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CONSTRUCTIVE APPROXIMATION
OF A BALL BY POLYTOPES

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ABSTRACT. In this paper, we give an explicit construction of \( m \) unit vectors in the \( n \)-dimensional Euclidean space such that the convex hull of them contains a ball of radius \( \sqrt{n^{-1} \log(m/n)} \), where \( 2n \leq m \leq e^n \). This construction is asymptotic optimal. Finally we discuss some algorithmical consequences of our result.

1. Introduction

Approximation of a ball by polytopes is a well-studied subject in convexity theory. Let

\[
V(n, m) = \frac{\max\{\text{vol}(\text{conv}\{x_1, \ldots, x_m\}); x_1, \ldots, x_m \in S_n(1)\}}{\text{vol}(S_n(1))},
\]

where \( S_n(\delta) \) denotes the \( n \)-dimensional ball of radius \( \delta \) with centre at the origin, and \( \text{conv}(K) \) denotes the convex hull of \( K \). The behavior of \( V(n, m)^{1/n} \) has been investigated in [2] (see also [1], [3], [4]). It was proved that, if \( m \) is a function of \( n \) (linear, polynomial, exponential) and \( n \to \infty \), then

\[
c_1 \sqrt{\frac{\log(m/n)}{n}} \leq V(n, m)^{1/n} \leq c_2 \sqrt{\frac{\log(m/n)}{n}},
\]

where \( c_1, c_2 \) are constants. For further details and information on approximation, see [2].

A similar question is to determine \( \rho(n, m) \), the maximal radius of a ball (with centre at the origin) which is contained in the convex hull of \( m \) points chosen


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from $S_n(1)$. Clearly, $p(n, m) \leq V(n, m)^{1/n}$, and one would expect asymptotic equality here. For this it is enough to show that
\[ c_3 \sqrt{\frac{\log(m/n)}{n}} \leq p(n, m). \tag{2} \]

In the second paragraph, we shall give an explicit construction that proves (2) for any $n \geq 1$ and $2n \leq m \leq c^n$ ($c > 1$ is a constant). This result is asymptotic optimal because the upper bound of (1) holds in fact for any $m > n$ and $n$ sufficiently large (it remains to set $k = \log(n/d)$ in [2; Theorem 3]). In the third paragraph, we shall apply this result and sketch how it can be used for an improvement of some algorithms from computational geometry.

2. The construction

Let $\|u\|$ denote the Euclidean norm for $u \in \mathbb{R}^r$ and $S^{r-1} = \{ x \in \mathbb{R}^r : \|x\| = 1 \}$. By a 1-net $N_r$ in $S^{r-1}$, we mean a subset of $S^{r-1}$ such that for any $x \in S^{r-1}$ there exists a $v \in N_r$ satisfying $\|x - v\| \leq 1$. We shall frequently use the following well-known fact:
\[ \text{If } x = (x_1, \ldots, x_r) \in S^{r-1}, \text{ then } \sum_{i=1}^{r} |x_i| \leq \sqrt{r}. \tag{3} \]

**Lemma 1.** If $N_r$ is a 1-net, then $S_r(1/2) \subseteq \text{conv}(N_r)$.

**Proof.** If $z \in S_r(1/2)$ does not belong to $\text{conv}(N_r)$, then separating $z$ from $\text{conv}(N_r)$ by a hyperplane $p_z$ we get a cap of $S_r(1)$ which is disjoint from $N_r$ and its “top” $t$ ($t$ is one of the unit vectors perpendicular to $p_z$) satisfies $\|t - v\| > 1$ for any $v \in N_r$ -- a contradiction. Thus $S_r(1/2) \subseteq \text{conv}(N_r)$.  

In the sequel, we shall need a 1-net in $S^{r-1}$ of cardinality at most $d^r$ for any integer $r$. The existence of such 1-nets can be proved in several ways: By random construction, or by greedy algorithm, or just choosing a subset $X$ which is maximal with respect to the property that two distinct elements of $X$ are at least 1 apart. Explicit constructions are, as usual, of greatest interest. This will be done in the following lemma.

**Lemma 2.** Let $r$ be a positive integer and
\[ A_r := \mathbb{Z}^r \cap S_r(3\sqrt{r}) , \quad B_r := \{ b = a/\|a\| ; \ a \in A_r , \ \|a\| \neq 0 \} . \]
Then $B_r$ is a 1-net in $S^{r-1}$ of cardinality at most $d^r$, where $d$ is a constant independent on $r$.

Proof. Let $x = (x_1, \ldots, x_r) \in S^{r-1}$, then, by (3), $\sum_{i=1}^r |x_i| \leq \sqrt{r}$. Denote $u_i := \lfloor 3\sqrt{r} x_i \rfloor$ for any $i \in \{1, \ldots, r\}$, and $u = (u_1, \ldots, u_r) \in \mathbb{R}^r$. Let $v = (v_1, \ldots, v_r) := u/\|u\|$. Then

\[
3\sqrt{r} x_i - 1 < u_i \leq 3\sqrt{r} x_i,
\]
\[
9r x_i^2 - 6\sqrt{r} x_i + 1 < u_i^2 \leq 9r x_i^2,
\]
\[
9r \sum_{i=1}^r x_i^2 - 6\sqrt{r} \sum_{i=1}^r |x_i| + r < \sum_{i=1}^r u_i^2 \leq 9r \sum_{i=1}^r x_i^2,
\]
\[
9r - 6r + r < \|u\|^2 \leq 9r,
\]
\[
2\sqrt{r} < \|u\| \leq 3\sqrt{r}.
\]

Thus $u \in A_r$ and $v \in B_r$. Furthermore, if $x_i \geq 0$, then $-\sqrt{r} x_i \leq 2\sqrt{r} x_i - [3\sqrt{r} x_i] < \|u\| x_i - [3\sqrt{r} x_i] \leq 3\sqrt{r} x_i - [3\sqrt{r} x_i] < 1$. Thus

\[
\|u\| x_i - u_i = \|u\| x_i - [3\sqrt{r} x_i] \leq \sqrt{r}|x_i| + 1.
\]

This inequality holds also if $x_i < 0$ because then $1 - \sqrt{r} x_i > 2\sqrt{r} x_i - [3\sqrt{r} x_i] > \|u\| x_i - [3\sqrt{r} x_i] \geq 3\sqrt{r} x_i - [3\sqrt{r} x_i] \geq 0$.

Let $y = (y_1, \ldots, y_r) := x - v$. Then

\[
|y_i| = |x_i - v_i| = \frac{\|u\| x_i - u_i}{\|u\|} \leq \frac{\sqrt{r}|x_i| + 1}{2\sqrt{r}},
\]
\[
\|y\|^2 = \sum_{i=1}^r y_i^2 \leq \frac{r \left( \sum_{i=1}^r x_i^2 \right) + 2\sqrt{r} \left( \sum_{i=1}^r |x_i| \right) + r}{4r} \leq \frac{4r}{4r} = 1.
\]

Thus $\|y\| \leq 1$, and therefore $B_r$ is a 1-net.

Now we show that $|B_r|$ is bounded by an exponential function of $r$. Let $H_r$ denote the cube in $\mathbb{R}^r$ whose vertices have all coordinates equal to $\pm 1/2$. $H_r$ has centre at the origin, and its volume is 1. Let $H_r + a$ denote the image of $H_r$ under the translation by $a \in \mathbb{R}^r$. If $a = (a_1, \ldots, a_r) \in A_r$, then we show that $H_r + a \subseteq S_r(7\sqrt{r}/2)$. Really, since $a \in A_r$, then $\sum_{i=1}^r a_i^2 \leq 9r$. 
and, by (3), \( \sum_{i=1}^{r} |a_i| \leq \sqrt{r} \|a\| \leq 3r \) and
\[
\sum_{i=1}^{r} (a_i + 1/2)^2 \leq \sum_{i=1}^{r} a_i^2 + \sum_{i=1}^{r} |a_i| + r/4 \leq 49r/4.
\]
i.e. all vertices of \( H_r + a \) are from \( S_r(7\sqrt{r}/2) \), therefore, if \( a \in A_r \), then \( H_r + a \subseteq S_r(7\sqrt{r}/2) \).

Clearly, if \( a, b \in \mathbb{Z}^r \) and \( a \neq b \), then the set \( (H_r + a) \cap (H_r + b) \) is not full-dimensional, i.e. its volume is 0. Then
\[
|A_r| = \sum_{a \in A_r} \text{vol}(H_r + a) \leq \text{vol}(S_r(7\sqrt{r}/2)) = (7/2)^{r/2} \text{vol}(S_r(1)).
\]

It is known that
\[
\text{vol}(S_r(1)) = \pi^{r/2}/\Gamma(r/2 + 1),
\]
where \( \Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} \, dt \) \((x > 0)\) is the gamma-function. By the Stirling formula,
\[
\Gamma(r/2 + 1) = \sqrt{\pi} r^{r/2} e^{\theta(r/2)},
\]
where \( 0 < \theta(x) < 1/12x \). Therefore \( |B_r| \leq |A_r| \leq d^r \), where \( d \) is a constant independent on \( r \).

Now we can formulate the main theorem.

**Theorem 1.** Let \( n, m \) be integers, \( 2n \leq m \leq c^n \), where \( c > 1 \) is a constant. Then there exist \( m \) unit vectors in \( \mathbb{R}^n \) such that the convex hull of them contains a ball with centre at the origin and of radius \( c_3 \sqrt{n^{-1}} \log(m/n) \), where the constant \( c_3 \) does not depend on \( n \).

**Proof.** Let \( m, n \) be integers such that \( 2n \leq m \leq c^n \). We also suppose that \( n \geq 2 \). By Lemma 2, for any integer \( r \) there exists a 1-net \( B_r \) in \( S_r^{-1} \) of cardinality at most \( d^r \), where \( d > 1 \) is a constant. If \( 2n \leq m \leq d^2n \), take \( C_n := \{ \pm e_i; \ i = 1, \ldots, n \} \). Then \( |C_n| = 2n \leq m \), and, by (3), \( S_n(1/\sqrt{n}) \subseteq \text{conv}(C_n) \), what proves (2) in this special case. If \( m > d^2n \), then choose
\[
r := \lceil \log_d(m/n) \rceil, \quad s := \lfloor n/r \rfloor.
\]
Clearly, \( r \geq 2 \). If \( n = rs \), then take \( s \) copies of \( B_r \) in pairwise orthogonal \( r \)-dimensional subspaces of \( \mathbb{R}^n \), the set we obtain is denoted by \( C_n \). Then
\[
|C_n| \leq sd^r \leq \left( \frac{n}{r} + 1 \right) \frac{m}{n} \leq \frac{m}{r} + \frac{m}{n} \leq m \text{ because } n, r \geq 2.
\]
We show that
\[
S_n(1/\sqrt{4s}) \subseteq \text{conv}(C_n).
\]
To prove this, let \( x \in S^{n-1} \). Then \( x \) can be expressed as sum of its \( s \) projections \( x_i \ (i \in \{1, \ldots, s\}) \) on the pairwise orthogonal \( r \)-dimensional subspaces of \( \mathbb{R}^n \). i.e. \( x = \sum_{i=1}^{s} x_i, \ 1 = \|x\|^2 = \sum_{i=1}^{s} \|x_i\|^2 \), and, by (3), \( \sum_{i=1}^{s} \|x_i\| \leq \sqrt{s} \).

By Lemma 1, \( S_r(1/2) \subseteq \text{conv}(B_r) \), therefore for any \( i \in \{1, \ldots, s\} \) there exist \( v_{i,1}, \ldots, v_{i,n_i} \in C_n \) and positive reals \( \alpha_{i,1}, \ldots, \alpha_{i,n_i} \) such that \( \sum_{j=1}^{n_i} \alpha_{i,j} \leq 2 \) and \( x_i/\|x_i\| = \sum_{j=1}^{n_i} \alpha_{i,j} v_{i,j} \). Then \( x = \sum_{i=1}^{s} \sum_{j=1}^{n_i} \|x_i\| \alpha_{i,j} v_{i,j} \) and \( \sum_{i=1}^{s} \sum_{j=1}^{n_i} \|x_i\| \alpha_{i,j} \leq 2\sqrt{s} \), thus \( (1/\sqrt{4s})x \in \text{conv}(C_n) \). Since this is true for any \( x \in S^{n-1} \), then \( S_n(1/\sqrt{4s}) \subseteq \text{conv}(C_n) \).

Concluding, \( C_n \) has cardinality at most \( m \) and contains a ball of radius \( 1/\sqrt{4s} \geq c_3 \sqrt{n^{-1} \log(m/n)} \), what proves (2). If \( n < rs \), then take \( s \) copies of \( B_r \) or \( B_{r-1} \) in pairwise orthogonal \( r \)- or \( (r-1) \)-dimensional subspaces of \( \mathbb{R}^n \) and continue analogously as if \( n = rs \).

Note that our construction gives in fact a constructive proof of the lower bound of (1).

Finally let us discuss the restriction \( 2n \leq m \leq c^n \). The upper bound is trivial because, if \( n \to \infty \), then the lower bounds given in (1), (2) are valid only if \( m \leq c^n \). On the other hand, the optimal ratios \( V(n, n + 1) \) and \( \rho(n, n + 1) \) occur if we deal with regular simplex. Then \( V(n, n + 1)^{1/n} \approx \text{const} \sqrt{1/n} \), and (1) remains true also if \( m > n \). It is known that any \( n \)-dimensional regular simplex with unit vertices contains a ball (with centre at the origin) of radius at most \( 1/n \) (see e.g. [7]). Thus \( \rho(n, n + 1) = 1/n \). But \( c_3 \sqrt{n^{-1} \log(1 + n^{-1})} \approx \text{const} \sqrt{1/n} > 1/n \) if \( n \to \infty \). Therefore (2) is not true for any \( m > n \) though it holds for any \( m \geq 2n \). It could be of some interest to study the behaviour of \( \rho(n, m) \) if \( n < m < 2n \). We show that, if \( r < n \), then

\[
\rho(n, n + r + 1) \geq 1/\sqrt{r + (n-r)^2}. \tag{4}
\]

To prove this, take the set \( B_1 = \{ \pm e_i; \ i = 1, \ldots, r \} \) of unit vectors in \( \mathbb{R}^r \) and the set \( B_2 \) consisting of unit vertices of a regular simplex in \( \mathbb{R}^{n-r} \). Take copies of \( B_1 \) and \( B_2 \) in two mutually orthogonal \( r \)- and \( (n-r) \)-dimensional subspaces of \( \mathbb{R}^n \) respectively. The set we obtain is denoted by \( B \). Then \( |B| = n + r + 1 \). Since \( \max \{ ax + b \sqrt{1 - x^2}; \ x \in (0, 1) \} = \sqrt{a^2 + b^2} \), then using the above methods we can check that \( S_n(1/\sqrt{r + (n-r)^2}) \subseteq \text{conv}(B) \), what proves (4).

3. Algorithmical consequences of the construction

Now we sketch an application of our result in computational geometry. From Theorem 1 it follows:
Corollary 1. There exist \( n^2 \) unit vectors in \( \mathbb{R}^n \) (\( n \geq 2 \)) such that the convex hull of them contains the ball \( S_n(c_3 \sqrt{n^{-1} \log n}) \), where the constant \( c_3 \) does not depend on \( n \).

Primarily, we suppose that any convex body \( K \subseteq \mathbb{R}^n \) is given by a membership oracle, i.e., we have an oracle that decides for any \( x \in \mathbb{Q}^n \) whether \( x \in K \) or not. Furthermore, we suppose that \( K \) is contained in a ball with centre at the origin and of radius \( R \), \( K \) contains a ball with centre \( a \in \mathbb{Q}^n \) and of radius \( r \) and the coordinates of \( a \), \( R \) and \( 1/r \) are bounded by a polynomial of \( 2^n \). This model coincides with that of Grolsche1, Lovasz and Schrijver [7].

It is well known (see [7], [5]) that every convex body \( K \subseteq \mathbb{R}^n \) is contained in a unique ellipsoid \( E \) of minimal volume. This ellipsoid is called the Löwner-John ellipsoid of \( K \). Moreover, \( K \) contains the ellipsoid \( (1/n)E \) (where \( (1/\delta)E \) will denote the ellipsoid obtained from \( E \) by shrinking it from its centre by a factor of \( \delta \)). If \( K \) is centrally symmetric, then the component \( 1/n \) can be improved on \( 1/\sqrt{n} \) (see [7] for more details).

In general, the Löwner-John ellipsoid of a convex body is hard to compute. Grolsche1, Lovasz and Schrijver [7] (see also [6]) presented an algorithm bounded by a polynomial of \( n \) that approximate the Löwner-John ellipsoid. This algorithm finds an ellipsoid \( E \) such that \( (c_4/n^{3/2})E \subseteq K \subseteq E \) for any convex set \( K \subseteq \mathbb{R}^n \), and, if \( K \) is centrally symmetric, then \( (c_4/n)E \subseteq K \subseteq E \). Using Corollary 1 and the methods of [7; Theorems 4.6.1 and 4.6.3] (see also [7; Remark 4.6.2]) we can asymptotically improve this algorithm such that the components \( c_4/n^{3/2} \) and \( c_4/n \) are replaced by \( c_5 \sqrt{\log n}/n^{3/2} \) and \( c_5 \sqrt{\log n}/n \), respectively.

It is easy to compare volumes of two concentrical ellipsoids. Thus, the algorithm of Grolsche1, Lovasz and Schrijver for approximation of the Löwner-John ellipsoid gives in fact an upper bound \( \overline{\text{vol}}(K) \) and a lower bound \( \underline{\text{vol}}(K) \) for the volume of the convex set \( K \) such that \( \overline{\text{vol}}(K)/\underline{\text{vol}}(K) \leq (n/c_4)^{3n/2} \) in general case, and, if \( K \) is centrally symmetric, then \( \overline{\text{vol}}(K)/\underline{\text{vol}}(K) \leq (n/c_4)^n \). Thus our improvement of the algorithm improves the ratio

\[
\frac{\overline{\text{vol}}(K)}{\underline{\text{vol}}(K)} \leq \left( \frac{n^{3/2}}{c_5 \sqrt{\log n}} \right)^n
\]

in general case, and, if \( K \) is centrally symmetric, then

\[
\frac{\overline{\text{vol}}(K)}{\underline{\text{vol}}(K)} \leq \left( \frac{n}{c_5 \sqrt{\log n}} \right)^n.
\]
Barány and Füredi [1] proved the following negative result. For any polynomial time algorithm which gives an upper bound $\overline{\text{vol}}(K)$ and a lower bound $\underline{\text{vol}}(K)$ for the volume of a convex set $K \subseteq \mathbb{R}^n$ the ratio $\overline{\text{vol}}(K)/\underline{\text{vol}}(K)$ is at least $(c_6 n/\log n)^n$ for some convex body $K \subseteq \mathbb{R}^n$, where $c_6$ is a constant independent of $n$. Thus our algorithm is very close to being asymptotically optimal for centrally symmetric convex bodies.

Other results from [7; Section 4.6] can be improved similarly.

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