)$, we have furthermore that

$$v_k - u_k \ge \frac{Y-X}{m(D'+\varepsilon)}\,,$$

and consequently

$$\int_{T_1}^{T_2} h\big(x(t)\big)\, dt \ge \frac{Y-X}{m(D'+\varepsilon)} \sum_{k=1}^{m} h(\xi_k) = \frac{1}{D'+\varepsilon} \sum_{k=1}^{m} h(\xi_k) \frac{Y-X}{m}\,.$$

Therefore, in view of the continuity of $h(x)$ and the definition of $\int_X^Y h(s)\, ds$, we arrive at (13) with respect to

$$\lim_{m\to\infty} \sum_{k=1}^{m} h(\xi_k) \frac{Y-X}{m} = \int_X^Y h(s)\, ds\,.$$

R e m a r k 6. One can readily check that the following assertion holds as well. If ε is a positive constant and

$$\liminf_{t\to\infty} x(t) = -\infty$$

for a solution $x(t)$ of (6), then

$$\int_{T_1}^{T_2} h\big(x(t)\big)\, dt \le \frac{1}{D'+\varepsilon} \int_X^Y h(s)\, ds\,,$$

where T_1, T_2 ($T_1 < T_2$) are suitable (in order that $h\big(x(t)\big) < 0$ for $t \in (T_1, T_2)$) sufficiently large positive reals with $h\big(x(T_1)\big) = h\big(x(T_2)\big) = 0$, $Y = x(T_1) > x(T_2) = X$.

Now, we can give the principal result of this paper.

THEOREM 1. *Let $h(x)$ be an oscillatory function in the above sense and let* (11) *be satisfied for all solutions of* (6). *If, furthermore, a nonnegative constant P_0 exists such that*

$$\left|\int_{t_0}^{t} p(s)\,\mathrm{d}s\right| = P_0 \qquad \text{for all} \quad t \geq t_0 \geq 0\,, \tag{14}$$

and

$$\begin{aligned}&\Delta > D' + P_0 \qquad \text{for} \quad n = 2\,,\\ &\Delta > A_{n-2}D' + 2\left(\sum_{k=1}^{n-3} A_k D^{(n-k-1)} + D^{(n-1)}\right) + P_0 \qquad \text{for} \quad n \geqq 3\,,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \Big(\sum_{k=1}^{0} \cdot = 0\Big)\,,\end{aligned} \tag{15}$$

where

$$\Delta := \inf_{i,j}\left(a|Y_{i,j}^{-,+} - X_{i,j}^{-,+}| + \frac{1}{D'}\int_{X_{i,j}^{-,+}}^{Y_{i,j}^{-,+}} |h(s)|\,\mathrm{d}s\right), \tag{16}$$

$$(\,0 <\,)\; a := \inf_{x\in\mathbb{R}^1} g_{n-1}(x)\,, \tag{17}$$

$$A_k := \sup_{|y|\leq D^{(n-k-1)}} g_k(y)\;(\,< \infty\,)\,, \qquad k = 1,\dots,n-2\,, \tag{18}$$

then all solutions of (6) *are bounded.*

P r o o f. Let $x(t)$ be a solution of (6) satisfying (12); the case

$$\liminf_{t\to\infty} x(t) = -\infty$$

can be treated quite analogously (see Remark 6). Let ε be a positive constant. According to (11), there exists a nonnegative real T_x such that

$$|x^{(j)}(t)| \leq D^{(j)} + \varepsilon \qquad \text{for} \quad t \geq T_x\,, \;\; j = 1,\dots,n-1\,.$$

Furthermore, there exist numbers T_1, $T_2 \geq T_x$ having the same properties as in Lemma 2 and $x'(T_1) \geq 0$, $x'(T_2) \geq 0$. Define

$$A_k^{(\varepsilon)} := \sup_{|y|\leq D^{(n-k-1)}+\varepsilon} g_k(y)\,, \qquad k = 1,\dots,n-2\,.$$

Substituting $x(t)$ into (6) and integrating the obtained identity from T_1 to T_2, we get

$$x^{(n-1)}(T_2) - x^{(n-1)}(T_1) + \sum_{k=1}^{n-1} \int\limits_{x^{(n-k-1)}(T_1)}^{x^{(n-k-1)}(T_2)} g_k(s)\ \mathrm{d}s + \int\limits_{T_1}^{T_2} h\big(x(t)\big)\ \mathrm{d}t = \int\limits_{T_1}^{T_2} p(t)\ \mathrm{d}t\,.$$

Since there is still

$$\begin{gathered} x(T_2) - x(T_1) = Y - X > 0\,, \qquad |x'(T_2) - x'(T_1)| \le D' + \varepsilon\,, \\ |x^{(j)}(T_2) - x^{(j)}(T_1)| \le 2\big(D^{(j)} + \varepsilon\big)\,, \qquad j = 2,\dots,n-1\,, \end{gathered}$$

we arrive at the inequality (cf. (13), (14), (17), (18))

$$\begin{aligned} & a\big[x(T_2) - x(T_1)\big] + \frac{1}{D'+\varepsilon} \int\limits_X^Y h(s)\ \mathrm{d}s \\ & \le \int\limits_{x(T_1)}^{x(T_2)} g_{n-1}(s)\ \mathrm{d}s + \int\limits_{T_1}^{T_2} h\big(x(t)\big)\ \mathrm{d}t \\ & \le A_{n-2}^{(\varepsilon)}(D'+\varepsilon) + 2\left[\sum_{k=1}^{n-3} A_k^{(\varepsilon)}\big(D^{(n-k-1)} + \varepsilon\big) + \big(D^{(n-1)} + \varepsilon\big)\right] + P_0\,, \end{aligned}$$

holding for all positive ε, and that is why

$$a\big[x(T_2)-x(T_1)\big] + \frac{1}{D'}\int\limits_X^Y h(s)\ \mathrm{d}s \le A_{n-2}D' + 2\left(\sum_{k=1}^{n-3} A_k D^{(n-k-1)} + D^{(n-1)}\right) + P_0\,.$$

However, this is a contradiction (cf. (16)) to (15), and consequently $x(t)$ must be bounded.

Since we have $\Delta = a\pi + 2b/D'$ (cf. (5)) for $h(x) = b\sin x$ in (16), where b is a positive constant, we can give immediately the following important consequence of Theorem 1 and Lemma 1.

THEOREM 2. *Equation*

$$x'' + g(x)x' + b\sin x = p(t)$$

is stable in sense of Lagrange, provided (2), (4), (14) *and*

$$a > \frac{(b+P)\{P_0 + [P_0^2 + 4(2b + \pi(b+P))]^{1/2}\}}{2(2b + \pi(b+P))} . \tag{19}$$

Remark 7. For $p(t) \equiv 0$ inequality (19) reduces to $a > \sqrt{b/(2+\pi)}$, which is not a necessary condition, because we know [1] that the same assertion can be proved for every positive a.

Convergence result for $n = 3$

At last, consider equation (7) for $n = 3$, i.e.

$$x''' + a_1 x'' + a_2 x' + h(x) = p(t) , \tag{20}$$

where a_1, a_2 are positive constants, $h(x) \in C(\mathbb{R}^1)$ satisfies (3) and $p(t) \in C(\langle 0, \infty))$ satisfies (2).

It follows from Remark 4 that

$$\limsup_{t\to\infty} |x'(t)| \le D_1 , \quad \limsup_{t\to\infty} |x''(t)| \le D_2 \quad (\Longrightarrow \limsup_{t\to\infty} |x'''(t)| \le D_3) , \tag{21}$$

where

$$D_1 := \frac{1}{a_2}(H+P) , \quad D_2 := \frac{2}{a_1}(H+P) \quad (\Longrightarrow D_3 := 4(H+P))$$

for all solutions $x(t)$ of (20), and consequently $x(t)$ must exist on the whole positive half-line.

Assume additionally that $h(x) \in C^1(\mathbb{R}^1)$, all the zero points $\overline{x}$ of $h(x)$ are isolated, $p(t)$ is differentiable and

$$a_1 a_2 - h'(x) \ge \varepsilon > 0 \quad (\varepsilon \text{ – constant}) , \qquad \liminf_{|x|\to\infty} h'(x) > -\infty , \tag{22}$$

$$\limsup_{|x|\to\infty} |h(x)| > 0 , \tag{23}$$

$$\liminf_{|x|\to\infty} \int_0^x h(s)\,\mathrm{d}s > -\infty , \tag{24}$$

as well as

$$\int_0^\infty |p(t)|\,\mathrm{d}t < \infty , \quad \limsup_{t\to\infty} |p'(t)| < \infty \quad (\overset{[19,p.211]}{\Longrightarrow} \lim_{t\to\infty} p(t) = 0) . \tag{25}$$

THEOREM 3. *Under the above assumptions, a zero point $\overline{x}$ of $h(x)$ exists to every solution $x(t)$ of* (20) *such that*

$$\lim_{t\to\infty}\left[x(t)-\overline{x}\right] = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0 .$$

Proof. Let $x(t)$ be a solution of (20). Substituting $x(t)$ into (20) and multiplying (20) by $x'(t)$, we obtain, after integration from t_1 to t_2, that

$$-\int_{t_1}^{t_2} x''^2(t)\,\mathrm{d}t + a_2\int_{t_1}^{t_2} x'^2(t)\,\mathrm{d}t + \int_{x(t_1)}^{x(t_2)} h(s)\,\mathrm{d}s$$

$$= \left[x'(t_1)x''(t_1) - x'(t_2)x''(t_2)\right] + \frac{a_1}{2}\left[x'^2(t_1) - x'^2(t_2)\right] + \int_{t_1}^{t_2} p(t)x'(t)\,\mathrm{d}t .$$

Similarly, substituting $x(t)$ into (20) and multiplying (20) by $x''(t)$, we obtain, after integration from t_1 to t_2, that

$$a_1\int_{t_1}^{t_2} x''^2(t)\,\mathrm{d}t - \int_{t_1}^{t_2} h'\big(x(t)\big)x'^2(t)\,\mathrm{d}t$$

$$= \frac{1}{2}\left[x''^2(t_1) - x''^2(t_2)\right] + \frac{a_2}{2}\left[x'^2(t_1) - x'^2(t_2)\right]$$

$$+ \left[h\big(x(t_1)\big)x'(t_1) - h\big(x(t_2)\big)x'(t_2)\right] + \int_{t_1}^{t_2} p(t)x''(t)\,\mathrm{d}t .$$

After the summation of the first identity multiplied by a_1 and the second, we arrive at the relation (cf. (25))

$$\int_{t_1}^{t_2} \left[a_1a_2 - h'\big(x(t)\big)\right]x'^2(t)\,\mathrm{d}t$$

$$\le -a_1\int_{x(t_1)}^{x(t_2)} h(s)\,\mathrm{d}s + \sup_{t\ge 0}\big(|x''(t)| + a_1|x'(t)|\big)\int_0^\infty |p(t)|\,\mathrm{d}t$$

$$+ \Big|\frac{1}{2}\left[x''^2(t_1) - x''^2(t_2)\right] + \frac{1}{2}(a_2 + a_1^2)\left[x'^2(t_1) - x'^2(t_2)\right]$$

$$+ a_1\left[x'(t_1)x''(t_1) - x'(t_2)x''(t_2)\right] + \left[h\big(x(t_1)\big)x'(t_1) - h\big(x(t_2)\big)x'(t_2)\right]\Big| ,$$

and consequently (observe that t_1 is fixed and so $x(t_1)$ is finite)

$$\varepsilon \int_{t_1}^{\infty} x'^2(t)\, dt \le \int_{t_1}^{\infty} \left[a_1 a_2 - h'\big(x(t)\big)\right] x'^2(t)\, dt < \infty$$

because of (21), (22), (24) and (25).

This implies, according to the well-known lemma of Barbalat (see e.g. [19, p. 211]) that

$$\lim_{t\to\infty} x'(t) = 0. \tag{26}$$

It follows, furthermore, from the first identity that

$$\int_{t_1}^{\infty} x''^2(t)\, dt < \infty,$$

and by the same reason (Barbalat's lemma)

$$\lim_{t\to\infty} x''(t) = 0. \tag{27}$$

Now, substituting $x(t)$ into (20) and derivating the obtained identity, we come to the inequality (cf. (25)–(27) and the second part in (22))

$$\begin{aligned} \limsup_{t\to\infty} |x''''(t)| &\le \limsup_{t\to\infty} a_1 |x'''(t)| + \limsup_{t\to\infty} |p'(t)| \\ &\le \limsup_{t\to\infty} a_1 \big|h\big(x(t)\big)\big| + \limsup_{t\to\infty} |p'(t)| < \infty. \end{aligned}$$

This implies (see [7, p. 161]) jointly with (26), (27) that also

$$\lim_{t\to\infty} x'''(t) = 0,$$

and consequently (cf. (25) again)

$$\lim_{t\to\infty} h\big(x(t)\big) = 0,$$

i.e. (see (23))

$$\lim_{t\to\infty} x(t) = \overline{x}.$$

This completes the proof.

COROLLARY. *For the same purpose, condition* (24) *together with* (23) *and the second part in* (22) *can be evidently replaced by the one ensuring the boundedness of all solutions, namely* (*see* (14) - (16) *for* $n = 3$)

$$\inf_{i,j}\left(a_2|Y_{i,j}^{-,+} - X_{i,j}^{-,+}| + \frac{1}{D_1}\int_{X_{i,j}^{-,+}}^{Y_{i,j}^{-,+}} |h(s)|\,\mathrm{d}s\right) > a_1D_1 + 2D_2 + P_0\,.$$

Remark 8. Observe that Theorem 3 applies, for example, if $a_1a_2 > b > 0$ and $h(x) := b\sin x$.

If (23) is, however, replaced by

$$\lim_{|x|\to\infty} h(x) = 0\,, \tag{28}$$

then we are not able to eliminate the case

$$\lim_{t\to\infty} |x(t)| = \infty\,, \tag{29}$$

as far as $h(x)\operatorname{sgn} x \le 0$ for all x, i.e. the Lagrange stability need not hold. Indeed, imagine for a moment that there are no zero points of $h(x)$, but (28) takes place. Then, necessarily (29) holds, and consequently also equation (20) having a finite number of zero points of $h(x)$ must have the same property (29) (as far as $h(x)\operatorname{sgn} x \le 0$ for all x and (28) are satisfied).

Remark 9. Although Theorem 3 does not apply to (20) for $h(x) := \sin x - k$, where $k \in (0,1)$, Corollary says that this difficulty can be overcome, when e.g. $p(t) \equiv 0$, $a_1 = 1 < a_2$ and especially

$$\pi - 2\arcsin k + \frac{1}{1+k}\big[2\cos(\arcsin k) - k(\pi - 2\arcsin k)\big] \ge 5(1+k)\,.$$

The last inequality is satisfied for $k \le 0.0117$.

Remark 10. If $h(x)\operatorname{sgn} x \ge 0$ for all x (see Corollary), then our result reduces to the one in [24]. In fact, K. E. Swick has not assumed that (2), (3), (23) and the second parts in (22) and (25) are satisfied.

Remark 11. One can readily check in the proof of Theorem 3 that $x'(t), x''(t) \in L_2\langle 0,\infty)$ for all solutions $x(t)$ of (20). If there is still $h'(\overline{x}) \ne 0$ for all the zero points $\overline{x}$ of $h(x)$, then it can be proved (see [3]) that also

$[x(t) - \overline{x}] \in L_2\langle 0, \infty)$ for each solution $x(t)$ and the appropriate zero point $\overline{x}$ of $h(x)$.

Remark 12. For $p(t) \equiv 0$, the zero points $\overline{x}$ of $h(x)$ with $h'(\overline{x}) > 0$ are asymptotically stable in sense of Liapunov with the basin of attractivity given by (cf. [5, p. 279])

$$a_1 \int\limits_{\overline{x}}^{x} h(s)\,\mathrm{d}s + h(x)x' + \frac{a_2}{2}x'^2 + \frac{1}{2}(x'' + a_1 x')^2 > 0\,.$$

It might be, therefore, expected that the desired convergence results are especially valid (not necessarily under (24)) when the first as well as the second derivatives of all solutions of (20) are ultimately very small in an absolute value. This is true (see (21)) for sufficiently large values of both a_1 and a_2.

For a more detailed analysis and other related methods see [23] (cf. also Example in [11]).

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