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D–POSETS

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ABSTRACT. This paper deals with partially ordered sets for which a difference (as a partial binary operation) is introduced. These structures, so-called D-posets, are a natural generalization of quantum logics, real vector lattices, orthoalgebras, MV algebras. At the same time they give a new look at the fuzzy quantum logics.

1. Introduction

A usual mathematical description of the quantum mechanics is a quantum logic [12], [15]. Recently there appeared many structures generalizing quantum logics, for example, quasi-orthocomplemented posets [1], weakly complemented posets [4], or orthoalgebras [6].

The fundamental notions of the quantum logics theory are observables and states.

If L is a quantum logic (σ-orthomodular poset) [12], then an *observable* x is a σ-homomorphism of logics, that is, a mapping x from the σ-algebra $\mathcal{B}(M)$ of Borel sets of a separable Banach space M into a given logic L such that

(i) $x(M) = 1$;
(ii) $x(M \setminus A) = x(A)^{\perp}$ for any $A \in \mathcal{B}(M)$;
(iii) if A_n, $n \in \mathbb{N}$, is a countable set of Borel sets in M, then

$$x\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigvee_{n=1}^{\infty} x(A_n).$$

A *state* on the logic L is a mapping $m\colon L \to [0,1]$ such that

(i) $m(1) = 1$;

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(ii) if a_n, $n \in \mathbb{N}$, is a sequence of mutually orthogonal elements in L, then

$$m\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} m(a_n).$$

There are also some alternative models for quantum mechanics based on fuzzy sets ideas: fuzzy quantum logics [13], F-quantum spaces [14], fuzzy logics [8], and h-fuzzy quantum logics [9].

2. D-POSETS

DEFINITION 1. *Let $(P, \leq)$ be a non-empty partially ordered set (poset). A partial binary operation $\searrow$ is called a difference on P, and an element $b \searrow a$ is defined in P if and only if $a \leq b$, and the following conditions are satisfied:*

(1) $b \searrow a \leq b$;

(2) $b \searrow (b \searrow a) = a$;

(3) *if* $a \leq b \leq c$, *then* $c \searrow b \leq c \searrow a$ *and* $(c \searrow a) \searrow (c \searrow b) = b \searrow a$.

Example 1. Let $\mathbb{R}^+$ be a set of all non-negative real numbers. The difference $b - a$ of real numbers $a, b \in \mathbb{R}^+$, $a \leq b$, satisfies the conditions (1)–(3).

Example 2. Let F be a family of all real functions from non-empty set X into the interval $[0, \infty)$. Let $\leq$ be a partial ordering on F such that $f \leq g$ if and only if $f(t) \leq g(t)$ for every $t \in X$. Let $\Phi\colon [0, \infty) \to [0, \infty)$ be a strongly increasing continuous function such that $\Phi(0) = 0$. A partial binary operation $\searrow$ defined by the formula

$$(g \searrow f)(t) = \Phi^{-1}\big(\Phi(g(t)) - \Phi(f(t))\big)$$

for every $f, g \in F$, $f \leq g$, $t \in X$, is a difference on F.

Specifically, if $\Phi(x) = x$, then $(g \searrow f)(t) = g(t) - f(t)$, if $\Phi(x) = x^2$, then

$$(g \searrow f)(t) = \sqrt{g^2(t) - f^2(t)}, \qquad \text{etc.}$$

If we restrict our considerations to the unit interval $[0, 1]$, $F = [0, 1]^X$, $\Phi\colon [0, 1] \to [0, \infty)$, $\Phi(1) = \infty$, then $f, g \in F$ are fuzzy subsets of X and the difference $g \searrow f$,

$$(g \searrow f)(t) = \Phi^{-1}\big(\Phi(g(t)) - \Phi(f(t))\big),$$

coincides with a strict fuzzy difference introduced by Weber [16].

Example 3. If F is the system of all constant functions, $F \subseteq [0,\infty)^X$, then the difference from Example 2 gives other examples of differences on $\mathbb{R}^+$.

PROPOSITION 1. *Let $(P,\leq)$ be a poset with the difference, and let $a,b,c,d \in P$. The following assertions are true.*

(i) *If $a \leq b \leq c$, then $b \setminus a \leq c \setminus a$ and $(c \setminus a) \setminus (b \setminus a) = c \setminus b$;*
(ii) *if $b \leq c$ and $a \leq c \setminus b$, then $b \leq c \setminus a$ and $(c \setminus b) \setminus a = (c \setminus a) \setminus b$;*
(iii) *if $a \leq b \leq c$, then $a \leq c \setminus (b \setminus a)$ and $\big(c \setminus (b \setminus a)\big) \setminus a = c \setminus b$;*
(iv) *if $a \leq c$ and $b \leq c$, then $c \setminus a = c \setminus b$ if and only if $a = b$;*
(v) *if $d \leq a \leq c$, $d \leq b \leq c$, then $c \setminus a = b \setminus d$ if and only if $c \setminus b = a \setminus d$.*

Proof.

(i) From (3) and (1) we get that $(c \setminus a) \setminus (c \setminus b) = b \setminus a \leq c \setminus a$ and

$$(c \setminus a) \setminus (b \setminus a) = (c \setminus a) \setminus \big((c \setminus a) \setminus (c \setminus b)\big) = c \setminus b\,.$$

(ii) From the assumptions it follows that $a \leq c \setminus b \leq c$, and from (3) we obtain

$$c \setminus (c \setminus b) \leq c \setminus a\,, \qquad \text{i.e.} \qquad b \leq c \setminus a\,.$$

Because, by (i), $(c \setminus b) \setminus a \leq c \setminus a$, we get from (i) $(c \setminus a) \setminus \big((c \setminus b) \setminus a\big) = c \setminus (c \setminus b) = b$, therefore

$$(c \setminus a) \setminus b = (c \setminus a) \setminus \big((c \setminus a) \setminus \big((c \setminus b) \setminus a\big)\big) = (c \setminus b) \setminus a\,.$$

(iii) According to (i), we have $b \setminus a \leq c \setminus a \leq c$ and, by (3), we obtain

$$c \setminus (c \setminus a) \leq c \setminus (b \setminus a)\,, \qquad \text{i.e.} \qquad a \leq c \setminus (b \setminus a) \leq c\,.$$

Using (ii) and (i), we get

$$\big(c \setminus (b \setminus a)\big) \setminus a = (c \setminus a) \setminus (b \setminus a) = c \setminus b\,.$$

(iv) If $c \setminus a = c \setminus b$, then $b = c \setminus (c \setminus b) = c \setminus (c \setminus a) = a$.
The converse assertion is evident.

(v) If $c \setminus a = b \setminus d$, then $c \setminus b = (c \setminus d) \setminus (b \setminus d) = (c \setminus d) \setminus (c \setminus a) = a \setminus d$.
The converse assertion can be proved by analogy. □

DEFINITION 2. *Let $(P, \leq, \searrow)$ be a poset with a difference, and let 1 be the greatest element in P. The structure $(P, \leq, \searrow, 1)$ is called a D-poset.*

A D-poset $(P, \leq, \searrow, 1)$ satisfying the condition:

$$\text{(4)} \quad \text{if } (a_n)_{n=1}^{\infty} \subseteq P, \ a_n \leq a_{n+1} \text{ for any } n \in \mathbb{N}, \text{ then } \bigvee_{n=1}^{\infty} a_n \in P.$$

is called a D-σ-poset.

Example 4. Let X be a non-empty set, and let $S(X)$ be the set of all subsets of X. Let Q be a subset of $S(X)$ containing X and closed with respect to the formation of the set-theoretic difference of sets which are in the inclusion relation. Then Q with $\leq$, being the inclusion relation, and $\searrow$, being the set-theoretic difference, forms a D-poset.

Example 5. Let $(L, \leq, \perp, 1, 0)$ be an orthomodular poset (see e.g. [12]). We put $b \searrow a = b \wedge a^{\perp}$ for every $a, b \in L$, $a \leq b$. Then L is a D-poset.

Example 6. Let T be a vector lattice (a real vector space which is a lattice). Let $e \in T$, $e > 0$, $V = \{a \in T : \ 0 \leq a \leq e\}$. The system V with usual difference of vectors is a D-poset.

Example 7. Let H be a Hilbert space. A positive Hermitian operator A on H such that $O \leq A \leq I$, where O and I are operators on H defined by the formulas $Ox = 0$, $Ix = x$ for any $x \in H$, is said to be an effect ([3]).

A system $E(H)$ of effects closed with respect to the difference $B - A$ of operators $A, B \in E(H)$, $A \leq B$, is a D-poset.

Example 8. Let X be a non-empty set and let F be a system of all real functions $f \colon X \to [0,1]$. Let $\Phi \colon [0,1] \to [0,\infty)$ be a strongly increasing continuous function such that $\Phi(0) = 0$. If we put

$$(g \searrow f)(t) = \Phi^{-1}\big(\Phi\big(g(t)\big) - \Phi\big(f(t)\big)\big)$$

for every $f, g \in F$, $f \leq g$, and for any $t \in X$ (see Example 2), then F becomes a D-poset (a D-poset of fuzzy sets, see [7]).

Note that, in this case, $g \searrow f$ coincides with a nilpotent fuzzy difference of Weber [16].

E x a m p l e 9. A set A containing two special elements 0, 1 with $0 \neq 1$ on which there is a partially defined binary operation $\oplus$ satisfying for all elements $p, q, r \in A$ the following four conditions:

(i) if $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$ (commutativity);

(ii) if $p \oplus q$ is defined and $(p \oplus q) \oplus r$ is defined, then $q \oplus r$ and $p \oplus (q \oplus r)$ are defined, and $(p \oplus q) \oplus r = p \oplus (q \oplus r)$ (associativity);

(iii) for each $p \in A$ there is a unique $q \in A$ such that $p \oplus q$ is defined and $p \oplus q = 1$ (orthocomplementation);

(iv) if $p \oplus p$ is defined, then $p = 0$ (consistency)

is called an orthoalgebra ([6]).

The unique element $q \in A$ satisfying the conditions in (iii) is denoted by $q = p'$ and called the orthocomplement of p.

If $p, q \in A$, we define $p \leq q$ to mean that there exists $r \in A$ such that $p \oplus r$ is defined and $p \oplus r = q$. It is not difficult to check that this element r is defined uniquely. Indeed, if there are $r, s \in A$ such that $p \oplus r = q = p \oplus s$, then $1 = (p \oplus r) \oplus q' = r \oplus (p \oplus q')$, which implies that $r' = p \oplus q'$ and $r = (p \oplus q')'$. Similarly, $s = (p \oplus q')'$, therefore $r = s$.

We put $q \setminus p = (p \oplus q')'$ for $p, q \in A$, $p \leq q$.

We prove that the partial binary operation $\setminus$ is the difference on the orthoalgebra A (in the sense of Definition 1).

(a) If $p \leq q$, then there exists $r \in A$, $r = (p \oplus q')'$, such that $q = p \oplus r = p \oplus (p \oplus q')'$, which gives $(p \oplus q')' \leq q$, i.e. $q \setminus p \leq q$.

(b) Let $p \leq q$. Because $1 = (p \oplus q') \oplus (p \oplus q')' = p \oplus \big(q' \oplus (p \oplus q')'\big)$, we have $p' = q' \oplus (p \oplus q')'$, which implies that $q \setminus (q \setminus p) = \big((p \oplus q')' \oplus q'\big)' = (p')' = p$.

(c) If $p \leq q \leq w$, then there exists $s \in A$ such that $q = s \oplus p$. From the equalities $1 = (q \oplus w') \oplus (q \oplus w')' = \big((s \oplus p) \oplus w'\big) \oplus (q \oplus w')' = \big(s \oplus (p \oplus w')\big) \oplus (q \oplus w')' = (p \oplus w') \oplus \big(s \oplus (q \oplus w')'\big)$ it follows that $(p \oplus w')' = s \oplus (q \oplus w')'$, which is equivalent with the inequality $(q \oplus w')' \leq (p \oplus w')'$, that is $w \setminus q \leq w \setminus p$.

Calculate,

$$(w \setminus p) \setminus (w \setminus q) = \big((q \oplus w')' \oplus (p \oplus w')\big)' = \big(((q \oplus w')' \oplus w') \oplus p\big)' = (q' \oplus p)' = q \setminus p .$$

We have proved that every orthoalgebra is a D-poset.

We note that the connection between D-posets and orthoalgebras was noticed firstly by N a v a r a and P t á k [11] [1].

[1] The authors are indebted to Dr. Navara who after the first version of the present paper called our attention to this fact.

Example 10. In [10], an MV algebra is defined as follows:

An *MV algebra* is an algebra $(M, \oplus, \odot, \star, 0, 1)$, where M is a non-empty set, 0 and 1 are constant elements of M, $\oplus$ and $\odot$ are binary operations, and $\star$ is a unary operation such that for all $x, y, z \in M$ the following axioms are satisfied:

(A1) $(x \oplus y) = (y \oplus x)$,
(A2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
(A3) $x \oplus 0 = x$,
(A4) $x \oplus 1 = 1$,
(A5) $(x^\star)^\star = x$,
(A6) $0^\star = 1$,
(A7) $x \oplus x^\star = 1$,
(A8) $(x^\star \oplus y)^\star \oplus y = (x \oplus y^\star)^\star \oplus x$,
(A9) $x \odot y = (x^\star \oplus y^\star)^\star$.

The lattice operations $\vee$ and $\wedge$ are defined by the formulas

$$x \vee y = (x \odot y^\star) \oplus y \quad \text{and} \quad x \wedge y = (x \oplus y^\star) \odot y .$$

We write $x \leq y$ if and only if $x \vee y = y$. The relation $\leq$ is a partial ordering over M and $0 \leq x \leq 1$ for every $x \in M$.

An MV algebra is a distributive lattice with respect to the operations $\vee$, $\wedge$. We put

$$y \searrow x = (x \oplus y^\star)^\star \qquad \text{for} \quad x, y \in M, \ \ x \leq y .$$

The partial binary operation $\searrow$ is the difference on M. Indeed:

(a) Let $x \leq y$. Then

$$\begin{aligned}(y \searrow x) \vee y = (x \oplus y^\star)^\star \vee y = \big((x \oplus y^\star)^\star \odot y^\star\big) \oplus y = \big((x \oplus y^\star) \oplus y\big)^\star \oplus y \\ = \big(x \oplus (y^\star \oplus y)\big)^\star \oplus y = (x \oplus 1)^\star \oplus y = 0 \oplus y = y .\end{aligned}$$

therefore $y \searrow x \leq y$.

(b) Let $x \leq y$. We calculate

$$y \searrow (y \searrow x) = \big((y \searrow x) \oplus y^\star\big)^\star = \big((x \oplus y^\star)^\star \oplus y^\star\big)^\star = (x \oplus y^\star) \odot y = x \wedge y = x .$$

(c) Let $x \leq y \leq z$. By a simple calculation, we get $z^\star \leq y^\star \leq x^\star$, $x^\star \oplus y = 1$ and $y^\star \oplus z = 1$.

Further,

$$\begin{aligned}(z \setminus y) \vee (z \setminus x) &= \left((y \oplus z^\star) \oplus (x \oplus z^\star)^\star\right)^\star \oplus (z \setminus x)\\ &= \left(y \oplus \left(z^\star \oplus (x \oplus z^\star)^\star\right)\right)^\star \oplus (z \setminus x)\\ &= \left(y \oplus \left(x^\star \ominus (x^\star \oplus z)^\star\right)\right)^\star \oplus (z \setminus x)\\ &= \left((y \oplus x^\star) \oplus 1^\star\right)^\star \oplus (z \setminus x) = (1 \oplus 0)^\star \oplus (z \setminus x) = z \setminus x\,,\end{aligned}$$

therefore $z \setminus y \leq z \setminus x$, and

$$\begin{aligned}(z \setminus x) \setminus (z \setminus y) &= (x \oplus z^\star)^\star \setminus (y \oplus z^\star)^\star = \left((y \oplus z^\star)^\star \oplus (x \oplus z^\star)\right)^\star\\ &= \left(\left((y^\star \oplus z)^\star \oplus y^\star\right) \oplus x\right)^\star = \left((1^\star \oplus y^\star) \oplus x\right)^\star\\ &= (y^\star \oplus x)^\star = y \setminus x\,.\end{aligned}$$

We have proved that every MV algebra is a D-poset.

PROPOSITION 2. *Every D-poset contains the least element* 0, *and* $0 = 1 \setminus 1$.

P r o o f . Let $a \in P$. Then $1 \setminus a \in P$, $1 \setminus a \leq 1 \leq 1$, and, by (3), we have $1 \setminus 1 \leq a$, which implies that $1 \setminus 1$ is the least element in P, and we denote it by 0. □

PROPOSITION 3. *Let P be a D-poset. Then the following assertions are true.*

(i) $a \setminus 0 = a$ *for any* $a \in P$;
(ii) $a \setminus a = 0$ *for any* $a \in P$;
(iii) *if* $a, b \in P$, $a \leq b$, *then* $b \setminus a = 0$ *if and only if* $b = a$;
(iv) *if* $a, b \in P$, $a \leq b$, *then* $b \setminus a = b$ *if and only if* $a = 0$.

P r o o f .

(i) For every $a \in P$ we have $0 \leq a \setminus a \leq a$. From (2) and (3) we get

$$a = a \setminus (a \setminus a) \leq a \setminus 0 \leq a\,,$$

which implies $a \setminus 0 = a$.

(ii) From the above we have $a \setminus a = a \setminus (a \setminus 0) = 0$.

The proof of (iii) and (iv) is evident. □

3. Observables and states on D-σ-posets

DEFINITION 3. *Let P and T be two D-σ-posets. A mapping $w \colon P \to T$ is called a morphism (of D-σ-posets) if the following conditions are satisfied:*

(7) $w(1_P) = 1_T$;

(8) *if* $(a_n)_{n=1}^{\infty} \subseteq P$, $a \in P$, $a_n \nearrow a$ ($a_n \le a_{n+1}$ *for any* $n \in \mathbb{N}$ *and* $a = \bigvee_{n=1}^{\infty} a_n$), *then* $w(a_n) \nearrow w(a)$;

(9) *if* $a, b \in P$, $a \le b$, *then* $w(b \setminus a) = w(b) \setminus w(a)$.

If P *is the* σ*-algebra of Borel sets of the real line* $\mathbb{R}$, *then the morphism* $x\colon \mathcal{B}(\mathbb{R}) \to T$ *is called an observable* (*on* T).

If T *is a D-poset of all real numbers from the interval* $[0,1]$ *with usual difference* (*and sum*) *of real numbers, then the morphism* $m\colon P \to [0,1]$ *is called a state* (*on* P).

If $m\colon P \to [0,1]$ is a state, then the conditions (8) and (9) are equivalent to the condition

(10) if $(a_n)_{n=1}^{\infty} \subseteq P$, $a \in P$, $a_n \nearrow a$, then

$$m(a) = m(a_1) + \sum_{n=2}^{\infty} m(a_n \setminus a_{n-1}).$$

Let us note that, if x is an observable and m is a state on a σ-orthomodular poset, then x is a σ-homomorphism and m is a σ-additive mapping.

Example 11. Let P be a D-σ-poset, $a \in P$. A mapping $x_a\colon \mathcal{B}(\mathbb{R}) \to P$ defined via

$$x_a(E) = \begin{cases} 1 & \text{if } \{0,1\} \cap E = \{0,1\}, \\ a & \text{if } \{0,1\} \cap E = \{1\}, \\ 1 \setminus a & \text{if } \{0,1\} \cap E = \{0\}, \\ 0 & \text{if } \{0,1\} \cap E = \emptyset \end{cases}$$

is an observable on P. The observable x_a is called an indicator of a.

The set $\mathcal{R}(x) = \{x(E) : E \in \mathcal{B}(\mathbb{R})\}$ is said to be a *range* of an observable x. In general, the range of an observable on a D-poset is not closed with respect to the difference of its elements (see the next example).

Example 12. Let F be the D-poset of fuzzy sets (see Example 8), where $\Phi(t) = t$ for every $t \in [0,1]$. Let x be the observable on F defined as that in Example 10, where $a \in F$ is the constant function, $a = 0,8$. Then $\mathcal{R}(x) = \{0;\ 0,2;\ 0,8;\ 1\}$, but $0,8 \setminus 0,2 = 0,6$ is not contained in $\mathcal{R}(x)$.

E x a m p l e 13. Every probability measure $p\colon \mathcal{B}(\mathbb{R}) \to [0,1]$ is an observable on a D-poset of all real numbers from the interval $[0,1]$ with usual difference of real numbers. More specifically, if (Ω, S, p) is a probability space, then the probability distribution p_ξ of a random variable ξ is an observable on the D-poset $[0,1]$.

If L is a quantum logic, x is an observable, and m is a state on L, then a probability distribution m_x of the observable x in the state m is an observable on the D-poset $[0,1]$, too.

It is easy to prove that the following proposition holds.

PROPOSITION 4. *Let x be an observable on a D-σ-poset P. Then the following assertions are true:*

(i) $x(A \cup B) \setminus x(B) = x(A) \setminus x(A \cap B)$ *for all* $A, B \in \mathcal{B}(\mathbb{R})$;
(ii) *if* $x(A) = 1$, *then* $\big(x(A) \setminus x(B)\big) \in \mathcal{R}(x)$, *and, moreover,* $x(A \cap B) = x(B)$ *for any* $B \in \mathcal{B}(\mathbb{R})$;
(iii) *if* $x(B) = 0$, *then* $\big(x(A) \setminus x(B)\big) \in \mathcal{R}(x)$, *and, moreover,* $x(A \cup B) = x(A)$ *for any* $A \in \mathcal{B}(\mathbb{R})$;
(iv) *if* $x(A) \le x(B)$, *then* $x(B) \setminus x(A) \le x(B \setminus A)$.

THEOREM 1. *Let x be an observable, and let m be a state on a D-σ-poset P. A mapping $m_x\colon \mathcal{B}(\mathbb{R}) \to [0,1]$ defined via*

$$m_x(E) = m\big(x(E)\big) \qquad \text{for any} \quad E \in \mathcal{B}(\mathbb{R}),$$

is a probability measure on $\mathcal{B}(\mathbb{R})$.

P r o o f. We prove only the σ-additivity of the mapping m_x. Let $(E_n)_{n=1}^\infty$ be a sequence of pairwise disjoint Borel subsets. Put $A_n = \bigcup\limits_{i=1}^{n} E_i$, $n = 1, 2, \dots$. The sequence $\big(A_n\big)_{n=1}^\infty$ is monotonic, and

$$\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty E_n .$$

Let us calculate

$$
\begin{aligned}
m_x\left(\bigcup_{n=1}^{\infty} E_n\right) &= m\left(x\left(\bigcup_{n=1}^{\infty} E_n\right)\right) = m\left(x\left(\bigcup_{n=1}^{\infty} A_n\right)\right) = m\left(\bigvee_{n=1}^{\infty} x(A_n)\right) \\
&= m\big(x(A_1)\big) + \sum_{n=2}^{\infty} m\big(x(A_n) \setminus x(A_{n-1})\big) \\
&= m\big(x(A_1)\big) + \sum_{n=2}^{\infty} m\big(x(A_n \setminus A_{n-1})\big) \\
&= m\big(x(E_1)\big) + \sum_{n=2}^{\infty} m\big(x(E_n)\big) = \sum_{n=1}^{\infty} m\big(x(E_n)\big) = \sum_{n=1}^{\infty} m_x(E_n) .
\end{aligned}
$$

□

The mapping m_x is said to be a *probability distribution* of the observable x in the state m and, by Example 13, the mapping m_x is an observable on the D-σ-poset $[0,1]$.

Now a *mean value* of the observable x in the state m can be defined by the integral

$$E(x) := \int_{\mathbb{R}} t \; m_x(\mathrm{d}t)$$

if it exists and is finite.

4. Representation of observables

The functional calculus for compatible observables in quantum logics is based on a representation of these observables by Borel measurable functions.

The functional calculus for observables in D-posets may be constructed in a similar way.

LEMMA 1. *Let $x\colon \mathcal{B}(\mathbb{R}) \to P$ be an observable on a D-σ-poset P and let $f\colon \mathbb{R} \to \mathbb{R}$ be a Borel measurable mapping. Then the mapping $y\colon \mathcal{B}(\mathbb{R}) \to P$ defined by the formula $y(E) = x\big(f^{-1}(E)\big)$ for any $E \in \mathcal{B}(\mathbb{R})$ is also an observable (and we write $y = x \circ f^{-1}$).*

The proof of this Lemma requires only a routine verification of the conditions in the definition of an observable.

THEOREM 2. **(Representation Theorem)** *Let x, y be two observables on a D-σ-poset P. Then the following two conditions are equivalent:*

(i) *There is a Borel measurable function $f\colon \mathbb{R} \to \mathbb{R}$ such that $x(E) = y\big(f^{-1}(E)\big)$ for any $E \in \mathcal{B}(\mathbb{R})$.*

(ii) *There is a chain M, $M \subseteq \mathcal{B}(\mathbb{R})$, such that $\big\{x\big((-\infty, r)\big) :\ r \in \mathbb{Q}\big\} \subseteq \big\{y(A) :\ A \in M\big\}$, where $\mathbb{Q}$ is the set of all rationals.*

P r o o f. Let M be a linear ordered set of the Borel subsets such that

$$\big\{x\big((-\infty, r)\big) :\ r \in \mathbb{Q}\big\} \subseteq \big\{y(A) :\ A \in M\big\}.$$

Then for every $r \in \mathbb{Q}$ there is a Borel subset $A_r \in M$ such that $x\big((-\infty, r)\big) = y(A_r)$.

We note that, if $y(A) \le y(B)$ for $A, B \in M$, then there are $C, D \in M$ such that $A \subseteq C$ and $y(B) = y(C)$, $D \subseteq B$ and $y(A) = y(D)$.

Indeed, it suffices to put $C = A \cup B$, $D = A \cap B$. Similarly, if $A, B, C \in M$, $A \subseteq C$ and $y(A) \le y(B) \le y(C)$, then there is $D \in M$ such that $A \subseteq D \subseteq C$ and $y(D) = y(B)$. It suffices to put $D = A \cup (B \cap C)$.

Now we can construct by induction a sequence $(B_n)_{n=1}^{\infty} \subseteq M$ such that $x\big((-\infty, r_n)\big) = y(B_n)$ for any $r_n \in \mathbb{Q}$ and, if $r_i < r_j$, then $B_i \subset B_j$.

Let $B = \bigcap\limits_{n=1}^{\infty} B_n$. Put $A_n = B_n \setminus B$. Because $y(B) = x(\emptyset) = 0$, we have

$$y(A_n) = y(B_n \setminus B) = y(B_n) \setminus y(B) = y(B_n) = x\big((-\infty, r_n)\big).$$

The sequence $(A_n)_{n=1}^{\infty}$ is constructed such that:

(i) $x\big((-\infty, r_n)\big) = y(A_n)$ for any $r_n \in \mathbb{Q}$, $n = 1, 2, \dots$;

(ii) $A_i \subseteq A_j$ if $r_i < r_j$;

(iii) $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$.

We define an $\mathcal{B}(\mathbb{R})$-measurable function $f\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f(t) = \begin{cases} 0 & \text{if } t \in \mathbb{R} \setminus \bigcup\limits_{n=1}^{\infty} A_n, \\ \inf\{r_i \in \mathbb{Q} :\ t \in A_i\} & \text{if } t \in \bigcup\limits_{n=1}^{\infty} A_n. \end{cases}$$

The function f is everywhere well-defined and finite. Moreover,

$$f^{-1}\big((-\infty, r_k)\big) = \begin{cases} \bigcup\limits_{r_i < r_k} A_i & \text{if } r_k \leqq 0\,, \\ \bigcup\limits_{r_i < r_k} A_i \cup \Big(\mathbb{R} \setminus \bigcup\limits_{n=1}^{\infty} A_n\Big) & \text{if } r_k > 0\,. \end{cases}$$

hence f is $\mathcal{B}(\mathbb{R})$-measurable.

Let $r \in \mathbb{Q}$, $r \leqq 0$. Then

$$\begin{aligned} y\big(f^{-1}((-\infty, r))\big) &= y\Big(\bigcup_{r_i < r} A_i\Big) = y\Big(\bigcup_{i=1}^{\infty} A_{j_i}\Big) = \bigvee_{n=1}^{\infty} y\Big(\bigcup_{i=1}^{n} A_{j_i}\Big) \\ &= \bigvee_{n=1}^{\infty} y(A_{K_n}) = \bigvee_{n=1}^{\infty} x\big((-\infty, r_{K_n})\big) \\ &= x\Big(\bigcup_{n=1}^{\infty} (-\infty, r_{K_n})\Big) = x\big((-\infty, r)\big)\,, \end{aligned}$$

where $(r_{j_i})_{n=1}^{\infty} = \{r_i \in \mathbb{Q} :\ r_i < r\}$, $r_{K_n} = \max\,\{r_{j_1}, r_{j_2}, \dots, r_{j_n}\}$.

Similarly, if $r > 0$.

It is clear that $y\big(f^{-1}(\mathbb{R})\big) = x(\mathbb{R})$ because $y\big(f^{-1}(\mathbb{R})\big) = 1$.

Let $[a, b)$ be an interval, $a, b \in \mathbb{Q}$, $a < b$. Then $[a, b) = (-\infty, b) \setminus (-\infty, a)$, therefore

$$y\big(f^{-1}([a, b))\big) = x\big([a, b)\big)\,.$$

Let us denote $\mathcal{S} = \big\{[a, b) :\ a, b \in \mathbb{Q},\ \ a < b\big\}$. It is not difficult to show that

$$y\big(f^{-1}([a, b) \cup [c, d))\big) = x\big([a, b) \cup [c, d)\big)\,,$$

and

$$y\big(f^{-1}([a, b) \setminus [c, d))\big) = x\big([a, b) \setminus [c, d)\big)\,.$$

Now we put

$$\mathcal{K} = \big\{A \in \mathcal{B}(\mathbb{R}) :\ y\big(f^{-1}(A)\big) = x(A)\big\}\,.$$

The system $\mathcal{K}$ contains the algebra $s(\mathcal{S})$ over the system $\mathcal{S}$. We show that $\mathcal{K}$ is a monotone system.

Let $(E_n)_{n=1}^{\infty} \subset \mathcal{K}$, $E_n \subseteq E_{n+1}$ for any $n \in \mathbb{N}$. Then

$$y\left(f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right)\right) = y\left(\bigcup_{n=1}^{\infty} f^{-1}(E_n)\right)$$
$$= \bigvee_{n=1}^{\infty} y(f^{-1}(E_n)) = \bigvee_{n=1}^{\infty} x(E_n) = x\left(\bigcup_{n=1}^{\infty} E_n\right).$$

There holds: $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{S}) \subseteq \mathcal{M}(s(\mathcal{S})) \subseteq \mathcal{K}$, where $\sigma(\mathcal{S})$ denotes the least σ-algebra over $\mathcal{S}$, and $\mathcal{M}(s(\mathcal{S}))$ denotes the least monotone system over $s(\mathcal{S})$, which implies that $\mathcal{K} = \mathcal{B}(\mathbb{R})$.

Conversely, let $f\colon \mathbb{R} \to \mathbb{R}$ be a Borel measurable function with $y(f^{-1}(E)) = x(E)$ for every $E \in \mathcal{B}(\mathbb{R})$. Then the system $M = \{f^{-1}(-\infty, r) : r \in \mathbb{Q}\}$ is a chain such that

$$\{x((-\infty, r)) : r \in \mathbb{Q}\} \subseteq \{y(A) : A \in M\}.$$

□

The representation theorem enables to define the compatible observables, the joint observable and to prove, for example, the weak law of large numbers in D-posets (see [2]), etc.

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