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ON MATRIX TRANSFORMATIONS OF SOME GENERALIZED SEQUENCE SPACE

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ABSTRACT. P. Schaefer [9] defined the concepts of \( \sigma \)-conservative, \( \sigma \)-regular, and \( \sigma \)-coercive matrices and characterized these classes of matrices, i.e. \((c, V_\sigma)\), \((c, V_\sigma)_{reg}\), and \((\ell_\infty, V_\sigma)\). Recently Mursaleen [5] determined the classes \((\ell(p), V_\sigma)\) and \((M_0(p), V_\sigma)\). The object of this paper is to obtain necessary and sufficient conditions to characterize the matrices of the classes \((c_0(p), V_0(q))\) and \((c_0(p), V_\sigma(q))\).

1. Preliminaries

Let \( \sigma \) be a mapping of the set of positive integers into itself. A continuous linear functional \( \varphi \) on \( \ell_\infty \), the space of bounded sequences, is said to be an invariant mean, or a \( \sigma \)-mean, if and only if

(i) \( \varphi(x) \geq 0 \) when the sequence \( x = (x_n) \) has \( x_n > 0 \) for all \( n \),

(ii) \( \varphi(e) = 1 \), where \( e = (1, 1, \ldots) \),

(iii) \( \varphi(x_{\sigma(n)}) = \varphi(x) \) for all \( x \in \ell_\infty \).

In case, \( \sigma \) is the translation mapping \( n \mapsto n + 1 \), a \( \sigma \)-mean is often called a Banach limit ([2]), and \( V_\sigma \), the set of bounded sequences all of whose invariant means are equal, is the set \( f \) of almost convergent sequences ([3]).

Let \( f_0 \) denote the space of almost convergent null sequences.

If \( x = (x_n) \), set \( T x = (T x_n) = (x_{\sigma(n)}) \). It is known that

\[
V_\sigma = \left\{ x \in \ell_\infty : \lim_{m \to \infty} d_{mn}(x) = L e, \ \text{uniformly in} \ n, \ \text{and} \ L = \sigma-lim x \right\},
\]

where

\[
d_{mn}(x) = \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n.
\]

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The special case of (1.1) in which \( a(n) = n + 1 \) was given by Lorentz [3; Theorem 1]; the general result can be proved in a similar way.

It is familiar that a Banach limit extends the limit functional on \( c \), the space of convergent sequences. It is known ([5]) that a \( \sigma \)-mean extends the limit functional on \( c \) in the sense that \( \varphi(x) = \lim x \) for all \( x \in c \) if and only if \( \sigma \) has no finite orbits, that is to say, if and only if for all \( n \geq 0, j \geq 1, \sigma^j(n) \neq n \).

P. Schaefer [9] defined the concepts of \( \sigma \)-concervative, \( \sigma \)-regular, and \( \sigma \)-coercive matrices and obtained conditions to characterize these classes of matrices.

Let \( V_{0\sigma} \) denote the set of all bounded sequences which are \( \sigma \)-convergent to zero.

Recently, in [5] and [7] the spaces \( V_\sigma, V_{0\sigma}, f, \) and \( f_0 \) were extended to \( V_\sigma(p), V_{0\sigma}(p), f(p), \) and \( f_0(p) \) in the following manner.

If \( p = (p_m) \) is a sequence of real numbers such that \( p_m > 0 \) and \( \sup p_m < \infty \), we define

\[
V_{0\sigma}(p) = \left\{ x : \lim_{m \to \infty} |d_{mn}(x)|^{p_m} = 0, \text{ uniformly in } n \right\},
\]

\[
V_\sigma(p) = \left\{ x : \lim_{m \to \infty} |d_{mn}(x - Le)|^{p_m} = 0, \text{ uniformly in } n, \sigma\text{-lim} x = L \right\},
\]

\[
f_0(p) = \left\{ x : \lim_{m \to \infty} \frac{1}{m+1} \left| \sum_{i=0}^{m} x_{i+n} \right|^{p_m} = 0, \text{ uniformly in } n \right\},
\]

\[
f(p) = \left\{ x : \lim_{m \to \infty} \frac{1}{m+1} \left| \sum_{i=0}^{m} (x_{i+n} - L) \right|^{p_m} = 0 \text{ for some } L, \text{ uniformly in } n \right\}.
\]

In particular, if \( p_m = p > 0 \) for all \( m \), we have \( V_{0\sigma}(p) = V_{0\sigma} \) and \( V_\sigma(p) = V_\sigma \). If \( \sigma(n) = n + 1 \), we get \( V_{0\sigma}(p) = f_0(p) \) and \( V_\sigma(p) = f(p) \).

S. M. Zaidi [10] has determined necessary and sufficient conditions for some matrix \( A = (a_{nk}), n, k = 1, 2, \ldots, \) such that the \( A \)-transform of \( x = (x_k) \) belongs to the set \( V_\sigma(q) \), where in particular \( x \in \ell_\infty(p) \).

Just as boundedness is related to convergence, it was quite natural to expect that the sequence space \( \ell_\infty^\sigma \) of \( \sigma \)-boundedness is related to \( \sigma \)-convergence.

We write

\[
\ell_\infty^\sigma = \left\{ x : \sup_{m,n} |d_{mn}(x)| < \infty \right\}.
\]

But in [8], Savas has observed that this concept coincides with \( \ell_\infty \), viz., \( \ell_\infty^\sigma = \ell_\infty \).

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2. Notation

If \( p = (p_k) \) is a sequence of real numbers such that \( p_k > 0 \) and \( \sup_k p_k < \infty \), we write
\[
\ell_\infty(p) = \left\{ x : \sup_k |x_k|^{p_k} < \infty \right\},
\]
\[
c(p) = \left\{ x : \lim_{k \to \infty} |x_k - L|^{p_k} = 0, \text{ for some } L \right\},
\]
\[
c_0(p) = \left\{ x : \lim_{k \to \infty} |x_k|^{p_k} = 0 \right\}.
\]

As special cases of the above, with \( p_k = 1 \) for all \( k \), we get \( \ell_\infty \), \( c \), and \( c_0 \). By the Köthe-Toeplitz dual of a set \( E \subset s \), the set of complex sequences, \( E \neq \emptyset \), we mean the linear space
\[
E^+ = \left\{ a : \sum_k a_k x_k \text{ convergent for all } x \in E \right\}.
\]

\( E^* \) denotes the dual space of the continuous linear functionals of \( E \).

We want to add that \( p = (p_k) \) and \( q = (q_k) \) in the sequel will denote sequences with \( p_k > 0 \) and \( q_k > 0 \).

We use the fact that \( c_0(p) \) is a complete paranormed space with paranorm
\[
g(x) = \left( \sup_k |x_k|^{p_k} \right)^\frac{1}{M}, \quad M = \max \left( 1, \sup_k p_k \right).
\]

The purpose of this paper is to obtain necessary and sufficient conditions to characterize the matrices of classes \( (c_0(p), V_{0\sigma}(q)) \) and \( (c_0(p), V_\sigma(q)) \).

3. Main results

If \( X \) and \( Y \) are two sequence spaces, let \((X, Y)\) denote the set of all matrices \( A = (a_{nk}) \), \( n, k = 1, 2, \ldots \), that transform \( x = (x_k) \in X \) into \( y = (y_n) = Ax = (A_n(x)) \in Y \), defined by \( y_n = \sum_k a_{nk} x_k \) \( (n = 1, 2, \ldots) \). Let us write for all integers \( n, m \geq 1 \),
\[
t_{mn} = t_{mn}(Ax) = \sum_k a(n, k, m) x_k,
\]
where \( a(n, k, m) = \frac{1}{m + 1} \sum_{j=0}^{m} a(\sigma^j(n), k) \).

In the sequel, we can assume \( p_k \leq 1 \) for all \( k \) without loss of generality because \( c_0(p) = c_0(p/M) \) for \( M = \max \left( 1, \sup_k p_k \right) \).

Now, let us quote some known results as the following.

We remark that \( \ell_\infty^\infty(q) = \ell_\infty(q) \) in the lemmas below.
LEMMA A. ([4]) Let $X$ be a complete paranormed space with Schauder basis $(b_k)$, and $(A_n)$ a sequence of elements of $X^*$ with $A_n(x) = \sum a_{nk}x_k$ for all $x \in X$ and $n \in \mathbb{N}$. Furthermore, let $q = (q_k)$ be a bounded sequence. Then

$$A \in (X, V_{0\sigma}(q)) \iff \begin{cases} (t_{mn}(b_k)) \in V_{0\sigma}(q) \text{ for all } k, \\ \lim_{M \to \infty} \limsup_m (\|t_{mn}\|_M)^{q_m} = 0. \end{cases}$$

LEMMA B. ([4]) Let $X$ be a complete paranormed space with Schauder basis $(b_k)$, and $(A_n)$ a sequence of elements of $X^*$ with $A_n(x) = \sum a_{nk}x_k$ for all $x \in X$ and $n \in \mathbb{N}$. Furthermore, let $q = (q_k)$ be a bounded sequence. Then

$$A \in (X, V_{\sigma}(q)) \iff \begin{cases} \text{there exists an } L \in X^* \text{ with } \\ (t_{mn}(b_k) - L(b_k)) \in V_{0\sigma}(q) \text{ for all } k, \\ \lim_{M \to \infty} \limsup_m (\|t_{mn}\|_M)^{q_m} = 0. \end{cases}$$

LEMMA C. ([4]) Let $p, q \in \ell_\infty$. Then

$$A \in (c_0(p), \ell_{\sigma}^\infty(q)) \iff \sup_{m,n} \left( \sum_k |a(n,k,m)|M^{\frac{1}{p_k}} \right)^{q_m} < \infty$$

for some $M > 1$.

Additionally, we use the characterization of the Köthe-Toeplitz dual of $c_0(p)$:

$$c_0^+(p) = \bigcup_{N>1} \left\{ a : \sum_k |a_k|N^{-\frac{1}{p_k}} < \infty \right\}$$

and the fact that $c_0^+(p) \cong c_0^*(p)$ (isometrically isomorphic for bounded sequences $p$).

We now establish the following theorems.

THEOREM 1. Let $p, q \in \ell_\infty$. Then

$$A \in (c_0(p), V_{0\sigma}(q)) \iff \begin{cases} \lim_{m \to \infty} |a(n,k,m)|^{q_m} = 0, \text{ uniformly in } n, \\ \lim_{M \to \infty} \limsup_m \left( \sum_k |a(n,k,m)|M^{\frac{1}{p_k}} \right)^{q_m} = 0. \end{cases}$$

Proof. Let $A \in (c_0(p), V_{0\sigma}(q))$. Since $V_{0\sigma}(q) \subset \ell_{\infty}^\sigma(q)$, we have $A \in (c_0(p), \ell_{\sigma}^\infty(q))$. Then $t_{mn}(Ax) = \sum_k a(n,k,m)x_k$ is defined for all $x \in c_0(p)$, $m$
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and \( n \). That is \( a(n, k, m) \in c_0^+(p) \) and \( t_{mn} \in c_0^+(p) \) for all \( m, n \), and \( \|t_{mn}\|_M = \sum_{k=1}^{\infty} |a(n, k, m)|M^{-\frac{1}{p_k}} \) if \( \|t_{mn}\| \) is defined. \( c_0(p) \) being complete, we obtain (ii) by Lemma A and (i) by using \( e^{(k)} \in c_0(p) \).

Conversely, suppose that the conditions (i) and (ii) hold and \( x \in c_0(p) \). By (ii) it follows that for some \( M > 1 \),

\[
\sup_{m,n} \left( \sum_k |a(n, k, m)|M^{-\frac{1}{p_k}} \right)^{q_m} < \infty.
\]

Due to the convergence of \( \sum_{k=1}^{\infty} |a(n, k, m)|M^{-\frac{1}{p_k}} \), we have \( a(n, k, m) \in c_0^+(p) \), and therefore \( t_{mn} \in c_0^+(p) \) and \( \|t_{mn}\|_M = \sum_{k=1}^{\infty} |a(n, k, m)|M^{-\frac{1}{p_k}} \) if \( \|t_{mn}\| \) is defined. Trivially \( (e^{(k)}) \) is a Schauder basis of \( c_0(p) \). By Lemma A, \( A \in (c_0(p), \ell_0(q)) \).

We have

**THEOREM 2.** Let \( p, q \in \ell_\infty \). Then \( A \in (c_0(p), \ell_0(q)) \) if and only if

(i) \( \sup_{n,m} \sum_k |a(n, k, m)|M^{-\frac{1}{p_k}} < \infty \) for some \( M > 1 \),

(ii) there exist \( \alpha_1, \alpha_2, \ldots \in C \) with \( |a(n, k, m) - \alpha_k|^{q_m} \to 0 \), as \( m \to \infty \), uniformly in \( n \), for each \( k \),

(iii) \( \lim_{M \to \infty} \limsup_m \left( \sum_k |a(n, k, m) - \alpha_k|M^{-\frac{1}{p_k}} \right)^{q_m} = 0 \).

**Proof.** Suppose that \( A \in (c_0(p), \ell_0(q)) \). Because of \( \ell_0(q) \subset \ell_\infty^0(q) \), we have \( A \in (c_0(p), \ell_\infty^0(q)) \), and so that \( t_{mn} \in c_0(p) \). By Lemma B, there exists an \( L \in c_0^+(p) \) with

1. \( (t_{mn}(e^{(k)}) - L(e^{(k)})) \in V_{0\alpha}(q) \) for all \( k \),
2. \( \lim_{M \to \infty} \limsup_m (\|t_{mn} - L\|_M)^{q_m} = 0 \).

This \( L \in c_0^+(p) \) can be written as

\[
L(x) = \sum_k \alpha_k x_k
\]

for all \( x \in c_0(p) \) with \( (\alpha_k) \in c_0^+(p) \). Then (1) reads as \( |a(n, k, m) - \alpha_k|^{q_m} \to 0 \), as \( m \to \infty \), uniformly in \( n \), for each \( k \), which is (ii).

By (2) and since \( \|t_{mn} - L\|_M = \sum_k |a(n, k, m) - \alpha_k|M^{-\frac{1}{p_k}} \) for all \( M \), for which \( \|t_{mn} - L\|_M \) is defined, (iii) follows.
Noting that $V_\sigma(q) \subseteq \ell_\infty = \ell_\infty$ and that therefore $A \in (c_0(p), \ell_\infty)$, we may apply Lemma C to obtain

$$\sup_{m,n} \left( \sum_k |a(n,k,m)|M^{-1}_{\frac{1}{p_k}} \right) < \infty.$$ 

For the converse, let (i), (ii), and (iii) hold. From (i), we have $a(n,k,m) \in c_0(p)$ for all $n,m$, and therefore $t_{mn} \in c_0^*(p)$ for all $n,m$. It follows from (i) and (iii) that for $n, m$ and $M$ large enough

$$\sum_k |\alpha_k|M^{-1}_{\frac{1}{p_k}} \leq \sum_k |a(n,k,m) - \alpha_k|M^{-1}_{\frac{1}{p_k}} + \sum_k |a(n,k,m)|M^{-1}_{\frac{1}{p_k}} < \infty.$$ 

Therefore

$$(\alpha_k) \in c_0^+(p),$$

and with $Lx = \sum_k \alpha_k x_k$:

$$L \in c_0^*(p).$$

So we have for $t_{mn}$, $L \in c_0^*(p)$

$$\|t_{mn} - L\|_M = \sum_k |a(n,k,m) - \alpha_k|M^{-1}_{\frac{1}{p_k}}.$$ 

By Lemma B, $A \in (c_0(p), V_\sigma(q))$. This completes the proof. 

4. Corollaries

We deduce the following corollaries.

**COROLLARY 1.** $A \in (c_0(p), V_0\sigma)$ if and only if

(i) $a(n,k,m) \to 0$ as $m \to \infty$, uniformly in $n$, for each $k$,

(ii) $\lim_{M \to \infty} \limsup_m \sum_k |a(n,k,m)|M^{-1}_{\frac{1}{p_k}} = 0$.

**Proof.** Take $q_k = 1$ for all $k$ in Theorem 1. \square

**COROLLARY 2.** $A \in (c_0(p), V_\sigma)$ if and only if

(i) $\sup_{n,m} \sum_k |a(n,k,m)|M^{-1}_{\frac{1}{p_k}} < \infty$ for some $M > 1$,

(ii) there exist $\alpha_1, \alpha_2, \ldots \in C$ with $|a(n,k,m) - \alpha_k| \to 0$, as $m \to \infty$, uniformly in $n$, for each $k$,

(iii) $\lim_{M \to \infty} \limsup_m \sum_k |a(n,k,m) - \alpha_k|M^{-1}_{\frac{1}{p_k}} = 0$.

**Proof.** Take $q_k = 1$ for all $k$ in Theorem 2. \square
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**Corollary 3.** Let \( p, q \in \ell_\infty \). Then \( A \in (c_0(p), f_0(q)) \) if and only if

(i) \( |b(n,k,m)|^{q_m} \to 0 \) as \( m \to \infty \), uniformly in \( n \), for each \( k \),

(ii) \[
\lim_{M \to \infty} \limsup_m \left( \sum_k |b(n,k,m)| M^{-\frac{1}{p_k}} \right)^{q_m} = 0, 
\]

where \( b(n,k,m) = \frac{1}{m+1} \sum_{j=0}^{m} a(n+j,k) \).

Taking \( \sigma(n) = n + 1 \) in Theorem 1, we close the proof.

**Corollary 4.** Let \( p, q \in \ell_\infty \). Then \( A \in (c_0(p), f(q)) \) if and only if

(i) \( \sup_{n,m,k} |b(n,k,m)| M^{-\frac{1}{p_k}} \) is finite for some \( M > 1 \),

(ii) there exist \( \alpha_1, \alpha_2, \ldots \in C \) with \( |b(n,k,m) - \alpha_k|^{q_m} \to 0 \), as \( m \to \infty \), uniformly in \( n \), for each \( k \),

(iii) \[
\lim_{M \to \infty} \limsup_m \left( \sum_k |b(n,k,m) - \alpha_k| M^{-\frac{1}{p_k}} \right)^{q_m} = 0, 
\]

where \( \alpha_k = L - \lim_n a_{nk} \).

**Proof.** Choosing the mapping \( \sigma(n) = n + 1 \) instead of mapping \( \sigma \) as the transformation mapping, the space \( V_\sigma(q) \) of Theorem 2 reduces to \( f(q) \). Hence it is proved. \( \square \)

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