Mathematica Slovaca

Ladislav Matejíčka

On approximate solutions of degenerate integrodifferential parabolic problems

Mathematica Slovaca, Vol. 45 (1995), No. 1, 91--103

Persistent URL: http://dml.cz/dmlcz/136638

Terms of use:

© Mathematical Institute of the Slovak Academy of Sciences, 1995

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Math. Slovaca, 45 (1995), No. 1, 91–103

Mathematica Slovaca
©1995
Mathematical Institute
Slovak Academy of Sciences

ON APPROXIMATE SOLUTIONS OF DEGENERATE INTEGRODIFFERENTIAL PARABOLIC PROBLEMS

LADISLAV MATEJÍČKA

(*Communicated by Jozef Kačúr*)

ABSTRACT. A solution of a nonlinear diffusion problem with Volterra operators by Rothe's method is obtained. The convergence of Rothe's functions to the solution, constructed by means of weak approximate solutions of approximation elliptic equations, is proved.

1. Introduction

In this paper, we shall deal with the following diffusion problem:

$$\begin{aligned} \partial_t u(t) - \Delta\beta\big(u(t)\big) &= f\left(t, \int_0^t K(t,s)\beta\big(u(s)\big)\,\mathrm{d}s\right) && \text{for} \quad (t,x) \in (0,T)\times\Omega\,, \\ \beta\big(u(0,x)\big) &= \beta\big(u_0(x)\big) && \text{on} \quad \Omega\,, \qquad (1) \\ \partial_\nu\beta\big(u(t)\big) &= g\left(t, \int_0^t M(t,s)\beta\big(u(s)\big)\,\mathrm{d}s\right) && \text{on} \quad (0,T)\times\Gamma\,, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz continuous boundary Γ, $0 < T < \infty$.

The solution of this problem will be obtained via solutions of linear approximation schemes. This way of solving nonlinear evolution equations has been introduced by Berger, Brezis, Rogers in [2]. They have dealt with the convergence of linear approximation schemes constructed for the problem (2′):

$$\partial_t u(t) - \Delta f\big(u(t)\big) = 0\,. \tag{2′}$$

Their results have been developed by W. Jäger and J. Kačúr ([6], [7], [10]). They have presented new approximation schemes for (2) and proved the convergence of these schemes:

AMS Subject Classification (1991): Primary 35K65, 35K55.
Key words: Integrodifferential parabolic equation, Rothe's method.

$$\begin{aligned} \partial_t u(t) - \Delta\beta\big(u(t)\big) &= f\big(t, \beta\big(u(t)\big)\big) && \text{in} \quad (t,x) \in (0,T) \times \Omega\,, \\ \beta\big(u(0,x)\big) &= \beta\big(u_0(x)\big) && \text{on} \quad \Omega\,, \\ \partial_\nu \beta\big(u(t)\big) &= g\big(t, \beta\big(u(t)\big)\big) && \text{on} \quad (0,T) \times \Gamma\,, \end{aligned} \tag{2}$$

where $\beta\colon \mathbb{R} \to \mathbb{R}$ is a nondecreasing Lipschitz continuous function.

The aim of this paper is to prove the convergence of the approximation schemes (introduced by W. Jäger and J. Kačúr) for (1). Problem (1) differs from (2) in the right-hand side. The equation and the boundary condition in (1) depend on Volterra operators. In this paper, we use the technique and methods which have been presented by W. Jäger and J. Kačúr ([6], [7], [10]). For a more complete survey of solutions of nonlinear degenerate parabolic problems, we refer the reader for example to [1], [5], [8], [13], [14], [17].

We also use the technique of memory terms for evolution integrodifferential equations, which has been presented by J. Kačúr in [9]. For another approach to the analysis of evolution integrodifferential equations, we refer the reader for example to [4], [12], [15], [16], [18].

Problem (1) shall be solved in the following way. We will divide the interval $I = (0,T)$ into n subintervals $\langle t_{i-1}, t_i\rangle$, $i = 1, \dots, n$, where $t_i = i\Big(\dfrac{T}{n}\Big)$. Then we shall find weak approximate solutions of the elliptic problem (3) on each subinterval $\langle t_{i-1}, t_i\rangle$ via weak solutions of the elliptic problem (4). From these solutions of (3) we shall construct Rothe's function $u_n(t,x)$. Finally, we shall prove that the weak limit u of u_n, in the functional space $L_2(I \times \Omega)$, is a weak solution of (1).

2. Notation and assumptions

We denote

$$(f,g) = \int_\Omega fg = \int_\Omega f(x)\cdot g(x)\,\mathrm{d}x\,, \qquad (f,g)_\Gamma = \int_\Gamma fg = \int_\Gamma f(x)\cdot g(x)\,\mathrm{d}x\,,$$

$$((f,g)) = (\nabla f, \nabla g)\,, \qquad H = W_2^1(\Omega) \text{ (Sobolev space)}\,,$$

$C^{0,\alpha}(\bar\Omega)$, $C(\bar\Omega)$, $L_2(I \times \Omega) = L_2\big(I, L_2(\Omega)\big) = L_2(I, L_2)$, $L_2(\Omega)$, $L_2(\Gamma)$ and $L_\infty(I,H)$ are the standard functional spaces. $\langle f, g\rangle$ is the duality between $f \in V^*$ and $g \in V$. $|\cdot|$, $|\cdot|_\Gamma$ and $|\cdot|_H$ are the norms in the functional spaces $L_2(\Omega)$, $L_2(\Gamma)$, H respectively. By C_i, we denote a generic positive constant.

We shall assume:

(P1) $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz continuous boundary Γ, $0 < T < \infty$.

(P2) $\beta\colon \mathbb{R} \to \mathbb{R}$ is a nondecreasing Lipschitz continuous function with $|\beta(s)| \ge C_1|s| - C_2$ for all $s \in \mathbb{R}$, $\beta(0) = 0$.

(P3) $g\colon I \times \Gamma \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in x and $|g(t,x,s) - g(t_1,x,s_1)| \le C\big(|t-t_1| + |t-t_1|\big(|s|+|s_1|\big) + |s-s_1|\big)$ for all $t, t_1 \in I$, $x \in \Gamma$, $s, s_1 \in \mathbb{R}$.

(P4) $f(t,x,s)$ is Lipschitz continuos in x and $|f(t_1,x,s_1) - f(t_2,x,s_2)| \le C\big(|t_1-t_2| + |t_1-t_2|\big(|s_1|+|s_2|\big) + |s_1-s_2|\big)$ for all $t_1, t_2 \in I$, $x \in \Omega$, $s_1, s_2 \in \mathbb{R}$.

(P5) $K(t,x,s), K_t(t,x,s) \in L_\infty(I \times \Omega \times I)$.

(P6) $M(t,x,s), M_t(t,x,s) \in L_\infty(I \times \Gamma \times I)$.

(P7) $u_0(x) \in L_\infty(\Omega)$ and $\beta\big(u_0(x)\big) \in W_2^1(\Omega)$.

3. Solution of the problem (1)

Definition 1. The function $u \in L_2(I, L_2)$ with $\partial_t u \in L_2(I, H^*)$ is called a *weak solution* of (1) if and only if

$$\int_I \langle \partial_t u(t), \varphi(t)\rangle + \int_I \big(\big(\beta\big(u(t)\big), \varphi(t)\big)\big) - \int_I \left(g\left(t, \int_0^t M(t,s)\cdot\beta\big(u(s)\big)\,\mathrm{d}s\right), \varphi(t)\right)_\Gamma = \int_I \left(f\left(t, \int_0^t K(t,s)\cdot\beta\big(u(s)\big)\,\mathrm{d}s\right), \varphi(t)\right)$$

for all $\varphi(t) \in L_2(I,H)$, $\beta\big(u(t)\big) \to \beta(u_0)$ in H^* for $t \to 0$, and $\beta(u) \in L_2(I,H)$.

Let n be a positive integer, $\tau = \dfrac{T}{n}$, $t_i = \tau \cdot i$, for $i = 1, \dots, n$. The linear approximation scheme corresponding to (1) can be written in the following way:

$$\begin{aligned} \mu_i(x)\cdot\big(\theta_i(x) - \beta\big(u_{i-1}(x)\big)\big) - \tau\Delta\theta_i(x) &= \tau f\left(t_i, x, \tau\sum_{j=0}^{i-1} K_{ij}(x)\theta_j(x)\right), \quad x \in \Omega, \\ \partial_\nu \theta_i(x) &= g\left(t_i, x, \tau\sum_{j=0}^{i-1} M_{ij}(x)\theta_j(x)\right), \quad x \in \Gamma, \end{aligned} \tag{3}$$

with the condition

$$\big|\beta\big(u_{i-1} + \mu_i\big(\theta_i - \beta(u_{i-1})\big)\big) - \beta(u_{i-1})\big| \le \alpha|\theta_i - \beta(u_{i-1})| + o\Big(\frac{1}{n}\Big), \tag{3.1}$$

where $u_i = u_{i-1} + \mu_i\big(\theta_i - \beta(u_{i-1})\big)$, $K_{ij}(x) = \dfrac{1}{\tau}\cdot\displaystyle\int_{t_j}^{t_{j+1}} K(t_i,x,s)\,\mathrm{d}s$, $M_{ij}(x) = \dfrac{1}{\tau}\cdot\displaystyle\int_{t_j}^{t_{j+1}} M(t_i,x,s)\,\mathrm{d}s$, $\mu_i(x) \in L_\infty(\Omega)$, $\theta_0 = \beta(u_0)$, $\displaystyle\lim_{n\to\infty} n\cdot o\Big(\frac{1}{n}\Big) = 0$, $i = 1, \dots, n$, $j = 0, \dots, i-1$.

There are many ways how to solve (3) with the condition (3.1). For example, we can put $\mu_i(x) = C$, $0 < C < \frac{\alpha}{L_\beta}$ and solve (3) by a numerical method. However, from the numerical point of view, these solutions are not satisfactory [11]. We shall show a way for finding better solutions. We use the idea of W. Jäger and J. Kačúr.

DEFINITION 2. We say that μ_i, θ_i are *weak approximate solutions* of (3) on $(t_{i-1}, t_i\rangle$ if there exist $q_i \in H^*$ and positive constants δ, K such that

(i) $\delta < \mu_i(x) < K$ for a.e. $x \in \Omega$, $\theta_i(x) \in H$, $|q_i(x)|_{H^*} < o\left(\frac{1}{n^2}\right)$,

(ii) $$\int_\Omega \mu_i(x)\cdot\big(\theta_i(x) - \beta(u_{i-1}(x))\big)\varphi(x) + \tau\int_\Omega \nabla\theta_i(x)\nabla\varphi(x) - \langle q_i(x), \varphi(x)\rangle - \tau\int_\Gamma g\Big(t_i, x, \tau\sum_{j=0}^{i-1} M_{ij}(x)\theta_j(x)\Big)\varphi(x) = \tau\int_\Omega f\Big(t_i, x, \tau\sum_{j=0}^{i-1} K_{ij}(x)\theta_j(x)\Big)\varphi(x) \text{ for all } \varphi(x) \in H.$$

The weak approximate solutions of (3) for given n will be obtained via solutions of (4). Similarly to M. Slodička [17], we shall consider the function $\beta_\varepsilon(s)$ instead of $\beta(s)$.

The scheme (4) reads as follows:

$$\begin{aligned} \beta_\varepsilon^{-1}\Big(\beta_\varepsilon(u_{i-1}) + \frac{\alpha}{2}\big(\theta_i - \beta(u_{i-1})\big)\Big) - u_{i-1} - \tau\Delta\theta_i &= \tau f\Big(t_i, \tau\sum_{j=0}^{i-1} K_{ij}\theta_j\Big), \quad x \in \Omega, \\ \partial_\nu\theta_i &= g\Big(t_i, \tau\sum_{j=0}^{i-1} M_{ij}\theta_j\Big), \qquad x \in \Gamma, \end{aligned} \tag{4}$$

where $\beta_\varepsilon(s) = \beta(s) + \varepsilon\cdot s$, and ε is a suitable constant.

Define $T_\varepsilon \colon H \to H^*$,

$$\langle T_\varepsilon(\theta), \varphi\rangle = \Big(\beta_\varepsilon^{-1}\Big(\beta_\varepsilon(u_{i-1}) + \frac{\alpha}{2}\big(\theta - \beta(u_{i-1})\big)\Big) - u_{i-1}, \varphi\Big) + \tau(\nabla\theta, \nabla\varphi).$$

Then

$$|T_\varepsilon(x) - T_\varepsilon(y)|_{H^*} < C_1(\varepsilon)|x-y|_H \quad \text{and}$$
$$\big\langle T_\varepsilon(x) - T_\varepsilon(y), x - y\big\rangle > C_2(\varepsilon)|x-y|_H^2, \quad C_2(\varepsilon) > 0, \qquad \text{for all } x, y \in H.$$

Now we make use of [3; p. 104, Theorem 3.4]. There exists $\{v_k\}_{k=1}^\infty$ such that $v_k \in H$ and $v_k \to \theta_i$ in H (θ_i is a weak solution of (4)).

Choose v_k so that $|v_k - \theta_i|_H < \varepsilon \frac{1}{n^2}$ and define

$$\mu_i(x) = \left(\frac{\beta_\varepsilon^{-1}\Big(\beta_\varepsilon\big(u_{i-1}(x)\big) + \frac{\alpha}{2}\big(v_k(x) - \beta(u_{i-1}(x))\big)\Big) - u_{i-1}(x)}{v_k(x) - \beta\big(u_{i-1}(x)\big)} \right).$$

Then μ_i, v_k satisfy

$$\begin{aligned} &\big(\mu_i\big(v_k - \beta(u_{i-1})\big), \varphi\big) + \tau(\nabla v_k, \nabla\varphi) \\ &= \tau\Big(g\Big(t_i, \tau\sum_{j=0}^{i-1} M_{ij}\theta_j\Big), \varphi\Big)_\Gamma + \tau\Big(f\Big(t_i, \tau\sum_{j=0}^{i-1} K_{ij}\theta_j\Big), \varphi\Big) + \langle q_i, \varphi\rangle, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \langle q_i, \varphi\rangle = \big(\mu_i\big(v_k - \beta(u_{i-1})\big), \varphi\big) + \tau\big(\nabla(v_k - \theta_i), \nabla\varphi\big) \\ - \Big(\beta_\varepsilon^{-1}\big(\beta_\varepsilon(u_{i-1}) + \frac{\alpha}{2}\big(\theta_i - \beta(u_{i-1})\big)\big) - u_{i-1}, \varphi\Big). \end{aligned}$$

Since $|q_i|_{H^*} < o\Big(\frac{1}{n^2}\Big)$, we have that v_k, μ_i are weak approximate solutions of (3). In addition, v_k, μ_i satisfy (3.1) and there exist $\delta(\varepsilon)$, $K(\varepsilon)$ such that

$$0 < \delta(\varepsilon) < \mu_i < K(\varepsilon) \qquad \text{for a.e. } x \text{ and for all } i.$$

So we can formulate the following theorem.

THEOREM 1. *Let* (P1)–(P7) *be satisfied. Let* $n \in \mathbb{N}$. *Then there exist* $\{\mu_i\}_{i=1}^n$, $\{\theta_i\}_{i=1}^n$ *such that* $\theta_i(x) \in W_2^1(\Omega)$, $\mu_i(x) \in L_\infty(\Omega)$, *and the functions* θ_i, μ_i *are weak approximate solutions of* (3) *and satisfy* (3.1).

Now we prove the following lemma.

LEMMA 1. *There exists* $n_0 \in \mathbb{N}$ *such that*

$$\max_{1\le i\le n} |\beta(u_{i-1})|_{L_2} + \sum_{i=1}^n |\theta_i|_H^2\tau + \sum_{i=1}^n |u_i - u_{i-1}|_{L_2}^2 \le C$$

for all $n \ge n_0$, *where* u_i, θ_i *are weak approximate solutions of* (3) *in* $\langle t_{i-1}, t_i\rangle$.

P r o o f. To prove this assertion, we put $\varphi = \tau\theta_i$ in the equation (5).

$$\begin{aligned} &\Big(\frac{u_i - u_{i-1}}{\tau}, \varphi\Big) + ((\theta_i, \varphi)) - \Big(g\Big(t_i, \tau\sum_{j=0}^{i-1} M_{ij}\theta_j\Big), \varphi\Big)_\Gamma \\ &\qquad = \Big(f\Big(t_i, \tau\sum_{j=0}^{i-1} K_{ij}\theta_j\Big), \varphi\Big) + \langle q_i, \varphi\rangle(\tau^{-1}), \end{aligned} \tag{3'}$$

which holds for all $\varphi \in H$, $i = 1, \dots, n$. Sum up them for $i = 1, \dots, l$. We will estimate only the terms

$$\left(g\left(t_i, \tau \sum_{j=0}^{i-1} M_{i,j}\theta_j\right), \varphi\right)_\Gamma, \qquad \langle q_i, \varphi\rangle(\tau^{-1}), \qquad \left(f\left(t_i, \tau \sum_{j=0}^{i-1} K_{i,j}\theta_j\right), \varphi\right).$$

The others can be estimated in the same way as in [6], [7].

$$\sum_{i=1}^{l}\left(g\left(t_i, \tau \sum_{j=0}^{i-1} M_{ij}\theta_j\right), \tau\theta_i\right)_\Gamma \le C_1\tau \sum_{i=1}^{l} |\theta_i|_\Gamma^2 + C_2\tau^2 \sum_{i=1}^{l}\sum_{j=0}^{i-1} \int_\Gamma |\theta_j| \cdot |\theta_i| + C_3$$

because $|g(t,s)| \le C_1 + C_2\big(|t| + |t| \cdot |s| + |s|\big)$. Now the estimate

$$|\varphi|_\Gamma \le C_1\left(\varepsilon|\nabla\varphi|_{L_2}^2 + \frac{1}{\varepsilon}|\varphi|_{L_2}^2\right) \tag{1.1}$$

can be used, and we conclude

$$\left|\sum_{i=1}^{l}\left(g\left(t_i, \tau \sum_{j=0}^{i-1} M_{ij}\theta_j\right), \tau\theta_i\right)_\Gamma\right|$$

$$\le C_1 + C_2\varepsilon \sum_{i=1}^{l} |\nabla\theta_i|_{L_2}^2\tau + C_3(\varepsilon)\tau \sum_{i=1}^{l} |\beta(u_i)|_{L_2}^2 + \tau C_4(\varepsilon) \sum_{i=1}^{l} |u_i - u_{i-1}|_{L_2}^2 .$$

The second term will be estimated as follows:

$$|\langle q_i, \varphi\rangle| \le C_1\varepsilon\tau|\theta_i|_H^2 + C_2(\varepsilon)\tau .$$

The last term will be estimated similarly:

$$\left|\sum_{i=1}^{l}\left(f\left(t_i, \tau \sum_{j=0}^{i-1} K_{ij}\theta_j\right), \tau\theta_i\right)\right| \le C_1\tau^2 \sum_{i=1}^{l}\sum_{j=0}^{i-1} |(\theta_j, \theta_i)|$$

because $|K_{ij}|_{L_\infty(\Omega\times\Omega)} \le C$.

Hence, we obtain

$$\sum_{i=1}^{l}\left(f\left(t_i, \tau \sum_{j=0}^{i-1} K_{ij}\theta_j\right), \tau\theta_i\right) \le C_1\tau \sum_{i=1}^{l} |u_i - u_{i-1}|_{L_2}^2 + C_2\tau \sum_{i=1}^{l} |\beta(u_{i-1})|_{L_2}^2 .$$

From Gronwall's Lemma we obtain the assertion of Lemma 1. □

Now we construct Rothe's functions $\theta^{(n)}$, $\bar{\theta}^{(n)}$:

$$\theta^{(n)}(t) = \theta_{i-1} + (t - t_{i-1})\left(\frac{\theta_i - \theta_{i-1}}{\tau}\right) \qquad \text{for} \quad t \in \langle t_{i-1}, t_i\rangle ,$$

$$\bar{\theta}^{(n)}(t) = \theta_i \qquad \text{for} \quad t \in (t_{i-1}, t_i\rangle , \quad i = 1, \dots, n$$

and similarly, we define $u^{(n)}$, $\bar{u}^{(n)}$, $\mu^{(n)}$, $\bar{\mu}^{(n)}$.

The following lemma guarantees the compactness of $\{\theta^{(n)}\}_{n=1}^\infty$ in $L_2(I \times \Omega)$.

Lemma 2. *Suppose that $\bar\theta^{(n)}$ are the functions constructed above. Then we have the estimate*

$$\int_0^{T-z} |\bar\theta^{(n)}(t+z) - \bar\theta^{(n)}(t)|_{L_2}^2 \,\mathrm{d}t \le C\big(z + n^{-\frac{1}{2}}\big)$$

for $n \ge n_0$ and $0 < z < z_0$.

Proof. Since $W_2^1(\Omega) \hookrightarrow L_2(\Omega)$ (continuous imbedding) and $W_2^1(\Omega) \hookrightarrow L_2(\Omega)$ is dense, then $L_2^* \hookrightarrow H$ is dense (H is reflexive). So we can identify $\forall \alpha \in L_2$ with $f_\alpha \in H^*$ for which $(\alpha, \varphi)_{L_2} = \langle f_\alpha, \varphi\rangle$ for all $\varphi \in H$. Hence, we obtain the estimate:

$$\begin{aligned}
|\partial_t u^{(n)}|_{H^*} &= \sup_{|\varphi|_H \le 1} \Big(\frac{u_i - u_{i-1}}{\tau}, \varphi\Big) \\
&\le \sup_{|\varphi|_H \le 1} \bigg\{ \int_\Gamma g\Big(t_i, \tau\sum_{j=0}^{i-1} M_{ij}\theta_j\Big)\varphi - \int_\Omega \nabla\theta_i\nabla\varphi \\
&\qquad + \int_\Omega f\Big(t_i, \tau\sum_{j=0}^{i-1} K_{ij}\theta_j\Big)\varphi + \langle q_i, \varphi\rangle \frac{1}{\tau}\bigg\} \\
&\le C_1 + C_2|\theta_i|_H
\end{aligned}$$

for all $n \ge n_0$ and for all $t \in (t_{i-1}, t_i\rangle$.

Due to Lemma 1, we have $|\partial_t u^{(n)}|_{L_2(I,H^*)} \le C$.

We estimate

$$\int_0^{T-z} |\bar\theta^{(n)}(t+z) - \bar\theta^{(n)}(t)|_{L_2}^2 \,\mathrm{d}t \le \frac{C_1}{n} + \int_\tau^{T-z} |\bar\theta^{(n)}(t+z) - \bar\theta^{(n)}(t)|_{L_2}^2 .$$

Furthermore, using $\bar\theta^{(n)}(t+\tau) = \beta\big(\bar u^{(n)}(t)\big) + \dfrac{1}{\bar\mu^{(n)}(t+\tau)}\big(\bar u^{(n)}(t+\tau) - \bar u^{(n)}(t)\big)$, we obtain

$$\begin{aligned}
&\int_0^{T-z} |\bar\theta^{(n)}(t+z) - \bar\theta^{(n)}(t)|_{L_2}^2 \,\mathrm{d}t \\
&\le C \int_0^{T-z-\tau} \big(|\bar\theta^{(n)}(t+\tau+z)|_H + |\bar\theta^{(n)}(t+\tau)|_H\big) \int_t^{t+z} |\partial_t u^{(n)}(s)|_{H^*} \,\mathrm{d}s\,\mathrm{d}t + \frac{C_1}{\sqrt{n}} \\
&\le C_2\Big(z + \frac{1}{\sqrt{n}}\Big).
\end{aligned}$$

The estimates (2.1) and $|\bar{\theta}^{(n)}|_{L_2(I,H)} \le C$ imply the compactness of $\{\bar{\theta}^{(n)}\}_{n=1}^{\infty}$ in $L_2(I \times \Omega)$. □

Next, subsequences of $\{n\}$ will be denoted again by $\{n\}$.

LEMMA 3. *There exists $u \in L_2(I, L_2)$ with $\beta(u) \in L_2(I, H)$ and subsequences $\{u^{(n)}\}_{n=1}^{\infty}$, $\{\theta^{(n)}\}_{n=1}^{\infty}$ such that $u^{(n)} \rightharpoonup u$ in $L_2(I, L_2)$, $\partial_t u^{(n)} \rightharpoonup \partial_t u$ in $L_2(I, H^*)$, $\theta^{(n)} \rightharpoonup \theta$ in $L_2(I, H)$, $\beta(\bar{u}^{(n)}) \to \beta(u)$, and $\theta^{(n)} \to \beta(u)$ in $L_2(I, L_2)$.*

Proof. There exists a function $b \in L_2(I \times \Omega)$ such that $\theta^{(n)} \to b$. Since $|\bar{\theta}^{(n)} - \theta^{(n)}|_{L_2(I,L_2)} < \dfrac{C}{\sqrt{n}}$, we obtain $\bar{\theta}^{(n)} \to b$ in $L_2(I, L_2)$, and $|\bar{\theta}^{(n)} - \beta(\bar{u}^{(n)})|_{L_2(I,L_2)} \le \dfrac{C}{\sqrt{n}}$ implies $\beta(\bar{u}^{(n)}) \to b$ in $L_2(I \times \Omega)$.

Since $|\bar{u}^{(n)}|^2_{L_2(I,L_2)} \le C$, we have $\bar{u}^{(n)} \rightharpoonup u$ in $L_2(I, L_2)$, and since $|\partial_t u^{(n)}|_{L_2(I,H^*)} \le C$, we deduce $\partial_t u^{(n)} \rightharpoonup \partial_t u$.

The monotonicity of β implies

$$\int_I \Big(\beta(\bar{u}^{(n)}) - \beta(\varphi), u^{(n)} - \varphi\Big) \, \mathrm{d}t \ge 0 \qquad \text{for all} \quad \varphi \in L_2(I, L_2).$$

Now we use the Minty-Browder trick. If we put $\varphi = u \pm \varepsilon r$ and $\varepsilon \to 0$, $n \to \infty$, then

$$\int_I \big(b - \beta(u), r\big) \ge 0 \qquad \text{for all} \quad r \in L_2(I, L_2).$$

Hence, $b = \beta(u)$ and $\theta^{(n)} \to \beta(u)$ in $L_2(I, L_2)$. Since $|\bar{\theta}^{(n)}|^2_{L_2(I,H)} \le C$, we have $\theta^{(n)} \rightharpoonup \theta$ in $L_2(I, H)$ and, from $L_2(I, L_2) \supset L_2(I, H)$, we obtain $\theta = \beta(u)$ and $\beta(u) \in L_2(I, H)$. □

LEMMA 4. *Let u be the same as in Lemma 3. Then there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that*

$$\lim_{n_k \to \infty} \int_0^t \langle \partial_t u^{(n_k)}, \bar{\theta}^{(n_k)} \rangle \ge \int_\Omega \Phi_\beta\big(u(t)\big) \, \mathrm{d}x - \int_\Omega \Phi_\beta(u_0) \, \mathrm{d}x, \qquad (1_4)$$

$$\int_0^t \langle \partial_t u, \beta(u) \rangle = \int_\Omega \Phi_\beta\big(u(t)\big) \, \mathrm{d}x - \int_\Omega \Phi_\beta(u_0) \, \mathrm{d}x, \qquad (2_4)$$

where $\Phi_\beta(x) = \int_0^x \beta(s) \, \mathrm{d}s$.

The proof is similar as in [6], [7].

LEMMA 5. *Let u be as in Lemma 3; then there exists a subsequence of $\{\bar\theta^{(n)}\}_{n=1}^{\infty}$ such that $\bar\theta^{(n)} \to \beta(u)$ in $L_2(I,H)$.*

P r o o f. We use the equation

$$\Big(\frac{u_i - u_{i-1}}{\tau}, \psi\Big) + ((\theta_i, \psi)) - \Big(g\Big(t_i, \tau\sum_{j=0}^{i-1} M_{ij}\theta_j\Big), \psi\Big)_\Gamma$$
$$= \Big(f\Big(t_i, \tau\sum_{j=0}^{i-1} K_{ij}\theta_j\Big), \psi\Big) + \langle q_i, \psi\rangle\frac{1}{\tau} \qquad \text{for all} \quad \psi \in H\,, \quad i = 1,\dots,n\,,$$

and we obtain

$$\int_I \langle \partial_t u^{(n)}, \psi\rangle + \int_I ((\bar\theta^{(n)}, \psi)) - \int_I \big(g_n\big(t, \bar\theta_\tau^{(n)}\big), \psi\big)_\Gamma$$
$$= \int_I \big(f_n\big(t, \bar\theta_\tau^{(n)}\big), \psi\big) + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \langle q_i, \psi\rangle \frac{1}{\tau}\,,$$

where

$$f_n\big(t, \bar\theta_\tau^{(n)}\big) = f\Big(t_i, \tau\sum_{j=0}^{i-1} K_{ij}\theta_j(x)\Big)\,, \qquad g_n\big(t, \bar\theta_\tau^{(n)}\big) = g\Big(t_i, \tau\sum_{j=0}^{i-1} M_{ij}\theta_j\Big)$$

for $t \in (t_{i-1}, t_i\rangle$, $i = 1,\dots n$.

If we put $\psi = \bar\theta^{(n)} - \beta(u)$ and consider a suitable subsequence of $\{n\}_{n=1}^{\infty}$, we obtain

$$\lim_{n\to\infty} \int_I \big\langle \partial_t u^{(n)}, \bar\theta^{(n)} - \beta(u)\big\rangle \ge 0\,.$$

We estimate the term

$$A = \int_I \Big(g_n\big(t, \bar\theta_\tau^{(n)}\big), \big(\bar\theta^{(n)} - \beta\big(u(t)\big)\big)\Big)\, \mathrm{d}t\,;$$

then

$$A \le \bigg| A - \int_I\int_\Gamma \Big(g\Big(t, \int_0^t M(t,s)\beta\big(u(s)\big)\, \mathrm{d}s\Big), \Big(\bar\theta^n(t) - \beta\big(u(t)\big)\Big)\, \mathrm{d}t\Big)\bigg|$$
$$+ \bigg| \int_I\int_\Gamma \Big(g\Big(t, \int_0^t M(t,s)\beta\big(u(s)\big)\, \mathrm{d}s\Big), \Big(\bar\theta^n - \beta\big(u(t)\big)\Big)\Big)\, \mathrm{d}t\, \bigg|\,.$$

Hence

$$|A| \le \tau C_1 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_{\Gamma} \sum_{j=0}^{i-1} |\theta_j||(\theta_i - \beta(u))| + o(1)$$

because $\theta^{(n)}(t) \rightharpoonup \beta\big(u(t)\big)$ in $L_2\big(I, L_2(\Gamma)\big)$, where

$$\left| A - \int_I \int_\Gamma \left(g\left(t, \int_0^t M(t,s)\beta\big(u(s)\big)\, \mathrm{d}s\right), \left(\bar\theta^n(t) - \beta\big(u(t)\big)\right)\right) \right| = o(1)\,.$$

Hence

$$A \le C_1 |\bar\theta^{(n)} - \beta(u)|^2_{L_2(I,L_2)} + \varepsilon C_2 \big|\nabla\big(\bar\theta^{(n)} - \beta(u)\big)\big|^2_{L_2(I,L_2)} + o(1)$$

(we used (1.1)).

Thus we conclude $A \le \varepsilon \big|\nabla\big(\bar\theta^{(n)} - \beta(u)\big)\big|^2_{L_2(I,L_2)} + o(1)$.

Finally, we estimate the term B of the equation (4).

$$B = \int_I \Big(f_n\big(t, \bar\theta^{(n)}(t-\tau)\big), \bar\theta^{(n)}(t) - \beta\big(u(t)\big)\Big)$$

$$\le \tau C_1 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sum_{j=0}^{i-1} \int_\Omega |\theta_j||\theta_i - \beta(u)| + C_2\,,$$

$$B \le o(1) + C_2 |\bar\theta^{(n)} - \beta(u)|_{L_2(I,L_2)} = o(1)\,.$$

The other terms can be estimated similarly as in [6], [7]. Summarizing the estimates we deduce the required assertion. □

THEOREM 2. *Suppose* P1–P7. *Then there exist a weak solution* u *of* (1) *and subsequences* $\{\theta^{(n)}\}_{n=1}^\infty$, $\{u^{(n)}\}_{n=1}^\infty$ *of weak approximate solutions of* (3) *such that* $\theta^{(n)} \to \beta(u)$, $u^{(n)} \rightharpoonup u$ *in* $L_2(I\times\Omega)$, *and* $\theta^{(n)} \to \beta(u)$ *in* $L_2(I,H)$. *If the weak solution* u *of* (1) *is unique, then the original sequences* $\{\theta^{(n)}\}_{n=1}^\infty$, $\{u^{(n)}\}_{n=1}^\infty$ *are convergent.*

Proof. If we take a suitable subsequence of $\{n\}_{n=1}^\infty$, we obtain:

$$\int_I \langle \partial_t u^{(n)}, \psi\rangle \to \int_I \langle \partial_t u, \psi\rangle \qquad \text{because } \partial_t u^{(n)} \rightharpoonup \partial_t u \text{ in } L_2(I, H^*)\,,$$

$$\int_I \big((\bar\theta^{(n)}, \psi)\big) \to \int_I \big((\beta(u), \psi)\big) \qquad \text{because } \bar\theta^{(n)} \to \beta(u) \text{ in } L_2(I, H)\,,$$

where $\psi \in H$, $n \to \infty$.

Now we show that

$$\int_I \int_\Gamma g_n\big(t, \bar\theta^{(n)}(t-\tau)\big)\psi(t) \to \int_I \int_\Gamma g\left(t, \int_0^t M(t,s)\beta\big(u(s)\big)\, \mathrm{d}s\right)\psi(t)$$

for all $\psi(t) \in H$ if $n \to \infty$. We have

$$\int_I \int_\Gamma \left| g_n\big(t, \bar\theta_\tau^{(n)}\big) - g\left(t, \int_0^t M(t,s)\beta\big(u(s)\big)\, \mathrm{d}s\right)\right|^2$$

$$\le \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Gamma \left| \tau \sum_{j=0}^{i-1} M_{ij}\theta_j - \int_0^t M(t,s)\beta\big(u(s)\big)\, \mathrm{d}s \right|^2 + o(1)$$

$$\le C|\bar\theta_\tau^{(n)} - \beta(u)|^2_{L_2(I,L_2(\Gamma))} + o(1)\,.$$

Since

$$\int_I \int_\Omega \left| f_n\big(t, \bar\theta_\tau^{(n)}\big) - \int_0^t K(t,s)\beta\big(u(s)\big)\, \mathrm{d}s \right|^2 \mathrm{d}x\, \mathrm{d}t \to 0 \qquad \text{for} \quad n \to \infty\,,$$

from the definition of Bochner's integral by step functions, we obtain that u fulfils (1) for all $\varphi(t) \in L_2(I,H)$. From $u^{(n)}(t) \to u(t)$ in $C(I,H^*)$, we conclude that $\beta\big(u^{(n)}\big) \to \beta(u_0)$ in H^*.

R e m a r k . The results can be extended to the nonlinear degenerate equation (5) if we assume (P1), (P2), (P3), (P4), (P7). We have

$$\partial_t u(t) - \nabla\Big(k\big(t,x,\beta(u(t))\big)\cdot \nabla\beta\big(u(t)\big)\Big)$$

$$= f\left(t, x, \beta\big(u(t)\big), \int_0^t K(t,s)\beta\big(u(s)\big)\, \mathrm{d}s, \int_0^t N(t,s)\cdot\nabla\beta\big(u(s)\big)\, \mathrm{d}s\right),$$

$$\beta\big(u(0,x)\big) = \beta\big(u_0(x)\big)\,, \tag{5}$$

$$\partial_\nu \beta\big(u(t)\big) = g\left(t, x, \int_0^t M(t,s)\beta\big(u(s)\big)\, \mathrm{d}s\right),$$

where the matrix k is supposed to satisfy $|k| < C_1$ and $C_2|\psi|^2 \le \big(k(t,x,\varphi)\psi, \psi\big) \le C_3|\psi|^2$ for all $t, x \in (0,T)\times\Omega$, $\psi \in \mathbb{R}^N$, $\varphi \in \mathbb{R}$, and where

$$K, K_t \in L_\infty\Big(I\times I,\ \Phi\big(L_2(I\times\Omega),\ L_2(I\times\Omega)\big)\Big)\,,$$

$$N, N_t \in L_\infty^N\Big(I\times I,\ \Phi\big(L_2(I\times\Omega),\ L_2(I\times\Omega)\big)\Big)\,,$$

$$M, M_t \in L_\infty\Big(I\times I,\ \Phi\big(L_2(I\times\Gamma),\ L_2(I\times\Gamma)\big)\Big)\,.$$

$\Phi(X,Y)$ is a space of linear continuos mappings from X to Y.

If we consider the following degenerate parabolic system (6), we can obtain the same results providing again the same assumptions. One has

$$\partial_t u_i(t) - \nabla\Big(\mathbf{D}_i\big(t,x,\boldsymbol{b}(\boldsymbol{u}(t))\big)\nabla b_i\big(u_i(t)\big)\Big)$$

$$= f_i\left(t,x,\boldsymbol{b}\big(\boldsymbol{u}(t)\big), \int_0^t \boldsymbol{k}_i(t,s)\boldsymbol{b}\big(\boldsymbol{u}(s)\big)\,\mathrm{d}s, \int_0^t \boldsymbol{N}_i(t,s)\nabla\boldsymbol{b}\big(\boldsymbol{u}(s)\big)\,\mathrm{d}s\right) \qquad \text{in} \quad I\times\Omega,$$

$$\beta_i\big(u_i(x,0)\big) = \beta_i\big(u_{i0}(x)\big) \qquad \text{on} \quad \Omega, \tag{6}$$

$$\mathbf{D}_i\Big(t,x,\boldsymbol{b}\big(\boldsymbol{u}(t)\big)\Big)\cdot\partial_\nu b_i\big(u^i(t)\big) = g_i\left(t,x,\int_0^t \boldsymbol{m}_i(t,s)\cdot\boldsymbol{b}\big(\boldsymbol{u}(s)\big)\,\mathrm{d}s\right) \qquad \text{on} \quad I\times\Gamma$$

for $i=1,\dots,m$, where $\boldsymbol{u}=(u_1,\dots,u_m)$, $\boldsymbol{b}(\boldsymbol{u}) = \big(b_1(u_1),\dots,b_m(u_m)\big)$, $\boldsymbol{g}=(g_1,\dots,g_m)$, $\boldsymbol{x}\cdot\boldsymbol{y} = \sum_{j=1}^{n} x_j y_j$, and the matrices $\mathbf{D}_i$ satisfy $|\mathbf{D}_i(t,x,s)| < C$ ($|\cdot|$ is the norm of $\mathbf{D}_i$ in $\mathbb{R}^n$),

$$C_1|\boldsymbol{v}|^2 \le \big(\mathbf{D}_i(t,x,\boldsymbol{s})\boldsymbol{v},\boldsymbol{v}\big) \le C_2|\boldsymbol{v}|^2 \qquad \forall i=1,\dots,m,$$

uniformly for $(t,x)\in I\times\Omega$, $\boldsymbol{s}\in\mathbb{R}^m$, $\boldsymbol{v}\in\mathbb{R}^n$. The members for the vectors $\boldsymbol{k}_i(t,s)$, $\boldsymbol{m}_i(t,s)\in L_\infty(\Omega)$.

REFERENCES

[1] AMIEZ, G.—GREMAUD, P. A.: *On a numerical approach to Stefan-like problems*, Numer. Math. **59** (1991), 71–89.

[2] BERGER, A. E.—BREZIS, H.—ROGERS, J. C. W.: *A numerical method for solving the problem $\partial_t u(t) - \Delta f(u(t)) = 0$*, RAIRO Modél. Math. Anal. Numér. **13** (1979), 297 312.

[3] GAJEWSKI, H.—GRÖGER, K.—ZACHARIAS, K.: *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademia-Verlag. Berlin, 1974.

[4] CHEN, C.—THOMÉE, V.—WAHLBIN, L. B.: *Finite element approximation of a parabolic integrodifferential equation with a weakly singular kernel*, Math. Comp. **58** (1992), 587–602.

[5] JEROME, J. W.—ROSE, M. E.: *Error estimates for the multidimensional two-phase Stefan Problem*, Math. Comp. **39** (1982), 377–414.

[6] JÄGER, W.—KAČÚR, J.: *Approximation of porous medium type systems by non degenerate elliptic systems.* Preprint, Universität Heilderberg, SFB 123 (1990).

[7] JÄGER, W.—KAČÚR, J.: *Solution of porous medium type systems by linear approximation schemes*, Numer. Math. **60** (1991), 407–427.

[8] KAČÚR, J.: *Method of Rothe in Evolution Equations*, BSB Teubner Verlag, Leipzig, 1985.

[9] KAČÚR, J.: *Application of Rothe's method to evolution integrodifferential equations*, J. Reine Angew. Math. **388** (1988), 73 105.

[10] KAČÚR, J.—HANDLOVIČOVÁ, A.—KAČÚROVÁ, M.: *Solution of nonlinear diffusion problems by linear approximation schemes.* Preprint, Comenius University, Bratislava (Accepted to SIAM J. Numer. Anal.).

[11] KAČÚROVÁ, M.: *Solution of porous medium type problems with nonlinear boundary conditions by linear approximation schemes.* (To appear).

[12] MacCAMY, R. C.—WONG, J. S. W.: *Stability theorems for some functional equations*, Trans. Amer. Math. Soc. **164** (1972), 1–37.

[13] MAGENES, E.—NOCHETTO, R. H.—VERDI, C.: *Energy error estimates for a linear scheme to approximate nonlinear parabolic equations*, RAIRO Modél. Math. Anal. Numér. **21** (1987), 655–678.

[14] MAGENES, E.—VERDI, C.—VISINTIN, A.: *Theoretical and numerical results on the two-phase Stefan problem*, SIAM J. Numer. Anal. **26** (1989), 1425–1438.

[15] McLEAN, W.—THOMÉE, V.: *Numerical solution of an evolution equation with a positive type memory term*, J. Austral. Math. Soc. Ser. B (Submitted).

[16] SLODIČKA, M.: *Application of Rothe's method to evolution integrodifferential systems*, Comment. Math. Univ. Carolin. **30** (1989), 57–70.

[17] SLODIČKA, M.: *On a numerical approach to nonlinear degenerate parabolic problems.* Preprint, Comenius University, M6 (1992).

[18] SLODIČKA, M.: *Numerical solution of a parabolic equation with a weakly singular positive-type memory term.* Preprint, Comenius University, M7 (1992).

Received July 3, 1992
Revised September 9, 1993

Nálepkova 248
SK - 019 01 Ilava
Slovakia