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IDENTITIES INVOLVING COVERING SYSTEMS II

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ABSTRACT. In this continuation of the previous paper we show how the notion of the covering system of congruences can be used to derive some identities involving Bernoulli and Euler numbers and polynomials.

1. Introduction

In [1] J. Beebee proved a formula for the Bernoulli numbers using the Raabe multiplication formula and the notion of disjoint covering system, thereby generalizing an earlier result of E. Y. Deeba and D. M. Rodriguez [3]. In [11], by replacing the disjoint covering system by general finite systems of congruences we proved a further extension of Beebee's recurrence relation and showed some of its further applications. In the present paper we shall combine ideas of [11] with those of [9] to obtain new recurrence relations involving in addition to Bernoulli numbers and polynomials also related Euler numbers and polynomials.

To make the paper selfcontained, let us recall the following notions.

Consider a system of congruence classes

$$a_i \pmod{n_i}, \qquad 0 \le a_i < n_i, \quad i = 1, 2, \dots, k, \quad k > 1. \tag{1}$$

Let a real (or possibly complex) valued function μ be defined on the system (1). Then the function

$$\mathfrak{m}(n) = \sum_{i=1}^{k} \mu_i \chi_i(n), \qquad n \in \mathbb{Z},$$

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where $\mu_i = \mu(i)$, χ_i is the indicator of the class $a_i \pmod{n_i}$ for $i = 1, \dots, k$, and $\mathbb{Z}$ is the set of all integers, is the so called *covering function* of system (1), and system (1) is called a $(\mu, \mathfrak{m})$*-covering*. The function $\mathfrak{m}$ is periodic and in what follows its (least nonnegative) period will be denoted by n_0. Note that n_0 always divides the l.c.m. $[n_1, n_2, \dots, n_k]$.

The notion of the $(\mu, \mathfrak{m})$-covering system includes some previously investigated notions. For instance, if $\mathfrak{m}(n) = 1$ for every $n \in \mathbb{Z}$, then the $(\mu, 1)$-covering systems are just the ε-covering systems from [16]. Further, if $\mathfrak{m}(n) \geq 1$ for every $n \in \mathbb{Z}$, then we obtain the so called *covering systems*. There are number of unproved conjectures concerning covering systems. Let us mention the three most famous of them:

Schinzel's conjecture: In every covering system (1) there are two classes $a_i \pmod{n_i}$ and $a_i \pmod{n_j}$ such that n_i divides n_j.
Erdős-Selfridge's conjecture: There is no covering system with all moduli n_i odd, distinct, and greater than one.
Erdős' conjecture: For arbitrarily large c there exists a covering system (1) with

$$c \leq n_1 < n_2 < \cdots < n_k .$$

The validity of each of these conjectures has some interesting consequences. For the present we know that the Erdős-Selfridge conjecture implies Schinzel's, [15]. In [9] it is proved that Schinzel's conjecture is true for every system (1) – not necessarily covering – provided the period n_0 of its covering function is a proper divisor of l.c.m. $[n_1, \dots, n_k]$. S. L. G. Choi [2] constructed a covering system with distinct moduli for $c = 20$ and R. Morikawa [5], [6] claims to have done the same for $c = 24$.

The simplest covering system with all moduli distinct must have $k = 5$, and one such system is

$$0 \pmod 2,\ 0 \pmod 3,\ 1 \pmod 4,\ 5 \pmod 6,\ 7 \pmod{12}. \tag{2}$$

Its covering function

$$\mathfrak{m}(t) = \begin{cases} 1, & \text{if } t = 1, 2, 3, 4, 7, 8, 10, 11, \\ 2, & \text{if } t = 0, 5, 6, 9 \end{cases}$$

has period $n_0 = 12$.

If the covering function of a covering system (1) is constant, say $\mathfrak{m}(n) = m$ for every $n \in \mathbb{Z}$, then the system (1) is called m *times covering*. m times covering systems with $m = 1$ are traditionally called *disjoint coverings* (or *exact coverings*). The system

$$0 \pmod m,\ 1 \pmod m,\ \dots,\ m-1 \pmod m \tag{3}$$

is perhaps the most trivial prototype of a disjoint covering system.

2. Identities

The Bernoulli polynomials $B_r(x)$ can be defined using the generating series

$$\frac{z\,\mathrm{e}^{xz}}{\mathrm{e}^z-1}=\sum_{r=0}^{\infty}B_r(x)\frac{z^r}{r!}\,,\qquad |z|<2\pi\,,\tag{4}$$

and the Bernoulli numbers B_r through

$$\frac{z}{\mathrm{e}^z-1}=\sum_{r=0}^{\infty}B_r\frac{z^r}{r!}\,,\qquad z<2\pi\,,$$

i.e. $B_r=B_r(0)$.

Closely related to the Bernoulli polynomials are the Euler polynomials $E_r(x)$ defined by the generating series

$$\frac{2\,\mathrm{e}^{xz}}{\mathrm{e}^z+1}=\sum_{r=0}^{\infty}E_r(x)\frac{z^r}{r!}\,,\qquad |z|<\pi\,.\tag{5}$$

For later purposes define $E_{-1}(x)$ identically equal to 0.

The Euler numbers E_r are defined by

$$E_r=2^nE_r\Big(\frac{1}{2}\Big)\,.$$

Unlike the Bernoulli numbers, the Euler numbers are integral: $E_0=1$, $E_1=0$, $E_2=-1$, $E_3=0$, $E_4=5$, $\dots$. The generating series for the Euler numbers is

$$\frac{2\,\mathrm{e}^{z}}{\mathrm{e}^{2z}+1}=\sum_{r=0}^{\infty}E_r\frac{z^r}{r!}\,,\qquad |z|<\frac{\pi}{2}\,.\tag{6}$$

For an analogue of [11; (10)] we can use the identity

$$\sum_{r=0}^{\infty}E_r(x)\frac{z^r}{r!}=\mathrm{e}^{(x-\frac{1}{2})z}\sum_{r=0}^{\infty}\frac{E_r}{2^r}\cdot\frac{z^r}{r!}$$

which gives

$$E_r(x)=\sum_{t=0}^{r}\binom{r}{t}\frac{E_t}{2^t}\Big(x-\frac{1}{2}\Big)^{r-t}\,.\tag{7}$$

Note that $E_0(x) = B_0(x) = 1$, $E_1(x) = B_1(x) = x - \frac{1}{2}$. Since $E_{2n+1} = 0$ for $n = 0, 1, \dots$, the identity

$$1 = \cosh(z) \sum_{r=0}^{\infty} E_{2r} \frac{z^{2r}}{(2r)!}$$

implies the recurrence formula

$$\sum_{t=0}^{r} \binom{2r}{2t} E_{2t} = 0, \qquad r \geq 1, \tag{8}$$

which is an analogue to the known formula for the Bernoulli numbers

$$\sum_{t=0}^{r-1} \binom{r}{t} B_t = 0. \tag{9}$$

There are number of relations between the Euler and Bernoulli polynomials, e.g.

$$E_{r-1}(x) = \frac{2}{r}\Big(B_r(x) - 2^r B_r\Big(\frac{x}{2}\Big)\Big) \tag{10}$$

for $r = 1, 2, \dots$.

Finally, if we define

$$E_r^{(1)} = E_r(0), \qquad r = -1, 0, 1, \dots,$$

then as in [11; (10)] we obtain (provided that the empty sum is considered equal to 0)

$$E_r(x) = \sum_{t=0}^{r} \binom{r}{t} x^{r-t} E_t^{(1)}, \qquad r = -1, 0, 1, \dots. \tag{11}$$

Note that (10) implies

$$(r+1)E_r^{(1)} = -2(2^{r+1} - 1)B_{r+1}, \qquad r = -1, 0, 1, \dots. \tag{12}$$

The analogy of the Raabe multiplication formula [11; (15)] for the Euler polynomials splits into two cases, depending on the parity of n. Namely, [12; pp. 23–28], [13], [8], for any real number x we have[1)]

$$E_r(x) = n^r \sum_{t=0}^{n-1} (-1)^t E_r\Big(\frac{x+t}{n}\Big), \qquad \text{if } n \text{ is odd}, \tag{13}$$

$$= -\frac{2n^r}{r+1} \sum_{t=0}^{n-1} (-1)^t B_{r+1}\Big(\frac{x+t}{n}\Big), \qquad \text{if } n \text{ is even}. \tag{14}$$

1) Note that the well-known folklore that the Bernoulli and Euler polynomials are the only – up to a constant multiple – polynomials satisfying such multiplication formulas was (to our knowledge) firstly proved in [8], compare with [4].

Motivated by Erdős–Selfridge's conjecture, a related result (for $x = 0$) was proved for general systems (1) in [9]. To formulate the corresponding analogue of [11; Theorem 1] we need the following symbol:

Let

$$\delta = \begin{cases} 1, & \text{if } n_0 \text{ is even,} \\ 0, & \text{if } n_0 \text{ is odd,} \end{cases}$$

i.e. $\delta = \frac{1}{2}\left(1 + (-1)^{n_0}\right)$.

Unless contrary is stated we shall always suppose that the classes in (1) are ordered in such a way that $n_1, n_2, \dots, n_q$ are the all even and $n_{q+1}, n_{q+2}, \dots, n_k$ the all odd moduli of congruences in (1). Then we have (recall our convention $E_{-1}(x) = 0$):

THEOREM 1. *Let x be any real number. Then a system (1) is $(\mu, \mathfrak{m})$-covering if and only if*

$$2\delta n_0^{r-1} \sum_{t=0}^{n_0-1} (-1)^t \mathfrak{m}(t) B_r\left(\frac{x+t}{n_0}\right) + (1-\delta) n_0^{r-1} r \sum_{t=0}^{n_0-1} (-1)^{t+1} \mathfrak{m}(t) E_{r-1}\left(\frac{x+t}{n_0}\right)$$
$$= 2\sum_{t=1}^{q} (-1)^{a_t} \mu_t n_t^{r-1} B_r\left(\frac{x+a_t}{n_t}\right) + r \sum_{t=q+1}^{k} (-1)^{a_t+1} \mu_t n_t^{r-1} E_{r-1}\left(\frac{x+a_t}{n_t}\right) \tag{15}$$

for every $r \in \mathbb{Z}^*$, the set of nonnegative integers.

P r o o f. It follows along the lines of the proof of Theorem 1 of [11], but with $-y$ instead of y. That is, (1) is $(\mu, \mathfrak{m})$-covering if and only if

$$\mathfrak{m}(0) + \mathfrak{m}(1)(-y) + \mathfrak{m}(2)(-y)^2 + \dots$$
$$= \sum_{t=1}^{k} \mu_t (-y)^{a_t} \left(1 + (-y)^{n_t} + (-y)^{2n_t} + \dots\right), \qquad |y| < 1,$$

and consequently if and only if

$$\sum_{t=0}^{n_0-1} \mathfrak{m}(t) \frac{(-y)^t}{1-(-y)^{n_0}} = \sum_{t=1}^{k} \mu_t \frac{(-y)^{a_t}}{1-(-y)^{n_t}}$$

or

$$\delta \sum_{t=0}^{n_0-1} (-1)^t \mathfrak{m}(t) \frac{y^t}{y^{n_0}-1} + (1-\delta) \sum_{t=0}^{n_0-1} (-1)^{t+1} \mathfrak{m}(t) \frac{y^t}{y^{n_0}+1}$$
$$= \sum_{t=1}^{q} (-1)^{a_t} \mu_t \frac{y^{a_t}}{y^{n_t}-1} + \sum_{t=q+1}^{k} (-1)^{a_t+1} \mu_t \frac{y^{a_t}}{y^{n_t}+1}.$$

Then the substitution $y = e^z$ and multiplication by $z\,e^{zx}$ gives the following equivalent relation

$$\delta \sum_{t=0}^{n_0-1} (-1)^t \mathfrak{m}(t) \frac{1}{n_0} \cdot \frac{n_0 z\, e^{\frac{x+t}{n_0} n_0 z}}{e^{n_0 z} - 1} + (1-\delta) \sum_{t=0}^{n_0-1} (-1)^{t+1} \mathfrak{m}(t) \frac{1}{n_0} \cdot \frac{n_0 z\, e^{\frac{x+t}{n_0} n_0 z}}{e^{n_0 z} + 1}$$
$$= \sum_{t=1}^{q} (-1)^{a_t} \mu_t \frac{1}{n_t} \cdot \frac{n_t z\, e^{\frac{x+a_t}{n_t} n_t z}}{e^{n_t z} - 1} + \sum_{t=q+1}^{k} (-1)^{a_t+1} \mu_t \frac{1}{n_t} \cdot \frac{n_t z\, e^{\frac{x+a_t}{n_t} n_t z}}{e^{n_t z} + 1}\,. \tag{16}$$

Using the generating series expansions (4) and (5), we see that the last equation is equivalent with the assertion of our theorem.

Note that using disjoint covering system (3) in (15), the formula (15) reduces and splits into (13) and (14).

As further corollaries we have the following analogues of Lemmas 1, 2 and 3 of [11]:

COROLLARY 1. *The following statements are equivalent*[2)] *:*

A. *The system* (1) *is* $(\mu, \mathfrak{m})$*-covering.*

B. *For every* $r \in \mathbb{Z}^*$ *we have*

$$2\delta n_0^{r-1} \sum_{t=0}^{n_0-1} (-1)^t \mathfrak{m}(t) B_r\Big(\frac{t}{n_0}\Big) + (1-\delta) n_0^{r-1} r \sum_{t=0}^{n_0-1} (-1)^{t+1} \mathfrak{m}(t) E_{r-1}\Big(\frac{t}{n_0}\Big)$$
$$= 2 \sum_{t=1}^{q} (-1)^{a_t} \mu_t n_t^{r-1} B_r\Big(\frac{a_t}{n_t}\Big) + r \sum_{t=q+1}^{k} (-1)^{a_t+1} \mu_t n_t^{r-1} E_{r-1}\Big(\frac{a_t}{n_t}\Big)\,.$$

The proof follows from Theorem 1 for $x = 0$. The next corollary follows from the previous one using (12) and the fact that the period n_0 of the covering function of an m times covering systems equals 1.

COROLLARY 2. *The system* (1) *is* m *times covering* ($m \in \mathbb{Z}$, $m \geq 1$) *if and only if*

$$2(2^r - 1) m B_r = 2 \sum_{t=1}^{q} (-1)^{a_t} n_t^{r-1} B_r\Big(\frac{a_t}{n_t}\Big) + r \sum_{t=q+1}^{k} (-1)^{a_t+1} n_t^{r-1} E_{r-1}\Big(\frac{a_t}{n_t}\Big)\,.$$

For $r = 0$ in **B** of Corollary 1 we get:

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2) Please add the case $r = -1$ and replace $n + 1$ by $r + 1$ in D_2 of [9].

COROLLARY 3. *If a system* (1) *is* $(\mu,\mathfrak{m})$*-covering, then*

$$\sum_{t=1}^{q}\frac{(-1)^{a_t}\mu_t}{n_t}=\frac{\delta}{n_0}\sum_{t=0}^{n_0-1}(-1)^t\mathfrak{m}(t)\,. \tag{17}$$

In [11] we derived the well-known Hermite identity involving the integer part function from Raabe's multiplication formula for the first Bernoulli polynomials. Simultaneously we generalized Hermite's identity to general systems of congruences. Our proof was based on the following relation

$$n_0^{r-1}\sum_{n=0}^{n_0-1}\mathfrak{m}(n)B_r\Big(\Big\{\frac{x+n}{n_0}\Big\}\Big)=\sum_{t=1}^{k}\mu_t n_t^{r-1}B_r\Big(\Big\{\frac{x+a_t}{n_t}\Big\}\Big)\,, \tag{18}$$

where $\{x\}$ denotes the fractional part of x. As in the meantime J. Beebee called our attention to a gap in our original proof of this relation[3], we give here another proof of this identity. Along the lines of [7] we can prove that for any positive integer M and any real y we have:

$$B_r(\{y\})=M^{r-1}\sum_{j=0}^{M-1}B_r\left(\Big\{\frac{y+j}{M}\Big\}\right). \tag{19}$$

If $N=\text{l.c.m.}\ [n_1,\dots,n_k]$ and $N_t=N/n_t$, $t=1,\dots,k$, then for $y=(x+a_t)/n_t$ and $M=N_t$ we get from (19)

$$n_t^{r-1}B_r\Big(\Big\{\frac{x+a_t}{n_t}\Big\}\Big)=N^{r-1}\sum_{j=0}^{N_t-1}B_r\Big(\Big\{\frac{x+a_t+jn_t}{N}\Big\}\Big)\,,$$

i.e.

$$\sum_{t=1}^{k}\mu_t n_t^{r-1}B_r\Big(\Big\{\frac{x+a_t}{n_t}\Big\}\Big)=N^{r-1}\sum_{n=0}^{N-1}\mathfrak{m}(n)B_r\Big(\Big\{\frac{x+n}{N}\Big\}\Big)\,.$$

Since $n_0\,|\,N$, we can reduce the right hand side of the last equality to the left hand side of (18) using (19) once again. Namely, if $N_0=N/n_0$ and $0\le n\le n_0-1$, then (18) and the periodicity of $\mathfrak{m}$ give

$$(n_0N_0)^{r-1}\sum_{t=0}^{N_0-1}\mathfrak{m}(n+tn_0)B_r\Big(\Big\{\frac{x+n+tn_0}{n_0N_0}\Big\}\Big)=n_0^{r-1}\mathfrak{m}(n)B_r\Big(\Big\{\frac{x+n}{n_0}\Big\}\Big)\,,$$

and (18) follows.

The corresponding even analogue of the mentioned Hermite's identity is contained in the next results:

[3] Two lines above this relation in [11] also a misprint appears. The formula here should be $\{x\}=\{x+a\}=\{\{(x+a)/n\}n\}$.

THEOREM 2. *If a system* (1) *is* $(\mu, \mathfrak{m})$*-covering, then for every real number* x *we have*

$$\delta\Bigg(\mathfrak{m}(0)\Big[\frac{x}{n_0}\Big] - \mathfrak{m}(1)\Big[\frac{x+1}{n_0}\Big] + \cdots + (-1)^{n_0-1}\mathfrak{m}(n_0-1)\Big[\frac{x+n_0-1}{n_0}\Big]\Bigg)$$
$$= \mu_1(-1)^{a_1}\Big[\frac{x+a_1}{n_1}\Big] + \mu_2(-1)^{a_2}\Big[\frac{x+a_2}{n_2}\Big] + \cdots + \mu_k(-1)^{a_q}\Big[\frac{x+a_q}{n_q}\Big].$$

COROLLARY 1. *If* (1) *is an* m *times covering system, then for every real number* x *we have*

$$(-1)^{a_1}\Big[\frac{x+a_1}{n_1}\Big] + (-1)^{a_2}\Big[\frac{x+a_2}{n_2}\Big] + \cdots + (-1)^{a_q}\Big[\frac{x+a_q}{n_q}\Big] = 0.$$

COROLLARY 2. *If* n *is an even positive integer, then for every real number* x *we have*

$$\Big[\frac{x+1}{n}\Big] - \Big[\frac{x+2}{n}\Big] + \cdots - \Big[\frac{x+n-1}{n}\Big] = 0.$$

THEOREM 3. *Let* $n_1, n_2, \dots, n_q$ *be the all even and* $n_{q+1}, n_{q+2}, \dots, n_k$ *the all odd moduli of congruences in* (1). *Then* (1) *is* $(\mu, \mathfrak{m})$*-covering if and only if*

$$\Bigg(\big(2\delta + (1-\delta)2^r - 1\big)n_0^{r-1}\sum_{t=0}^{n_0-1}(-1)^t\mathfrak{m}(t)$$
$$- \sum_{t=1}^{q}(-1)^{a_t}\mu_t n_t^{r-1} - (2^r-1)\sum_{t=q+1}^{k}(-1)^{a_t}\mu_t n_t^{r-1}\Bigg)B_r$$
$$= \sum_{j=0}^{r-1}\binom{r}{j}B_j\cdot\Bigg(\sum_{t=1}^{q}(-1)^{a_t}\mu_t n_t^{j-1}a_t^{r-j} + (2^j-1)\sum_{t=q+1}^{k}(-1)^{a_t}\mu_t n_t^{j-1}a_t^{r-j}$$
$$- \big(2\delta + (1-\delta)2^j - 1\big)n_0^{j-1}\sum_{t=0}^{n_0-1}(-1)^t\mathfrak{m}(t)t^{r-j}\Bigg)$$

for every $r \in \mathbb{Z}^*$.

P r o o f. Take $x = 0$ in (15) and firstly consider its left-hand side. Expanding the Eulerian polynomials using (11) we obtain

$$
\begin{aligned}
&(1-\delta)n_0^{r-1}r\sum_{t=0}^{n_0-1}(-1)^{t+1}\mathfrak{m}(t)E_{r-1}\Big(\frac{t}{n_0}\Big)\\
&=(1-\delta)n_0^{r-1}r\sum_{t=0}^{n_0-1}(-1)^{t+1}\mathfrak{m}(t)\sum_{j=0}^{r-1}\binom{r-1}{j}E_j^{(1)}\Big(\frac{t}{n_0}\Big)^{r-1-j}\\
&\overset{(12)}{=}(1-\delta)n_0^{r-1}r\sum_{j=0}^{r-1}\binom{r-1}{j}\Big(-\frac{2(2^{j+1}-1)}{j+1}\Big)B_{j+1}\sum_{t=0}^{n_0-1}(-1)^{t+1}\mathfrak{m}(t)\Big(\frac{t}{n_0}\Big)^{r-1-j}\\
&=(1-\delta)n_0^{r-1}2\sum_{j=1}^{r}\binom{r}{j}(2^j-1)B_j\sum_{t=0}^{n_0-1}(-1)^t\mathfrak{m}(t)\Big(\frac{t}{n_0}\Big)^{r-j}\\
&=(1-\delta)2\sum_{j=0}^{r}\binom{r}{j}(2^j-1)B_jn_0^{j-1}\sum_{t=0}^{n_0-1}(-1)^t\mathfrak{m}(t)t^{r-j}\,.
\end{aligned}
$$

For terms containing Bernoulli polynomials, similarly we get

$$
\begin{aligned}
2\delta n_0^{r-1}\sum_{t=0}^{n_0-1}(-1)^t\mathfrak{m}(t)B_r\Big(\frac{t}{n_0}\Big)&=2\delta n_0^{r-1}\sum_{t=0}^{n_0-1}(-1)^t\mathfrak{m}(t)\sum_{j=0}^{r}\binom{r}{j}B_j\cdot\Big(\frac{t}{n_0}\Big)^{r-j}\\
&=2\delta\sum_{j=0}^{r}\binom{r}{j}B_jn_0^{j-1}\sum_{t=0}^{n_0-1}(-1)^t\mathfrak{m}(t)t^{r-j}\,.
\end{aligned}
$$

On the right-hand side we obtain

$$
\begin{aligned}
2\sum_{t=1}^{q}(-1)^{a_t}\mu_tn_t^{r-1}B_r\Big(\frac{a_t}{n_t}\Big)&=2\sum_{t=1}^{q}(-1)^{a_t}\mu_tn_t^{r-1}\sum_{j=0}^{r}\binom{r}{j}B_j\cdot\Big(\frac{a_t}{n_t}\Big)^{r-j}\\
&=2\sum_{j=0}^{r}\binom{r}{j}B_j\sum_{t=1}^{q}(-1)^{a_t}\mu_tn_t^{j-1}t^{r-j},
\end{aligned}
$$

and

$$\begin{aligned} & r\sum_{t=q+1}^{k}(-1)^{a_t+1}\mu_t n_t^{r-1}E_{r-1}\Big(\frac{t}{n_0}\Big) \\ &= r\sum_{t=q+1}^{k}(-1)^{a_t+1}\mu_t n_t^{r-1}\sum_{j=0}^{r-1}\binom{r-1}{j}E_j^{(1)}\cdot\Big(\frac{a_t}{n_t}\Big)^{r-1-j} \\ &= r\sum_{j=0}^{r-1}\binom{r-1}{j}\Big(-\frac{2(2^{j+1}-1)}{j+1}\Big)B_{j+1}\sum_{t=q+1}^{k}(-1)^{a_t+1}\mu_t n_t^{r-1}\Big(\frac{a_t}{n_t}\Big)^{r-1-j} \\ &= 2\sum_{j=0}^{r}\binom{r}{j}(2^j-1)B_j\sum_{t=q+1}^{k}(-1)^{a_t}\mu_t n_t^{j-1}a_t^{r-j}, \end{aligned}$$

and the theorem follows.

The above result again simplifies for m times covering systems.

Corollary 1. *Let $n_1, n_2, \dots, n_q$ be the all even and $n_{q+1}, n_{q+2}, \dots, n_k$ the all odd moduli of congruences in* (1). *Then* (1) *is m times covering if and only if*

$$\begin{aligned} &\Big((2^r-1)m - \sum_{t=1}^{q}(-1)^{a_t}\mu_t n_t^{r-1} - (2^r-1)\sum_{t=q+1}^{k}(-1)^{a_t}\mu_t n_t^{r-1}\Big)B_r \\ &= \sum_{j=0}^{r-1}\binom{r}{j}B_j\cdot\Big(\sum_{t=1}^{q}(-1)^{a_t}\mu_t n_t^{j-1}a_t^{r-j} \\ &\qquad + (2^j-1)\sum_{t=q+1}^{k}(-1)^{a_t}\mu_t n_t^{j-1}a_t^{r-j} - (2^j-1)m\Big) \end{aligned}$$

for every $r \in \mathbb{Z}^*$.

Further simplification can be obtained for disjoint covering system (3) with even or odd m:

Corollary 2.

a) *For each even positive integer m we have*

$$(2^r-1)mB_r = \sum_{j=0}^{r-1}\binom{r}{j}B_j m^j\sum_{t=1}^{m-1}(-1)^t t^{r-j}.$$

b) *For every odd positive integer* m *we have*

$$(1-2^r)(m^r-m)B_r=\sum_{j=1}^{r-1}\binom{r}{j}(2^j-1)m^jB_j\sum_{t=0}^{m-1}(-1)^t t^{r-j}.$$

The proof follows from the previous Corollary using (9).

Note that the first formula reduces for $m=2$ to [14; p. 14, formula VIII]. Note also the difference on the left hand side caused by using alternating signs in the last sum in formula a) in comparison with the original formula

$$m(1-m^r)B_r=\sum_{j=0}^{r-1}\binom{r}{j}B_jm^j\sum_{t=1}^{m-1}t^{r-j},$$

proved in [3], which was the starting point for the above mentioned Beebee's generalization [1].

3. Identities for systems with odd moduli

One way to attack the Erdős-Selfridge conjecture may consist in proving the impossibility of some consequences of identities of the above type. Motivated by this idea we prove the following result:

THEOREM 4. *Let all the moduli of* (1) *be odd. Then* (1) *is* $(\mu,\mathfrak{m})$*-covering if and only if*

$$\left(n_0^r\sum_{t=0}^{n_0-1}(-1)^t\mathfrak{m}(t)-\sum_{t=1}^{k}(-1)^{a_t}\mu_tn_t^r\right)E_r$$
$$=\sum_{s=0}^{r-1}\binom{r}{s}E_s\cdot\left(\sum_{t=1}^{k}(-1)^{a_t}\mu_tn_t^s(2a_t-n_t)^{r-s}-n_0^s\sum_{t=0}^{n_0-1}(-1)^t\mathfrak{m}(t)(2t-n_0)^{r-s}\right)$$

for every $r\in\mathbb{Z}^*$.

Proof. The left-hand side of (15) reduces for $x=0$ to

$$n_0^{r-1}r\sum_{t=0}^{n_0-1}(-1)^{t+1}\mathfrak{m}(t)E_{r-1}\left(\frac{t}{n_0}\right)$$
$$\overset{(7)}{=}n_0^{r-1}r\sum_{t=0}^{n_0-1}(-1)^{t+1}\mathfrak{m}(t)\sum_{s=0}^{r-1}\binom{r-1}{s}\frac{E_s}{2^s}\left(\frac{t}{n_0}-\frac{1}{2}\right)^{r-1-s}$$
$$=\frac{r}{2^{r-1}}\sum_{s=0}^{r-1}\binom{r-1}{s}E_sn_0^s\sum_{t=0}^{n_0-1}(-1)^{t+1}\mathfrak{m}(t)(2t-n_0)^{r-1-s}.$$

On the right-hand side we have

$$\begin{aligned}
&r\sum_{t=1}^{k}(-1)^{a_t+1}\mu_t n_t^{r-1}E_{r-1}\Big(\frac{t}{n_0}\Big)\\
&=r\sum_{t=1}^{k}(-1)^{a_t+1}\mu_t n_t^{r-1}\sum_{s=0}^{r-1}\binom{r-1}{s}\frac{E_s}{2^s}\Big(\frac{a_t}{n_t}-\frac{1}{2}\Big)^{r-1-s}\\
&=\frac{r}{2^{r-1}}\sum_{s=0}^{r-1}\binom{r-1}{s}E_s\sum_{t=1}^{k}(-1)^{a_t+1}\mu_t n_t^{s}(2a_t-n_t)^{r-1-s};
\end{aligned}$$

these formulae together give

$$\begin{aligned}
&\frac{r}{2^{r-1}}\sum_{s=0}^{r-1}\binom{r-1}{s}E_s n_0^s\sum_{t=0}^{n_0-1}(-1)^{t+1}\mathrm{m}(t)(2t-n_0)^{r-1-s}\\
&=\frac{r}{2^{r-1}}\sum_{s=0}^{r-1}\binom{r-1}{s}E_s\sum_{t=1}^{k}(-1)^{a_t+1}\mu_t n_t^{s}(2a_t-n_t)^{r-1-s}.
\end{aligned}$$

Since after division by $-\frac{r}{2^{r-1}}$ and after substitution $r-1\mapsto r$ the resulting identity remains true for $r=0$, the theorem follows.

Note that the Erdős-Selfridge conjecture applies only to covering systems with all the moduli distinct. Since in m times covering systems this is not the case [10], the following results do not apply to the conjecture.

COROLLARY 1. *Let all the moduli of congruences be odd in* (1). *Then* (1) *is an m times covering if and only if*

$$\begin{aligned}
&\Big(m-\sum_{t=1}^{k}(-1)^{a_t}n_t^r\Big)E_r\\
&=\sum_{s=0}^{r-1}\binom{r}{s}E_s\cdot\Big(\sum_{t=1}^{k}(-1)^{a_t}n_t^s(2a_t-n_t)^{r-s}-(-1)^{r-s}m\Big)
\end{aligned}$$

for every $r\in\mathbb{Z}^*$.

Substituting the disjoint covering system (3) with odd m into the last Corollary we get:

COROLLARY 2. *For every odd positive integer m and $r \in \mathbb{Z}^*$ we have*

$$(1-m^r)E_r = \sum_{s=0}^{r-1}\binom{r}{s}E_s \cdot \left(m^s \sum_{t=0}^{m-1}(-1)^t(2t-m)^{r-s} - (-1)^{r-s}\right).$$

For recurrence purposes the following form of the last result is more appropriate:

COROLLARY 3. *For every odd positive integer m and $r \in \mathbb{Z}^*$ we have*

$$m^{2r}E_{2r} = \sum_{s=0}^{r-1}\binom{2r}{2s}m^{2s}E_{2s}\sum_{t=1}^{m-1}(-1)^{t+1}(2t-m)^{2(r-s)}.$$

To the proof note that (8) and the fact that $E_{2n+1} = 0$ for $n = 0, 1, \dots$ imply for even $r = 2f$

$$-\sum_{s=0}^{r-1}\binom{r}{s}E_s(-1)^{r-s} = -\sum_{s=0}^{f-1}\binom{2f}{2s}E_{2s}(-1)^{2(f-s)} = E_{2f}.$$

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