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REGULARIZATION OF CLOSED-VALUED MULTIFUNCTIONS IN A NON-METRIC SETTING

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ABSTRACT. In this paper, the existence of a regularization of multifunctions \( \Phi: T \rightarrow Z \) and \( F: T \times X \rightarrow Y \) is stated if \( T \) is a topological measurable space, and \( X, Y \) and \( Z \) are topological spaces with a countable base (Theorems 1 and 3). Utilizing Sainte-Beuve's selection theorem ([6]), uniqueness theorems (Theorems 2 and 4) are also derived. The obtained results generalize those of Rzeżuchowski in [5].

1. Introduction

Scorza-Dragoni type theorems for multifunctions \( F: T \times X \rightarrow Y \) of Carathéodory type are useful for the study of the set of solutions of Cauchy problems associated with the differential inclusion

\[
\dot{x} \in F(t,x).
\]

This occurs because the separated regularity of \( F \) with respect to \( t \) and \( x \) (i.e. the Carathéodory type property) implies (through the Scorza-Dragoni type theorem) an almost regularity with respect to \( (t,x) \).

For example, if \( T = [0,1] \), \( X = Y = \mathbb{R} \), \( F: T \times X \rightarrow Y \) has closed values, \( F(\cdot,x) \) is weakly measurable, and \( F(t,\cdot) \) is continuous, then for each \( \varepsilon > 0 \) there exists a closed set \( C_\varepsilon \subset [0,1] \), whose Lebesgue measure is \( > 1 - \varepsilon \), such that \( F|_{C_\varepsilon} \) is lower semicontinuous and has closed graph.

A tool for the study of the set of solutions of (1) when \( F \) is not of Carathéodory type can be the “regularization” of \( F \).

Key words: Multifunctions, Regularization, Measurability, Semicontinuity.

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Roughly speaking, a regularization of $F$ is a multifunction $G : T \times X \to Y$ which has the following properties:

\begin{align*}
  &i_1) \quad G(t, x) \subset F(t, x), \\
  &i_2) \quad q(t) \in F(t, p(t)) \implies q(t) \in G(t, p(t)) \text{ whenever } p : T \to X \text{ and } q : T \to Y \text{ are measurable functions},
\end{align*}

plus some Scorza-Dragoni type property.

In virtue of the properties $i_1)$ and $i_2)$, the set of solutions of (1) is the same as that of

$$
\dot{x} \in G(t, x). \tag{2}
$$

Existence and uniqueness theorems for regularizations $G$ of certain multifunctions $F$ have been given in [1], [4], [5].

In particular, Rzeźuchowski proved in [5] an existence theorem for regularizations of a given closed-valued multifunction $F : T \times X \to Y$ such that $F(t, \cdot)$ has closed graph when $T$ is a locally compact metric space endowed with a Borel, $\sigma$-finite, regular and complete measure $\mu$, and $X$ and $Y$ are separable metric spaces ([5; Theorem 1]). A uniqueness theorem was also presented when $X$ and $Y$ are even complete ([5; Theorem 4]).

The aim of this paper is essentially to show that the Rzeźuchowski existence and uniqueness theorems above remain still valid without assuming that $X$ and $Y$ are metric and without strengthening of other hypotheses: that is, existence and uniqueness theorems for regularizations proved here extend in a more general framework the results of [5].

The main idea for doing this, in existence theorems, is to replace the point-set distance function in the range-space (which plays a key role in the Rzeźuchowski existence proof) by a function $c$, taking only two values, which flags when two suitable sets, one coming from a basis of the topology of the range-space, the other from the values of the multifunction, intersect.

This idea has been tested also in a previous paper [5], concerning Lusin and Scorza-Dragoni type theorems; some results of [3] are also useful here.

The extension in uniqueness theorems essentially carries out in virtue of Sainte-Beuve’s selection theorem.

Moreover, the results of this paper not only extend but also improve those of [5], establishing indeed further properties for the regularization $G$.
2. Preliminaries

Let $S$ be a non-empty set and $(Z, \tau_Z)$ be a topological space. $\mathcal{P}(Z)$ (resp. $\text{Cl}(Z)$) denotes the family of all subsets (resp. closed subsets) of $Z$, while $\mathcal{B}(Z)$ denotes the Borel $\sigma$-algebra on $Z$. Let $\Phi: S \to \mathcal{P}(Z)$. If the values of $\Phi$ are closed subsets of $Z$, we write $\Phi: S \to \text{Cl}(Z)$. $\text{Gr}(\Phi)$ denotes the graph of $\Phi$, i.e. the set $\{(s, z) \in S \times Z : z \in \Phi(s)\}$. If $E \subseteq S$, we call $\Phi|_E$ the restriction of $\Phi$ to $E$. If $W \subseteq Z$, we put $\Phi^-(W) = \{s \in S : \Phi(s) \cap W \neq \emptyset\}$ and $\Phi^+(W) = \{s \in S : \Phi(s) \subseteq W\}$. We have the fundamental relations $\Phi^-(W) = S - \Phi^+(Z - W) = \text{proj}_S(\text{Gr}(\Phi) \cap (S \times W))$, where $\text{proj}_S$ denotes the projection map of $S \times Z$ onto $S$, and for each family $\{W_\alpha : \alpha \in A\} \subseteq \mathcal{P}(Z)$, $\Phi^{-}\left(\bigcup_{\alpha \in A} W_\alpha\right) = \bigcup_{\alpha \in A} \Phi^{-}(W_\alpha)$.

If $\Phi_1, \Phi_2: S \to Z$ are two multifunctions, we denote by $\Phi_1 \triangle \Phi_2$ the symmetric difference multifunction, that is the multifunction defined by $(\Phi_1 \triangle \Phi_2)(s) = \Phi_1(s) \Delta \Phi_2(s)$ for each $s \in S$.

If $(S, \tau_S)$ is a topological space, we say that $\Phi$ is lower (resp. upper) semicontinuous at $s_0 \in S$ if for each $W \in \tau_Z$ such that $s_0 \in \Phi^-(W)$ (resp. $I \subseteq \Phi^+(W)$) there exists an open neighbourhood $I$ of $s_0$ such that $I \subseteq \Phi^-(W)$ (resp. $I \subseteq \Phi^+(W)$). We say that $\Phi$ is lower (resp. upper) semicontinuous if it is lower (resp. upper) semicontinuous at every $s \in S$, or equivalently, if for each $W \in \tau_Z$ the set $\Phi^-(W)$ (resp. $\Phi^+(W)$) is open in $S$. We say that $\Phi$ is continuous if it is simultaneously lower and upper semicontinuous.

If $(S, \Sigma_S, \mu)$ is a measure space, we denote by $\Sigma_S^\mu$ the completion of $\Sigma_S$ with respect to $\mu$ and with $\mu^*$ the completion measure. $(S, \Sigma_S^\mu, \mu^*)$ is a complete measure space. Recall that $E \in \Sigma_S^\mu$ if and only if there exist $E', E'' \in \Sigma_S$ such that $E' \subseteq E \subseteq E''$ and $\mu(E') = \mu^*(E) = \mu(E'')$.

If $\Sigma$ is a $\sigma$-algebra of subsets of $S$, we say that $\Phi$ is $\Sigma$-weakly measurable (resp. $\Sigma$-measurable) if for each $W \in \tau_Z$ (resp. $W \in \text{Cl}(Z)$) $\Phi^-(W) \in \Sigma_S$. The definitions of lower and upper semicontinuity for real valued functions and those of measurability and continuity for functions with values in a topological space are the usual ones.

If $(S, \tau)$ is a topological space, $(S, \Sigma)$ is a measurable space, and $E \subseteq S$, then $\tau_E = \tau|_E$ and $\Sigma_E = \Sigma|_E$ denote respectively the induced topology and the induced $\sigma$-algebra on $E$. If $E \subseteq S$, and $E$ has the induced topology, then $\mathcal{B}(S)|_E = \mathcal{B}(E)$. If $E \subseteq \Sigma$, then $\Sigma|_E = \{A \in \Sigma : A \subseteq E\}$, so we speak of $\Sigma$-weak measurability (resp. $\Sigma$-measurability) of a multifunction (resp. function) instead of $\Sigma|_E$-weak measurability (resp. $\Sigma|_E$-measurability) whenever the multifunction (resp. function) is defined on $E$. 

115
If \( S \) is a structured space, and we want a structure on \( E \subset S \) when it is not specified, we refer to the induced structure; i.e., if \( S \) is a topological (resp. measurable) space, then \( E \) is a topological (resp. measurable) space with the induced topology (resp. \( \sigma \)-algebra).

If \( S \) and \( S' \) are two sets and \( E \subset S \times S' \), then for \( s \in S \), \( E_s = \{ s' \in S' : (s, s') \in E \} \) denotes the \( s \)-section of \( E \), and for \( s' \in S' \), \( E_{s'} = \{ s \in S : (s, s') \in E \} \) denotes the \( s' \)-section of \( E \).

If \( S \) and \( S' \) are two structured spaces, and we want a structure on \( S \times S' \) when it is not specified, we refer to the product structure; i.e., if \( S \) and \( S' \) are topological (resp. measurable) spaces, then \( S \times S' \) is a topological (resp. measurable) space with the product topology (resp. \( \sigma \)-algebra). We notice that if \( S \) and \( S' \) are topological spaces, in general, \( B(S) \times B(S') \subset B(S \times S') \), and the inclusion can be proper; \( B(S) \times B(S') = B(S \times S') \) if, for example, \( S \) and \( S' \) are second-countable topological spaces or Suslin spaces.

Moreover, when in the sequel we deal with the product of three sets \( S \), \( S' \), and \( S'' \), we always identify \( S \times (S' \times S'') \) with \( S \times S' \times S'' \), even if the structure is essentially that of \( S \times (S' \times S'') \). So, for example, if \( \Sigma_S \) and \( \Sigma_{S' \times S''} \) are \( \sigma \)-algebras on \( S \) and \( S' \times S'' \) respectively, when we say that \( E \subset S \times S' \times S'' \) lies in \( \Sigma_S \times \Sigma_{S' \times S''} \), we mean that \( \{ (s, (s', s'')) : \in S \times (S' \times S'') \} \in \Sigma_S \times \Sigma_{S' \times S''} \).

As in [7], we say that a topological space is Polish if it is separable and metrizable by a complete metric, Suslin if it is Hausdorff and a continuous image of a Polish space.

### 3. Regularization of closed-valued multifunctions

We begin with the following proposition in measure theory.

**Lemma 1.** Let \((T, \Sigma_T)\) be a measurable space, and \(\mu\) be a \(\sigma\)-finite measure on \(\Sigma_T\).

For each subset \(E\) of \(T\) there exists \(M \in \Sigma_T\) such that:

\[\alpha_1\) \quad M \subset E;\]

\[\alpha_2\) \quad for each \(L \in \Sigma_T^*\) such that \(L - E \in \Sigma_T^*\) and \(\mu^*(L - E) = 0\), then \(\mu^*(L - M) = 0\).

**Proof.** We prove Lemma 1 for \(\mu(T) < +\infty\) because it is obvious how to extend it to the \(\sigma\)-finite case.

Let \(\alpha = \sup\{ \mu(A) : A \in \Sigma_T, \ A \subset E \} < +\infty\). Then there exists a sequence \((A_n)_n\) of sets in \(\Sigma_T\) such that, for each \(n \in \mathbb{N}\), \(A_n \subset E\) and \(\mu(A_n) > \alpha - 1/n\).

The set \(M = \bigcup_n A_n\) is the requested set.
In fact, obviously, $M \in \Sigma_T$ and satisfies $\alpha_1$.

Moreover, let $L \in \Sigma_T^*$ be such that $L - E \in \Sigma_T^*$ and $\mu^*(L - E) = 0$; then 

$$(L - M) \cap E = (L - M) - (L - E) \in \Sigma_T^*$$

and $\mu^*((L - M) \cap E) = \mu^*(L - M)$.

Obviously, $((L - M) \cap E) \cup M \in \Sigma_T^*$; so let $L' \in \Sigma_T$ be such that $L' \subset ((L - M) \cap E) \cup M$ and $\mu(L') = \mu^*((((L - M) \cap E) \cup M)$. We have $L' \subset E$ and $\alpha \geq \mu(L') = \mu^*(L - M) + \mu(M) \geq \alpha$, from which $\mu^*(L - M) = 0$. Hence $M$ verifies $\alpha_2$.

We need, for the sequel, to reformulate Lemma 1 in terms of functions with only two values.

**COROLLARY 1.** Let $(T, \Sigma_T)$ be a measurable space, $\mu$ be a $\sigma$-finite measure on $\Sigma_T$, and let $\{0,1\}$ be endowed with the discrete topology.

If $\varphi: T \to \{0,1\}$ is a function, then there exists a $\Sigma_T$-measurable function $\psi: T \to \{0,1\}$ such that:

- $\beta_1$) $\psi(t) \leq \varphi(t)$ for each $t \in T$;
- $\beta_2$) for each $\Sigma_T$-measurable function $\vartheta: T \to \{0,1\}$ such that $\vartheta(t) \leq \varphi(t)$ a.e. in $T$, there holds $\vartheta(t) \leq \psi(t)$ a.e. in $T$.

From now on, unless otherwise stated, $(T, \tau_T)$ is a topological space, $\Sigma_T$ is a $\sigma$-algebra of subsets of $T$ such that $\tau_T \subset \Sigma_T$ (equivalently $\mathcal{B}(T) \subset \Sigma_T$), $\mu$ is a $\sigma$-finite measure on $\Sigma_T$ such that for every $A \in \Sigma_T$ and every $\epsilon > 0$ there exists a closed set $C_\epsilon \subset A$ with $\mu(A - C_\epsilon) < \epsilon$. Obviously, $\Sigma_T^*$ and $\mu^*$ have also these properties.

**LEMMA 2.** Let $Z$ be a topological space and $\mathcal{B}(T \times Z) = \mathcal{B}(T) \times \mathcal{B}(Z)$.

Let $\Psi: T \to Z$ be a multifunction such that for each $\epsilon > 0$ there exists a closed set $C_\epsilon \subset T$ with $\mu(T - C_\epsilon) < \epsilon$ such that $\text{Gr}(\Psi|_{C_\epsilon})$ is closed in a $\Sigma_T^* \times \mathcal{B}(Z)$-measurable set $\Omega$ which contains $\text{Gr}(\Psi)$.

Then there exists $T_0 \in \Sigma_T$ with $\mu(T_0) = 0$ such that $\text{Gr}(\Psi|_{T - T_0}) \in \Sigma_T^* \times \mathcal{B}(Z)$.

**Proof.** For each $k \in \mathbb{N}$ there exists a closed set $C_k \subset T$ with $\mu(T - C_k) < 1/k$ such that $\text{Gr}(\Psi|_{C_k}) = \text{Cl}(\text{Gr}(\Psi|_{C_k})) \cap \Omega$, where $\text{Cl}(\text{Gr}(\Psi|_{C_k}))$ denotes the closure of $\text{Gr}(\Psi|_{C_k})$ in $T \times Z$. But $\mathcal{B}(T \times Z) = \mathcal{B}(T) \times \mathcal{B}(Z)$, so $\text{Gr}(\Psi|_{C_k}) \in \Sigma_T^* \times \mathcal{B}(Z)$. Put $T_0 = \bigcap_{k} (T - C_k)$; then $\mu(T_0) = 0$ and $\text{Gr}(\Psi|_{T - T_0}) = \bigcup_{k} \text{Gr}(\Psi|_{C_k}) \in \Sigma_T^* \times \mathcal{B}(Z)$. 

We need also the following proposition, whose proof we give for completeness.
**Lemma 3.** Let $Z$ be a Suslin space. If $\Psi : T \to Z$ is a multifunction such that $\text{Gr}(\Psi) \in \Sigma_T \times \mathcal{B}(Z)$, then for every $\varepsilon > 0$ there exists a closed set $C_{\varepsilon} \subset T$ with $\mu(T - C_{\varepsilon}) < \varepsilon$ such that $\Psi|_{C_{\varepsilon}}$ is lower semicontinuous.

**Proof.** Let $g$ be a continuous function from a Polish space $Z'$ onto $Z$. Define the multifunction $\Psi' : T \to Z$ by putting $\Psi'(t) = g^{-1}(\Psi(t))$ for all $t \in T$.

We claim that $\text{Gr}(\Psi') \in \Sigma_T \times \mathcal{B}(Z')$. Indeed, it is easily seen that $(1_T, g) : T \times Z' \to T \times Z$, defined by $(1_T, g)(t, z') = (t, g(z'))$ for all $(t, z') \in T \times Z'$, is continuous, and $(1_T, g)^{-1}(\Omega) \in \Sigma_T \times \mathcal{B}(Z')$ for each $\Omega \in \Sigma_T \times \mathcal{B}(Z)$. Thus the claim follows from the fact that $\text{Gr}(\Psi') = (1_T, g)^{-1}(\text{Gr}(\Psi))$.

By Sainte-Beuve's projection theorem [6; Theorem 4], $\Psi'$ is $\Sigma_T^\ast$-$\mathcal{B}$-measurable since $\Psi'^{-}(W') = \text{proj}_T(\text{Gr}(\Psi') \cap (T \times W'))$ for each $W' \subset Z'$. Hence, a fortiori, $\Psi'$ is $\Sigma_T^\ast$-weakly measurable.

Now, by Theorem 1 of [3], for every $\varepsilon > 0$ there exists a closed set $C_{\varepsilon} \subset T$ with $\mu(T - C_{\varepsilon}) < \varepsilon$ such that $\Psi'|_{C_{\varepsilon}}$ is lower semicontinuous.

Since $g$ is surjective, $\Psi(t) = g(\Psi'(t))$ for every $t \in T$, and so $\Psi^{-}(W) = \Psi'^{-}(g^{-1}(W))$ for each $W \subset Z'$; thus $\Psi|_{C_{\varepsilon}}$ is lower semicontinuous.

As in [3], if $B, B' \subset Z$, we define

$$\delta(B, B') = \begin{cases} 1 & \text{if } B \cap B' \neq \emptyset, \\ 0 & \text{if } B \cap B' = \emptyset. \end{cases}$$

The following theorem is the key result of this paper.

**Theorem 1.** Let $Z$ be a second-countable topological space and $\Phi : T \to \text{Cl}(Z)$ a multifunction.

Then there exists a multifunction $\Psi : T \to \text{Cl}(Z)$ such that:

1. $\Psi(t) \subset \Phi(t)$ for each $t \in T$;
2. for each $\Delta \in \Sigma_T^\ast$ and for each $\Sigma_T^\ast$-weakly measurable multifunction $\Theta : \Delta \to Z$ such that $\Theta(t) \subset \Phi(t)$ a.e. in $\Delta$, there holds $\Theta(t) \subset \Psi(t)$ a.e. in $\Delta$;
3. for each $\varepsilon > 0$ there exists a closed set $C_{\varepsilon} \subset T$ with $\mu(T - C_{\varepsilon}) < \varepsilon$ such that $\text{Gr}(\Psi|_{C_{\varepsilon}})$ is closed in $T \times Z$;
4. $\text{Gr}(\Psi) \in \Sigma_T \times \mathcal{B}(Z)$.

If, moreover, we assume that $Z$ is also a Suslin space, then:

1. for each $\varepsilon > 0$ there exists a closed set $C_{\varepsilon} \subset T$ with $\mu(T - C_{\varepsilon}) < \varepsilon$ such that $\Psi|_{C_{\varepsilon}}$ is lower semicontinuous.
\[ \gamma_6 \) for each \( \Delta \in \Sigma_T^* \) and for each multifunction \( \Theta: \Delta \to Z \) with \( \text{Gr}(\Theta) \in \Sigma_T^* \times \mathcal{B}(Z) \) such that \( \Theta(t) \subset \Phi(t) \) a.e. in \( \Delta \), we have \( \Theta(t) \subset \Psi(t) \) a.e. in \( \Delta \).

**Proof.** Let \( \mathfrak{B} = \{B_n : n \in \mathbb{N}\} \) be a countable basis for \( \tau_Z \). For each \( n \in \mathbb{N} \) define \( \varphi_n : T \to \{0,1\} \) by putting, for each \( t \in T \), \( \varphi_n(t) = \delta(B_n, \Phi(t)) \).

By Corollary 1, there exists a \( \Sigma_T \)-measurable function \( \psi_n : T \to \{0,1\} \) such that:

1) \( \psi_n(t) \leq \varphi_n(t) \) for each \( t \in T \);
2) for each \( \Sigma_T^* \)-measurable function \( \vartheta : T \to \{0,1\} \) such that \( \vartheta(t) \leq \varphi_n(t) \) a.e. in \( T \), we have \( \vartheta(t) \leq \psi_n(t) \) a.e. in \( T \).

Let us define \( \Psi : T \to \text{Cl}(Z) \) by putting, for each \( t \in T \),

\[ \Psi(t) = \bigcap \{Z - B_n : \psi_n(t) = 0\}. \]

\( \Psi \) verifies \( \gamma_1 \). In fact, for each \( z \in \Psi(t) \) and each \( n \in \mathbb{N} \) such that \( \varphi_n(t) = 0 \), it follows that \( z \in Z - B_n \) in virtue of 1). Thus we obtain \( z \in \Phi(t) \), taking into account that, \( \Phi(t) \) being closed, \( \Phi(t) = \bigcap \{Z - B_n : \varphi_n(t) = 0\} \).

\( \Psi \) verifies \( \gamma_2 \). Let \( \Delta \in \Sigma_T^* \) and \( \Theta: \Delta \to Z \) be a \( \Sigma_T^* \)-weakly measurable multifunction such that \( \Theta(t) \subset \Phi(t) \) a.e. in \( \Delta \). For each \( n \in \mathbb{N} \) define \( \vartheta_n : T \to \{0,1\} \) by putting:

\[ \vartheta_n(t) = \begin{cases} \delta(B_n, \Theta(t)) & \text{if } t \in \Delta, \\ 0 & \text{if } t \notin \Delta. \end{cases} \]

Then, by using [3; Lemma 2.3] (\( \Rightarrow \)), it follows that \( \vartheta_n \) is \( \Sigma_T^* \)-measurable; moreover, since \( \Theta(t) \subset \Phi(t) \) a.e. in \( \Delta \), \( \vartheta_n(t) \leq \varphi_n(t) \) a.e. in \( T \). Hence, by 2), \( \vartheta_n(t) \leq \psi_n(t) \) a.e. in \( T \), and thus \( \Theta(t) \subset \Psi(t) \) a.e. in \( \Delta \).

\( \Psi \) verifies \( \gamma_3 \). Fix \( \varepsilon > 0 \). By Lusin’s theorem (see also [3; Lemma 1]) and by a standard argument which takes into account the countability of the family \( \{\psi_n : n \in \mathbb{N}\} \), we can find a closed set \( C_\varepsilon \subset T \) with \( \mu(T - C_\varepsilon) < \varepsilon \) such that \( \psi_n |_{C_\varepsilon} \) is continuous for each \( n \in \mathbb{N} \). We claim that \( \text{Gr}(\Psi |_{C_\varepsilon}) \) is closed in \( T \times Z \). Indeed, if \( z_0 \notin \Psi(t_0) \), \( t_0 \in C_\varepsilon \), then there is \( \bar{n} \in \mathbb{N} \) such that \( \psi_{\bar{n}}(t_0) = 0 \) and \( z_0 \in B_{\bar{n}} \). By the upper semicontinuity of \( \psi_n |_{C_\varepsilon} \) at \( t_0 \), there exists an open neighbourhood \( I \) of \( t_0 \) such that \( \psi_{\bar{n}}(t) = 0 \) for each \( t \in I \cap C_\varepsilon \). Thus

\[ (I \cap C_\varepsilon) \times B_{\bar{n}}) \cap \text{Gr}(\Psi |_{C_\varepsilon}) = \emptyset. \]

\( \Psi \) verifies \( \gamma_4 \). In fact, \( (T \times Z) - \text{Gr}(\Psi) = \bigcup_n (\psi_n^{-1}(\{0\}) \times B_n) \).

Finally, under the additional hypothesis on \( Z \), \( \gamma_5 \) is a direct consequence of \( \gamma_1 \), by Lemma 3, while \( \gamma_6 \) is a consequence of \( \gamma_2 \) because, using the equality...
**DIERG NOVERN**

\( \Theta^-(W) = \text{proj}_T(\text{Gr}(\Theta) \cap (\Delta \times W)) \) for \( W \in \tau_Z \), it follows that \( \Theta \) is \( \Sigma_T^\ast \)-weakly measurable by Sainte-Beuve’s projection theorem.

**Remark 1.** Obviously \( \gamma_1 \) and \( \gamma_2 \) of Theorem 1 imply respectively the following:

- \( \gamma_1' \) \; \( \Psi(t) \subseteq \Phi(t) \) a.e. in \( T \);
- \( \gamma_2' \) \; for each \( \Delta \in \Sigma_T^\ast \) and for each \( \Sigma_T^\ast \)-measurable function \( \theta: \Delta \to Z \) such that \( \theta(t) \in \Phi(t) \) a.e. in \( \Delta \), there holds \( \theta(t) \in \Psi(t) \) a.e. in \( \Delta \).

Hence Theorem 1 extends and improves [5; Theorem 2], in which \( \gamma_1' \), \( \gamma_2' \) and \( \gamma_3 \) are proved when \( T \) is a locally compact metric space, \( \mu \) is a Borel, \( \sigma \)-finite, regular and complete measure on \( T \), and \( Z \) is a separable metric space.

Moreover, if \( \Omega \) is a \( \Sigma_T^\ast \times \mathcal{B}(Z) \)-measurable set, with \( \text{Gr}(\Psi) \subseteq \Omega \), then \( \gamma_3 \) of Theorem 1 implies the following:

- \( \gamma_3' \) \; for each \( \varepsilon > 0 \) there exists a closed set \( C_\varepsilon \subseteq T \) with \( \mu(T - C_\varepsilon) < \varepsilon \) such that \( \text{Gr}(\Psi|_{C_\varepsilon}) \) is closed in \( \Omega \).

Now we prove the following uniqueness result, whose part 1) extends [5; Theorem 5].

**Theorem 2.** Let \( Z \) be a Suslin space and \( \Phi, \Psi_1, \Psi_2: T \to Z \) be three multifunctions.

Let us consider the following properties for \( i = 1, 2 \):

- \( \gamma_1' \) \; \( \Psi_1(t) \subseteq \Phi(t) \) a.e. in \( T \);
- \( \gamma_2' \) \; for each \( \Delta \in \Sigma_T^\ast \) and for each \( \Sigma_T^\ast \)-measurable function \( \theta: \Delta \to Z \) such that \( \theta(t) \in \Phi(t) \) a.e. in \( \Delta \), we have \( \theta(t) \in \Psi_i(t) \) a.e. in \( \Delta \);
- \( \gamma_3' \) \; for some \( \Omega \in \Sigma_T^\ast \times \mathcal{B}(Z) \) with \( \text{Gr}(\Phi) \subseteq \Omega \), and for each \( \varepsilon > 0 \), there exists a closed set \( C_\varepsilon \subseteq T \) with \( \mu(T - C_\varepsilon) < \varepsilon \) such that \( \text{Gr}(\Psi_i|_{C_\varepsilon}) \) is closed in \( \Omega \);
- \( \gamma_4' \) \; there is \( T_0 \in \Sigma_T \) with \( \mu(T_0) = 0 \) such that \( \text{Gr}(\Psi_i|_{T - T_0}) \in \Sigma_T^\ast \times \mathcal{B}(Z) \);
- \( \gamma_5' \) \; for each \( \varepsilon > 0 \) there exists a closed set \( C_\varepsilon \subseteq T \) with \( \mu(T - C_\varepsilon) < \varepsilon \) such that \( \Psi_i|_{C_\varepsilon} \) is lower semicontinuous.

Then:

1) \( \gamma_1' \), \( \gamma_2' \) and \( \gamma_4' \) imply that \( \Psi_1(t) = \Psi_2(t) \) a.e. in \( T \).
2) If \( \mathcal{B}(T \times Z) = \mathcal{B}(T) \times \mathcal{B}(Z) \), then \( \gamma_1' \), \( \gamma_2' \), and \( \gamma_3' \) imply that \( \Psi_1(t) = \Psi_2(t) \) a.e. in \( T \).
3) If \( Z \) is also a second-countable topological space and \( \Psi_1 \) and \( \Psi_2 \) are closed-valued, then \( \gamma_1' \), \( \gamma_2' \) and \( \gamma_5' \) imply that \( \Psi_1(t) = \Psi_2(t) \) a.e. in \( T \).
REGULARIZATION OF CLOSED-VALUED MULTIFUNCTIONS . . .

Proof.

1) Gr\((\Psi_1 \Delta \Psi_2) | T - T_0) = Gr(\Psi_1 | T - T_0) \cup Gr(\Psi_2 | T - T_0) \subseteq \Sigma_T \times B(Z).$

Put $\Delta = \text{proj}_T(Gr((\Psi_1 \Delta \Psi_2) | T - T_0));$ thus $\Delta \subseteq \Sigma_T$ by Sainte-Beuve's projection theorem, and $\Delta \subseteq T - T_0.$

Define $\Gamma: T \to Z$ by putting

$$\Gamma(t) = \begin{cases} 
(\Psi_1 \Delta \Psi_2)(t) & \text{if } t \in \Delta, \\
Z & \text{if } t \not\in \Delta.
\end{cases}$$

Gr\((\Gamma) = Gr((\Psi_1 \Delta \Psi_2) | T - T_0) \cup ((T - \Delta) \times Z) \subseteq \Sigma_T \times B(Z);$ thus, by Sainte-Beuve's selection theorem [6; Theorem 3], there exists a $\Sigma_T^*$-measurable selection $\theta$ of $\Gamma.$ By $\gamma'(\cdot),$ $\theta(t) \in \Phi(t)$ a.e. in $\Delta,$ thus by $\gamma'(\cdot),$ $\theta(t) \in \Psi_1(t) \cap \Psi_2(t)$ a.e. in $\Delta.$ It follows that $\mu^*(\Delta) = 0.$

2) By Lemma 2, $\gamma'_1$ implies $\gamma'_d.$ So the conclusion follows by 1).

3) $\Psi_1$ and $\Psi_2$ are $\Sigma_T^*$-weakly measurable. In fact, for $i = 1, 2$ and for each $k \in \mathbb{N}$ there exists a closed set $C_k \subseteq T$ with $\mu(T - C_k) < 1/k$ such that $\Psi_i|_{C_k}$ is lower semicontinuous, thus $\Sigma_{C_k}$-weakly measurable (see $\Sigma_{C_k} = \{A \in \Sigma_T : A \subseteq C_k\}).$ Hence, for $W \subseteq \tau_Z,$ we have $\Psi_i^-(W) = \bigcup_k (\Psi_i|_{C_k}^{-}(W)) \cap N,$ where $\mu^*(N) = 0,$ so $\Psi_i^-(W) \subseteq \Sigma_T^*.$

Then, thanks to [2; Theorem 2.4] (see also [2; Remarks 2.1 and 2.4]), we have that $Gr(\Psi_1), Gr(\Psi_2) \subseteq \Sigma_T^* \times B(Z);$ thus the conclusion follows again by 1). $\square$

The following Theorem 3 is the two-variables version of Theorem 1.

**Theorem 3.** Let $X$ and $Y$ be two second-countable topological spaces and $D \subseteq T \times X.$

If $F: D \to \text{Cl}(Y)$ is a multifunction such that there is a $T_0 \in \Sigma_T$ with $\mu(T_0) = 0$ such that $\text{Gr}(F(t, \cdot))$ is closed in $D_t \times Y$ for each $t \in \text{proj}_T(D) - T_0,$ then there exists a multifunction $G: D \to \text{Cl}(Y)$ such that:

- $\text{i}_0)$ $\text{Gr}(G(t, \cdot))$ is closed in $D_t \times Y$ for each $t \in \text{proj}_T(D);$ 
- $\text{i}_1)$ $G(t, x) \subseteq F(t, x)$ for each $(t, x) \in D;$ 
- $\text{i}_2)$ for each $\Delta \subseteq \Sigma_T^*,$ for each $\Sigma_T^*$-weakly measurable multifunction $Q: \Delta \to Y,$ and for each $\Sigma_T^*$-measurable function $p: \Delta \to X$ such that $(t, p(t)) \in D$ and $Q(t) \subseteq F(t, p(t))$ a.e. in $\Delta,$ there holds $Q(t) \subseteq G(t, p(t))$ a.e. in $\Delta;$ 
- $\text{i}_3)$ for each $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subseteq T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $\text{Gr}(G|_{D \cap (C_\varepsilon \times X)})$ is closed in $D \times Y;$ 
- $\text{i}_4)$ $\text{Gr}(G) \in (\Sigma_T \times B(X \times Y))|_{D \times Y}.$
Moreover, if we assume that $Y$ is also a Suslin space and that $\mathcal{B}(T \times Y) = \mathcal{B}(T) \times \mathcal{B}(Y)$, then

i) for each $\Delta \in \Sigma_T^*$, for each $\Sigma_T^*$-measurable function $p: \Delta \to X$ with $(t, p(t)) \in D$ a.e. in $\Delta$, and for each $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $G(\cdot, p(\cdot))|_{\Delta \cap C_\varepsilon}$ is lower semicontinuous.

Finally, if $X$ and $Y$ are also two Suslin spaces and $D \in \Sigma_T^* \times \mathcal{B}(X)$, then

ii) for each $\Delta \in \Sigma_T^*$ and for each multifunction $H: D \cap (\Delta \times X) \to Y$ with $\text{Gr}(H) \in \Sigma_T^* \times \mathcal{B}(X \times Y)$ such that $H(t, x) \subset F(t, x)$ for almost all $t \in \text{proj}_T(D) \cap \Delta$ and for each $x \in D_1$, there holds $H(t, x) \subset G(t, x)$ for almost all $t \in \text{proj}_T(D) \cap \Delta$ and for each $x \in D_1$.

Proof. First suppose $D = T \times X$. Consider the multifunction $\Phi: T \to \text{Cl}(X \times Y)$ defined by

$$\Phi(t) = \begin{cases} \text{Gr}(F(t, \cdot)) & \text{if } t \in T - T_0, \\ \emptyset & \text{if } t \in T_0. \end{cases}$$

By Theorem 1, there exists a multifunction $\Psi: T \to \text{Cl}(X \times Y)$ satisfying $\gamma_1), \gamma_2), \gamma_3), \gamma_4)$ and $\gamma_6).$ We claim that the multifunction $G: T \times X \to \text{Cl}(Y)$ defined by $G(t, x) = (\Psi(t))_x$ is the required multifunction.

In fact, it is easily seen that $G$ verifies $i_0), i_1), i_2)$ and $i_4)$.

$G$ verifies $i_2)$. Let $\Delta \in \Sigma_T^*$, $Q: \Delta \to Y$ be a $\Sigma_T^*$-weakly measurable multifunction and $p: \Delta \to X$ be a $\Sigma_T^*$-measurable function such that $Q(t) \subset F(t, p(t))$ a.e. in $\Delta$. The multifunction $\Theta: \Delta \times X \to Y$ defined by $\Theta(t) = \{(p(t), y) : y \in Q(t)\}$ is $\Sigma_T^*$-weakly measurable because for $U \in \tau_X$ and $V \in \tau_Y$, $\Theta^{-1}(U \times V) = p^{-1}(U) \cap Q^{-1}(V) \in \Sigma_T^*.$ Moreover, $\Theta(t) \subset \Phi(t)$ a.e. in $\Delta$, thus by $\gamma_2),$ $\Theta(t) \subset \Psi(t)$ a.e. in $\Delta,$ from which it follows that $Q(t) \subset G(t, p(t))$ a.e. in $\Delta$.

$G$ verifies $i_5)$. Let $\Delta \in \Sigma_T^*$ and $p: \Delta \to X$ be a $\Sigma_T^*$-measurable function. Extend $p$ to the $\Sigma_T^*$-measurable function $\hat{p}: T \to X$ defined by putting

$$\hat{p}(t) = \begin{cases} p(t) & \text{if } t \in \Delta, \\ \text{constant} & \text{if } t \notin \Delta. \end{cases}$$

If we show that $G(\cdot, \hat{p}(\cdot))$ is $\Sigma_T^*$-weakly measurable, then by [3; Theorem 1], it follows that for each $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset T$ with $\mu(T - C_\varepsilon) < \varepsilon$ such that $G(\cdot, \hat{p}(\cdot))|_{C_\varepsilon}$, and thus also $G(\cdot, p(\cdot))|_{\Delta \cap C_\varepsilon}$, is lower semicontinuous.
To prove this, it suffices to show that for each \( \varepsilon > 0 \) there exists a closed set \( C_\varepsilon \subset T \) with \( \mu(T - C_\varepsilon) < \varepsilon \) such that \( \text{Gr}(G(\cdot, \hat{p}(\cdot))|_{C_\varepsilon}) \) is closed in \( T \times Y \). In fact, this last condition being verified, by Lemma 2 there exists \( T_0 \in \Sigma_T \) with \( \mu(T_0) = 0 \) such that \( \text{Gr}(G(\cdot, \hat{p}(\cdot))|_{T - T_0}) \in \Sigma_T^* \times B(Y) \); so, from the equality

\[
G(\cdot, \hat{p}(\cdot))^{-1}(V) = \text{proj}_T \left( \text{Gr}(G(\cdot, \hat{p}(\cdot))|_{T - T_0}) \cap ((T - T_0) \times V) \right) \cup N,
\]

where \( N \subset T_0 \), and by Sainte-Beuve’s projection theorem, it follows that \( G(\cdot, \hat{p}(\cdot)) \) is \( \Sigma_T^* \)-weakly measurable.

So fix \( \varepsilon > 0 \). By (i3) and using [3; Theorem 1], there exists a closed set \( C_\varepsilon \subset T \) with \( \mu(T - C_\varepsilon) < \varepsilon \) such that \( \text{Gr}(G|_{C_\varepsilon \times X}) \) is closed in \( T \times X \times Y \), and \( \hat{p}|_{C_\varepsilon} \) is continuous. \( \text{Gr}(G(\cdot, \hat{p}(\cdot))|_{C_\varepsilon}) \) is closed in \( T \times Y \). Indeed, if we take \( (t_0, y_0) \notin \text{Gr}(G(\cdot, \hat{p}(\cdot))|_{C_\varepsilon}) \), then \( (t_0, \hat{p}(t_0), y_0) \notin \text{Gr}(G(\cdot, \hat{p}(\cdot))|_{C_\varepsilon \times X}) \); hence, by this and by the continuity of \( \hat{p}|_{C_\varepsilon} \), there are two open neighbourhoods \( I \) and \( V \) of \( t_0 \) and \( y_0 \) respectively such that, for \( t \in I \cap C_\varepsilon \) and \( y \in V \), \( (t, y) \notin \text{Gr}(G(\cdot, \hat{p}(\cdot))|_{C_\varepsilon}) \).

Finally, we prove (i6). Define \( \Theta: \Delta \to X \times Y \) by putting, for each \( t \in \Delta \),

\[
\Theta(t) = \text{Gr}(H(t, \cdot)), \quad \text{Gr}(\Theta) = \text{Gr}(H) \in \Sigma_T \times B(X \times Y).
\]

Moreover, \( \Theta(t) \subset \Phi(t) \) a.e. in \( \Delta \); then, by \( \gamma_6 \) \( (X \times Y \) is Suslin), \( \Theta(t) \subset \Psi(t) \) a.e. in \( \Delta \), hence \( H(t, x) \subset G(t, x) \) for almost all \( t \in \Delta \) and for each \( x \in X \).

Now we sketch the proof when \( D \subset T \times X \).

Define \( \hat{F}: T \times X \to \text{Cl}(Y) \) by putting

\[
\hat{F}(t, x) = \begin{cases} 
(\text{Gr}(F(t, \cdot)))_x & \text{if } t \in T - T_0, \\
\emptyset & \text{if } t \in T_0,
\end{cases}
\]

where the closure is taken in \( X \times Y \).

\[
\text{Gr}(F(t, \cdot)) = \overline{\text{Gr}(F(t, \cdot))} \quad \text{for each } t \in T - T_0, \text{ and taking into account that}
\]

\[
\text{Gr}(F(t, \cdot)) = \overline{\text{Gr}(F(t, \cdot)) \cap (D \times Y)} \quad \text{for each } t \in \text{proj}_T(D) - T_0, \text{ then we obtain}
\]

\[
F(t, x) = F(t, x) \quad \text{for all } (t, x) \in D - (T_0 \times X).
\]

Let \( G: T \times X \to Y \) be as in the first part of the proof with respect to \( \hat{F} \); it is not difficult to verify that \( G = G|_D \) is the required multifunction. \( \square \)

Remark 2. Obviously (i1) and (i2) of Theorem 3 imply respectively the following

(1) \( G(t, x) \subset F(t, x) \) for almost every \( t \in \text{proj}_T(D) \) and for each \( x \in D_t ; \)

(2) for each \( \Delta \in \Sigma^*_T \) and for all \( \Sigma^*_T \)-measurable functions \( q: \Delta \to Y \) and \( p: \Delta \to X \) such that \( (t, p(t)) \in D \) and \( q(t) \in F(t, p(t)) \) a.e. in \( \Delta \), we have \( q(t) \in G(t, p(t)) \) a.e. in \( \Delta \).
Hence Theorem 3 extends and improves [5; Theorem 1], in which i^1), i^2) and i^3) are proved when T is a locally compact metric space, \( \mu \) is a Borel, \( \sigma \)-finite, regular and complete measure on T, X and Y are two separable metric spaces.

Moreover, if \( \Omega \) is a \( \Sigma^*_T \times \mathcal{B}(X \times Y) \)-measurable set with \( \text{Gr}(G) \subseteq \Omega \), then i^3) in Theorem 3 implies the following:

i^3) for each \( \varepsilon > 0 \) there exists a closed set \( C_\varepsilon \subseteq T \) with \( \mu(T - C_\varepsilon) < \varepsilon \) such that \( \text{Gr}(G \big| D \cap (C_\varepsilon \times X)) \) is closed in \( (D \times Y) \cap \Omega \).

The following is a uniqueness theorem for the two-variables case; its part 2) extends [5; Theorem 4].

**THEOREM 4.** Let \( X \) be a topological space, \( Y \) be a Suslin space. \( D \in \Sigma^*_T \times \mathcal{B}(X \times Y) \), and \( F, G_1, G_2 : D \rightarrow Y \) be three multifunctions.

Let us consider the following properties for \( i = 1, 2 \):

1. G^i_t(t, x) \subseteq F(t, x) for almost every \( t \in \text{proj}_T(D) \) and for each \( x \in D_t \);
2. for each \( \Delta \in \Sigma^*_T \), for all \( \Sigma^*_T \)-measurable functions \( q: \Delta \rightarrow Y \) and \( p: \Delta \rightarrow X \) such that \( (t, p(t)) \in D \) and \( q(t) \in F(t, p(t)) \) a.e. in \( \Delta \), we have \( q(t) \in G^i_t(t, p(t)) \) a.e. in \( \Delta \);
3. for some \( \Sigma^*_T \times \mathcal{B}(X \times Y) \)-measurable set \( \Omega \) with \( \text{Gr}(F) \subseteq \Omega \), and for each \( \varepsilon > 0 \) there exists a closed set \( C_\varepsilon \subseteq T \) with \( \mu(T - C_\varepsilon) < \varepsilon \) such that \( \text{Gr}(G^i \big| D \cap (C_\varepsilon \times X)) \) is closed in \( (D \times Y) \cap \Omega \);
4. there is \( T_0 \in \Sigma_T \) with \( \mu(T_0) = 0 \) such that \( \text{Gr}(G^i \big| D \cap ((T - T_0) \times X)) \in \Sigma^*_T \times \mathcal{B}(X \times Y) \);
5. for each \( \Delta \in \Sigma^*_T \), for each \( \Sigma^*_T \)-measurable function \( p: \Delta \rightarrow X \) with \( (t, p(t)) \in D \) a.e. in \( \Delta \), and for each \( \varepsilon > 0 \), there exists a closed set \( C_\varepsilon \subseteq T \) with \( \mu(T - C_\varepsilon) < \varepsilon \) such that \( G^i(\cdot, p(\cdot)) \big| \Delta \cap C_\varepsilon \) is lower semicontinuous.

Then:

1. If \( X \) is a Suslin space, then i^1), i^2) and i^4) imply that \( G_1(t, x) = G_2(t, x) \) for almost every \( t \in \text{proj}_T(D) \) and for each \( x \in D_t \).
2. If \( X \) is a Suslin space and \( \mathcal{B}(T \times X \times Y) = \mathcal{B}(T) \times \mathcal{B}(X \times Y) \), then i^1), i^2) and i^3) imply that \( G_1(t, x) = G_2(t, x) \) for almost every \( t \in \text{proj}_T(D) \) and for each \( x \in D_t \).
3. If \( Y \) is also a second-countable topological space, and \( G_1 \) and \( G_2 \) are closed-valued, then i^1), i^2) and i^5) imply that for each \( \Delta \in \Sigma^*_T \) and for each \( \Sigma^*_T \)-measurable function \( p: \Delta \rightarrow X \) with \( (t, p(t)) \in D \) a.e. in \( \Delta \), it is \( G_1(t, p(t)) = G_2(t, p(t)) \) a.e. in \( \Delta \).
Sketch of the proof. First we prove the assertion 1) for $D = T \times X$. The multifunctions $\Phi, \Psi_1, \Psi_2 : T \to X \times Y$ defined respectively by $\Phi(t) = \text{Gr}(F(t, \cdot))$, $\Psi_1(t) = \text{Gr}(G_1(t, \cdot))$, and $\Psi_2(t) = \text{Gr}(G_2(t, \cdot))$ satisfy 1) of Theorem 2; then $\Psi_1(t) = \Psi_2(t)$ a.e. in $T$, from which $G_1(t, x) = G_2(t, x)$ for almost every $t \in T$ and for each $x \in X$.

The assertion 2) can be proved as above, taking into account 2) of Theorem 2.

To prove 3) when $D = T \times X$, extend $p$ to all of $T$ by putting $p(t) = \text{constant outside of } \Delta$. Then apply 3) of Theorem 2 to $\Phi(\cdot) = F(\cdot, p(\cdot))$, $\Psi_1(\cdot) = G_1(\cdot, p(\cdot))$, and $\Psi_2(\cdot) = G_2(\cdot, p(\cdot))$; so we obtain $\Psi_1(t) = \Psi_2(t)$ a.e. in $T$. Now return to the original $p$ defined in $\Delta$, so we obtain $G_1(t, p(t)) = G_2(t, p(t))$ a.e. in $\Delta$.

For the general case $D \subset T \times X$, extend $F$, $G_1$ and $G_2$ to all of $T \times X$ by putting their values empty outside of $D$; then apply the already proved uniqueness theorem for the case $D = T \times X$ and finally return to $D$. \hfill \Box

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