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A SUFFICIENT CONDITION FOR HAMILTONIAN GRAPHS

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ABSTRACT. Let G be a simple graph of order n, and let $\langle N(u)\rangle$ denote the subgraph of G induced by the neighbourhood of a vertex u. For a nonadjacent pair of vertices u and v we define an invariant $\omega(u,v)$ as the number of components of $\langle N(u)\rangle$ containing no neighbour of v. We prove that, if $d(u)+d(v)+\max\{\omega(u,v),\omega(v,u)\} \ge n$ for each pair of nonadjacent vertices u and v, then G is hamiltonian.

1. Introduction

In this paper, we consider simple graphs with the vertex set $V(G)$ and the edge set $E(G)$. The degree of a vertex v is denoted by $d_G(v)$. The neighbourhood $N_G(v)$ of v is $\{x:\ xv \in E(G)\}$. For $U \subseteq V(G)$ we denote the graph induced by U as $\langle U\rangle$.

Let G be a graph, and let u, v be two nonadjacent vertices. Then $\omega_G(u,v)$ will denote the number of components of the graph $\langle N_G(u)\rangle$ which contain no vertex of $N_G(v)$.

To simplify the text, we usually omit the subscripts in symbols $d_G(v)$, $N_G(v)$ and $\omega_G(u,v)$ if there is no ambiguity.

A graph is *hamiltonian* if it contains a cycle through all its vertices. Such a cycle is called a *hamiltonian cycle.*

In 1960, Ore proved this sufficient condition for hamiltonian graphs:

THEOREM 1. ([5]) *If G is a graph of order n such that $d(u)+d(v) \ge n$ for every pair of nonadjacent vertices u, $v \in V(G)$, then G is hamiltonian.*

Asratyan and Khachatryan proved a generalization of this theorem based on a property of the neighbourhoods of nonadjacent vertices u and v. They considered the subgraph $G_2(u)$ of a graph G induced by those vertices at distance at most 2 from u.

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Theorem 2. ([1]) *Let G be a graph of order n. Suppose that whenever $d_G(u) \leq (n-1)/2$ and v is a vertex at distance 2 from u, $d_G(u) + d_{G_2(u)}(v) \geq \left|V\left(G_2(u)\right)\right|$; then G is hamiltonian.*

Tian gave in [6] a sufficient condition using the cardinalities of neighbourhood unions of independent sets of vertices. This condition generalized the condition of Ore as well as the condition of Fraisse (see [3]) and the condition of Faudree, Gould, Jacobson, and Schelp (see [2]).

The degree $d(S)$ of a set S is defined to be $\left|\bigcup\limits_{v \in S} N(v)\right|$. Tian proved the following:

Theorem 3. ([6]) *Let G be a graph of order n and connectivity k. Suppose that there exists some t, $t \leq k$, such that for every independent set $S = \{v_1, v_2, \dots, v_{t+1}\}$ of cardinality $t+1$ we have $\sum\limits_{i=1}^{t+1} d\left(S - \{v_i\}\right) > t(n-1)$; then G is hamiltonian.*

2. Main result

Theorem 4. *Let G be a graph of order n. If $d(u) + d(v) + \max\left\{\omega(u,v), \omega(v,u)\right\} \geq n$ for each pair of nonadjacent vertices u and v of G, then G is hamiltonian.*

The proof of Theorem 4 is based on the following two lemmas.

Lemma 1. *Let G be a graph with a hamiltonian path $P = v_1 v_2 \dots v_n$, where v_1 and v_n are nonadjacent vertices such that $d(v_1) + d(v_n) + \max\left\{\omega(v_1, v_n), \omega(v_n, v_1)\right\} \geq n$. Then there exists an integer m $(1 \leq m \leq n-1)$ such that $v_1 v_{m+1}, v_m v_n \in E(G)$.*

Proof. We prove the case $\max\left\{\omega(v_1, v_n), \omega(v_n, v_1)\right\} = \omega(v_1, v_n)$.

Suppose the contrary. Then v_n is not adjacent to any vertex of the set A defined as $\left\{v_m : v_1 v_{m+1} \in E(G)\right\}$. Let B be $\left\{v_m : v_1 v_m \in E(G),\ v_1 v_{m+1} \notin E(G),\ v_m v_n \notin E(G)\right\}$. Note that the last condition says v_n is not adjacent to any vertex contained in B. These sets are obviously disjoint, and now we determine their cardinalities to obtain an upper bound for the degree of v_n.

The set A has as many vertices as the neighbourhood of v_1, therefore $|A| = d(v_1)$. To show that $|B| \geq \omega(v_1, v_n)$, consider the components of $\langle N(v_1) \rangle$ containing no neighbour of v_n. Let C_k, $1 \leq k \leq \omega(v_1, v_n)$, be one of them. Choose a vertex from $V(C_k)$, the closest to v_n along the path P, and denote its subscript by i. The vertex v_{i+1} cannot be adjacent to v_1; otherwise it would belong to the same component C_k of $\langle N(v_1) \rangle$, and v_i would not be the closest to v_n along P. Clearly, $v_1 v_i \in E(G)$ and $v_i v_n \notin E(G)$, therefore v_i belongs to B.

Since we can choose a vertex contained in B from each such component, we have $|B| \geq \omega(v_1, v_n)$. Then

$$\begin{aligned} d(v_n) &\leq |V(G)| - |\{v_n\}| - |A| - |B| \leq n - 1 - d(v_1) - \omega(v_1, v_n) \\ &= n - 1 - d(v_1) - \max\{\omega(v_1, v_n),\, \omega(v_n, v_1)\}\,, \end{aligned}$$

which is a contradiction.

To prove the other case, $\max\{\omega(v_1, v_n),\, \omega(v_n, v_1)\} = \omega(v_n, v_1)$, only relabel the vertices of P in reverse order and use the same argument. □

Lemma 2. *Let u, v be a pair of nonadjacent vertices of a graph G. Let H be the graph induced by a set S of vertices satisfying $\{u\} \cup N_G(u) \cup \{v\} \cup N_G(v) \subseteq S \subseteq V(G)$. Then $\omega_G(u, v) = \omega_H(u, v)$, $\omega_G(v, u) = \omega_H(v, u)$.*

P r o o f. The neighbourhoods of the vertex u are the same in both graphs G and $H = \langle S \rangle$ for any available set S. Since so are the neighbourhoods of v, the numbers $\omega(u, v)$ (and $\omega(v, u)$ too) must be identical in both G and H. □

P r o o f o f T h e o r e m 4. First of all, we show that a graph satisfying the hypothesis of the theorem is connected.

Let G be disconnected, and let G_1 be a component of G. Denote $G_2 = G - V(G_1)$, $k = |V(G_1)|$, $l = |V(G_2)|$. Clearly, $k + l = n$.

Let u and v be vertices of maximum degree in G_1 and G_2, respectively. Obviously, $m_1 = d(u) \leq k-1$ and $m_2 = d(v) \leq l-1$. Now we find upper bounds for the numbers $\omega_G(x, y)$ and $\omega_G(y, x)$, where x is an arbitrary neighbour of u, and y is an arbitrary neighbour of v.

Since $d(u) = m_1$, there exist $k - m_1 - 1$ vertices of G_1 that are not adjacent to the vertex u. Then the number of components of $\langle N_G(x) \rangle$ is at most $(k - m_1 - 1) + 1 = k - m_1$. Therefore $\omega_G(x, y) \leq k - m_1$. Similarly, $\omega_G(y, x) \leq l - m_2$.

Obviously, $d(x) \leq m_1$ and $d(y) \leq m_2$. Then

$$d(x) + d(y) + \max\{\omega_G(x, y),\, \omega_G(y, x)\} \leq m_1 + m_2 + \max\{k - m_1, l - m_2\}\,.$$

Since $k - m_1 \geq 1$ and $l - m_2 \geq 1$, we have $\max\{k - m_1, l - m_2\} < (k - m_1) + (l - m_2)$, and so

$$d(x) + d(y) + \max\{\omega_G(x, y),\, \omega_G(y, x)\} < k + l = n\,,$$

which is a contradiction.

We have proved that G is connected. Now assume that G is nonhamiltonian. Let $P = v_1 v_2 \dots v_k$ be a longest path in G. Consider the graph $H = \langle V(P) \rangle$.

Clearly, H cannot be hamiltonian, because for $k = n$ we have $H = G$ and for $k < n$, from the hamiltonicity of H and the connectedness of G, we would obtain a contradiction to the maximality of the path P.

So v_1 and v_k are nonadjacent. Since P is a longest path in G, neither v_1 nor v_k can be adjacent in G to a vertex not in $V(H)$. Obviously, $d_H(v_1) = d_G(v_1)$, $d_H(v_k) = d_G(v_k)$ and, from Lemma 2, $\omega_H(v_1, v_k) = \omega_G(v_1, v_k)$ and $\omega_H(v_k, v_1) = \omega_G(v_k, v_1)$. Then

$$\begin{aligned} &d_H(v_1) + d_H(v_k) + \max\{\omega_H(v_1, v_k),\, \omega_H(v_k, v_1)\} \\ &= d_G(v_1) + d_G(v_k) + \max\{\omega_G(v_1, v_k),\, \omega_G(v_k, v_1)\} \ge n \ge k\,. \end{aligned}$$

This enables us to use Lemma 1 with the hamiltonian path P in the graph H. We obtain that there exists some m $(1 \le m \le k-1)$ such that $v_1 v_{m+1}, v_m v_k \in E(H)$. But then $v_1 v_2 \dots v_m v_k v_{k-1} \dots v_{m+1}$ is a hamiltonian cycle in H, which is a contradiction. □

Finally we show that there exist infinitely many hamiltonian graphs satisfying neither the assumption of Theorem 2 nor those of Theorem 3, whose hamiltonicity can be proved by means of Theorem 4. Let G be the union of two graphs $H_1 \cup H_2$, where $H_1 = K_{n,n} - u_1 v_1$, $n \ge 3$, with the vertex sets $\{u_1, u_2, \dots, u_n\}$, $\{v_1, v_2, \dots, v_n\}$, and H_2 is an arbitrary graph with the vertex set $\{u_2, \dots, u_n\}$. Then the vertices v_1, v_2 are at distance 2 in G, and $d_G(v_1) = n-1 \le (2n-1)/2$, but the degree sum condition in Theorem 2 does not hold. Neither the theorem of Tian applies to G because, for each $t \le n-1$, the set $S = \{v_1, v_2, \dots, v_{t+1}\}$ does not satisfy the inequality in the theorem. However, $\omega_G(u_1, v_1) = n-1$; $\omega_G(v_j, v_1) = 1$ for $j > 1$ and $d_G(u_j) + \omega_G(u_j, u_1) \ge n+1$ in both cases, $d_{H_2}(u_j) = 0$ and $d_{H_2}(u_j) > 0$, for $j > 1$. This means the condition in Theorem 4 holds for each pair of nonadjacent vertices of G, and this theorem can be used to determine that G is hamiltonian.

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