Péter Kiss
Pure powers and power classes in recurrence sequences


Persistent URL: [http://dml.cz/dmlcz/136626](http://dml.cz/dmlcz/136626)

---

**Terms of use:**

© Mathematical Institute of the Slovak Academy of Sciences, 1994

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://project.dml.cz](http://project.dml.cz)
In memory of Professor Štefan Znám

PURE POWERS AND POWER CLASSES
IN RECURRENCE SEQUENCES

PÉTER KISS

(Communicated by Stanislav Jakubec)

ABSTRACT. Let $G$ be a linear recursive sequence of order $k$ satisfying the recursion $G_n = A_1G_{n-1} + \cdots + A_kG_{n-k}$. In case $k = 2$ it is known that there are only finitely many perfect powers in such a sequence. Ribenboim and McDaniel proved for sequences with $k = 2$, $G_0 = 0$ and $G_1 = 1$ that in general for a term $G_n$ there are only finitely many terms $G_m$ such that $G_mG_n = x^2$ for some integer $x$. In the general case, with some restrictions, we show that for any $n$ there exists a number $q_0$, depending on $G$ and $n$, such that the equation $G_nG_x = w^q$ in integers $x$, $w$, $q$ has no solution with $x > n$ and $q > q_0$.

Let $R = R(A, B, R_0, R_1)$ be a second order linear recursive sequence defined by

$$R_n = AR_{n-1} + BR_{n-2} \quad (n > 1),$$

where $A$, $B$, $R_0$ and $R_1$ are fixed rational integers. In the sequel we assume that the sequence is not a degenerate one, i.e. $\alpha/\beta$ is not a root of unity, where $\alpha$ and $\beta$ denote the roots of the polynomial $x^2 - Ax - B$.

The special cases $R(1, 1, 0, 1)$ and $R(2, 1, 0, 1)$ of the sequence $R$ are called the Fibonacci and the Pell sequence, respectively.

The squares and other pure powers in sequences $R$ were investigated by many authors. For the Fibonacci sequence Cohn [2] and Wylie [22] showed that a Fibonacci number $F_n$ is a square only when $n = 0, 1, 2$, or 12. Pethő [11], London and Finkelstein [8], [9] proved that $F_n$ is a full cube.

AMS Subject Classification (1991): Primary 11B37.
Key words: Recursive sequences, Perfect powers in sequences.

1 Research (partially) supported by Hungarian National Foundation for Scientific Research, grant No. 1641.

525
only if \( n = 0, 1, 2, \) or 6. From a result of Ljunggren [7] it follows that a Pell number is a square only if \( n = 0, 1, \) or 7, and Pethő [12] showed that these are the only perfect powers in the Pell sequence. Similar, but more general results were shown by McDaniel and Ribenboim [10], Robbins [18], [19] Cohn [3], [4], [5], and Pethő [14]. A general result was obtained by Shorey and Stewart [20]:

Any non degenerate binary recurrence sequence contains only finitely many pure powers which can be effectively determined.

This result also follows from a result of Pethő [13].

Another type of problems was studied by Ribenboim and McDaniel. For a sequence \( R \) we say that the terms \( R_m, R_n \) are in the same square-class if there exists a non zero integer \( x \) such that

\[
R_m R_n = x^2.
\]

A square-class is called trivial if it contains only one element.

Ribenboim [15] proved that in the Fibonacci sequence the square-class of a Fibonacci number \( F_m \) is trivial, i.e. the equation

\[
F_m F_y = x^2
\]

has no solution in non-zero integers \( x \) and \( y \neq m \), if \( m \neq 1, 2, 3, 6, \) or 12 and for the Lucas sequence \( L(1,1,2,1) \) the square-class of a Lucas number \( L_m \) is trivial if \( m \neq 0, 1, 3 \) or 6. For more general sequences \( R(A,B,0,1) \), with \((A,B) = 1\), Ribenboim and McDaniel [16] obtained that each square-class is finite and its elements can be effectively computable (see also Ribenboim [17]).

For general recursive sequences of order larger than two we have fewer results.

Let \( G = G(A_1, \ldots, A_k, G_0, \ldots, G_{k-1}) \) be a \( k \)th order linear recursive sequence of rational integers defined by

\[
G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} \quad (n > k - 1),
\]

where \( A_1, \ldots, A_k \) and \( G_0, \ldots, G_{k-1} \) are not all zero integers. Denote by \( \alpha = \alpha_1, \alpha_2, \ldots, \alpha_s \) the distinct zeros of the polynomial 

\[
x^k - A_1 x^{k-1} - A_2 x^{k-2} - \cdots - A_k.
\]

Assume that \( \alpha, \alpha_2, \ldots, \alpha_s \) has multiplicity \( 1, m_2, \ldots, m_s \) respectively, and \( |\alpha| > |\alpha_i| \) for \( i = 2, \ldots, s \). In this case, as it is known, the terms of the sequence can be written in the form

\[
G_n = a\alpha^n + r_2(n)\alpha_2^n + \cdots + r_s(n)\alpha_s^n \quad (n \geq 0),
\]

(1)
where \( r_i \) \((i = 2, \ldots, s)\) are polynomials of degree \( m_i - 1 \) and the coefficients of the polynomials and \( a \) are elements of the algebraic number field \( \mathbb{Q}(\alpha, \alpha_2, \ldots, \alpha_s) \). Under some natural conditions Shorey and Stewart [20] proved that the sequence \( G \) does not contain \( q \)th powers if \( q \) is large enough. This result follows also from [6] and [21], where more general theorems are presented.

The purpose of this note is to show a result, similar to those mentioned above, for general sequences.

**Theorem.** Let \( G \) be a \( k \)th order linear recursive sequence satisfying the above conditions. Assume that \( a \neq 0 \) and \( G_i \neq ax^i \) for \( i > n_0 \). Then for any integer \( n \), with \( G_n \neq 0 \), there exists a number \( q_0 \), depending only on \( n \) and the sequence, such that the equation

\[
G_n G_x = w^q
\]

in positive integers \( x, w, q \) has no solution with \( x > n \) and \( q > q_0 \).

For the proof of our theorem we need a result due to Baker [1].

**Lemma.** Let \( \gamma_1, \ldots, \gamma_v \) be non-zero algebraic numbers. Let \( M_1, \ldots, M_v \) be upper bounds for the heights of \( \gamma_1, \ldots, \gamma_v \), respectively. We assume that \( M_v \) is at least 4. Further let \( b_1, \ldots, b_{v-1} \) be rational integers with absolute values at most \( B \) and let \( b_v \) be a non-zero rational integer with absolute value at most \( B' \). We assume that \( B' \) is at least three. Let \( L \) be defined by

\[
L = b_1 \log \gamma_1 + \cdots + b_v \log \gamma_v,
\]

where the logarithms are assumed to have their principal values. If \( L \neq 0 \), then

\[
|L| > \exp(-C(\log B' \log M_v + B/B'))
\]

where \( C \) is an effectively computable positive number depending only on the numbers \( M_1, \ldots, M_{v-1}, \gamma_1, \ldots, \gamma_v, \) and \( v \) (see [1; Theorem 1] with \( \delta = 1/B' \)).

**Proof of the theorem.** We can suppose that \( n > n_0 \) and \( n \) is sufficiently large since by [20] or [6] it follows that for any given \( d \) the equation

\[
dG_x = w^q
\]

implies that \( q < q_0 \). We can also assume, without loss of generality, that the terms of the sequence \( G \) are positive.
Let $x$, $w$ and $q$ be integers satisfying (2). Then by (1)

$$w^q = a\alpha^x \left(1 + r_2(x) \frac{1}{a} \left(\frac{\alpha_2}{\alpha}\right)^x + \ldots\right) G_n,$$

and so

$$c_1 \frac{x}{q} < \log w < c_2 \frac{x}{q}$$

follows with some $c_1, c_2 > 0$, which depend on the sequence $G$, since $r_2(x)(\alpha_2/\alpha)^x \to 0$ as $x \to \infty$ and $\log G_n \approx n \log |\alpha| + \log |a| < c_3 x$. Using that $x > n_0$ and the properties of the logarithm function by (3), with some $c_4 > 0$, we have

$$L = \left|\log \frac{w^q}{G_n a\alpha^x}\right| < e^{-c_4 x}.$$  

On the other hand, by Lemma with $v = 4$, $M_4 = w$ and $B' = q$, we obtain the estimate

$$L = |q \log w - \log G_n - \log a - x \log \alpha| > e^{-C(\log q \log w + x/q)},$$

where $C > 0$ depends on $n$. By (5) and (6), using (4) we obtain

$$c_4 x < C(\log q \log w + c_5 \log w) < c_6 \log q \log w,$$

from which

$$x < c_7 \log q \log w$$

follows with some $c_5, c_6, c_7 > 0$. By (4) and (7), it follows that

$$q \log w < c_2 x < c_8 \log q \log w,$$

and so

$$q < c_8 \log q,$$

which is impossible if $q > q_0 = q_0(n)$.

This contradiction proves our theorem.

REFERENCES

PURE POWERS AND POWER CLASSES IN RECURRENCE SEQUENCES

[17] RIBENBOIM, P.: Square classes of \( (a^n-1)/(a-1) \) and \( a^n+1 \), Sichuan Daxue Xuneban. 26 (1989), 196–199.
[18] ROBBINS, N.: On Fibonacci numbers of the form \( px^2 \), where \( p \) is prime, Fibonacci Quart. 21 (1983), 266–271.
[22] WYLIE, O.: In the Fibonacci series \( F_1 = 1, F_2 = 1, F_{n+1} = F_n + F_{n-1} \) the first, second and twelfth terms are squares, Amer. Math. Monthly 71 (1964), 220–222.

Received January 17, 1994

Esztherházy K. Teacher's Training College
Department of Mathematics
Leányka u. 4
H–3301 Eger
Hungary