Surjit Singh Khurana; Sadoon Ibrahim Othman

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COMPLETENESS AND SEQUENTIAL COMPLETENESS IN CERTAIN SPACES OF MEASURES

SURJIT SINGH KHURANA* — SADOON IBRAHIM OTHMAN**

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ABSTRACT. Let X be a completely regular Hausdorff space, E a Banach space over K, the field of real or complex numbers, $C(X,E)$ ($C(X)$ if $E = K$) the space of all E-valued continuous functions on X, and $C_b(X,E)$ ($C_b(X)$ if $E = K$) the space of all E-valued bounded continuous functions on X. Put $F_z = (C_b(X,E), \beta_z)$ (β_z the so called strict topologies), and $F = (C(X,E), \beta_{\infty C})$. It is proved that $(F'_z, \sigma(F'_z, F_z))$ is sequentially complete for $z = \sigma, \infty, g$; if, in addition, X is meta-compact and normal, then the result is also true for $z = \tau$. Also it is proved that $(F', \sigma(F', F))$ is sequentially complete. For the Mackey topology it is proved that $(F'_z, \tau(F'_z, F_z))$ is complete for $z = \sigma, \infty, g$ and for $z = \tau(t)$ it is complete if and only if $M_g(X) = M_\tau(X)$ ($M_t(X)$). Further it is proved that $(F', \tau(F', F))$ is complete. Some additional results are proved for sequential convergence.

In this paper, X is a completely regular Hausdorff space, E a Banach space over K, the field of real or complex numbers, $C(X,E)$ ($C(X)$ if $E = K$) the space of all E-valued continuous functions on X, and $C_b(X,E)$ ($C_b(X)$ if $E = K$) the space of all E-valued bounded continuous functions on X. For locally convex spaces, the notations and results of [11] will be used. For topological measure theory, notations and results of [5], [7], [8] and [14] will be used. All locally convex spaces are assumed to be Hausdorff and over K. The topologies β_0, β_1, β, β_∞, β_g are defined on $C_b(X,E)$ in [5], [7], [8] (see also [1], [2], [3], [4], [12], [13]). We will also write β_σ for β_1, β_τ for β, and β_t for β_0. $\tilde{X}$ (νX) will denote the Stone-Čech compactification (real-compactification) of X. For a function $f \in C(X)$, $\bar{f}$ and $\tilde{f}$ denote its unique continuous extensions to νX and $\tilde{X}$ (extension to $\tilde{X}$ may be infinite-valued), respectively. For an f in $C(X,E)$, $\|f\|$ will denote an element of $C(X)$, $\|f\|(x) = \|f(x)\|$. For $\mu \in M_\sigma(X)$, we get $\tilde{\mu} \in M(\tilde{X})$, $\tilde{\mu}(g) = \mu(g|_X)$, $g \in C(\tilde{X})$; for $\tilde{\mu} \in M(\tilde{X})$, $\operatorname{supp}(\tilde{\mu})$ is the smallest

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compact set C in $\tilde{X}$ such that $|\tilde{\mu}|(C) = |\tilde{\mu}|(\tilde{X})$. For $\mu \in (C_b(X,E), \|\,\|)'$, $|\mu|(g) = \sup\{|\mu(h)| : h \in C_b(X,E), \|h\| \le g\}$, $g \in C_b(X)$, $g \ge 0$. ([5], [7], [8]); $|\mu| \in (C_b(X), \|\,\|)'$ (in [8], notation $\bar{\mu}$ is used). $\mathbb{N}$ will denote the set of natural numbers.

When $E = K = \mathbb{R}$, it is well known that $(M_\sigma, \sigma(M_\sigma(X), C_b(X)))$ and $(M_\infty, \sigma(M_\infty(X), C_b(X)))$ are sequentially complete [13]. In this paper, we consider some extensions of this result to the vector case and also case when we take Mackey topology.

LEMMA 1. *Let $\lambda_n \colon 2^{\mathbb{N}} \to K$ ($2^{\mathbb{N}}$ being all subsets of $\mathbb{N}$, the set of natural numbers) be a sequence of countably additive measures (this implies continuity in $2^{\mathbb{N}}$, with product topology) such that $\lambda_n(M)$ exists for all $M \subset \mathbb{N}$. Then the convergence is uniform on $2^{\mathbb{N}}$.*

Proof. The result follows easily from classical Philips' lemma ([5]). □

LEMMA 2. *A net $f_\alpha \to 0$ in $(C_b(X,E), \mathscr{F})$ if and only if $\|f_\alpha\| \to 0$ in $(C_b(X), \mathscr{F})$, where $\mathscr{F} = \beta_0, \beta_1, \beta, \beta_\infty$, or β_g; in the dual sense, $A \subset (C_b(X,E), \mathscr{F})'$ is $\mathscr{F}$-equicontinuous if and only if $|A|$ is equicontinuous. The result also holds in $(C(X,E), \beta_{\infty C})$.*

Proof. For β_g, it is proved in [8]; the proof for others is similar. The main result used is that these topologies are locally solid ([5]). □

THEOREM 3. *Let E be a Banach space and $F_z = (C_b(X,E), \beta_z)$. Then $(F_z', \sigma(F_z', F_z))$ is sequentially complete for $z = \sigma, \infty$, or g. If X is also metacompact and normal, then the result is also true for $z = \tau$.*

Proof.

The case $z = \sigma$.

Let $\{\mu_n\}$ be a Cauchy sequence in $(F_z', \sigma(F_z', F_z))$, and define $\mu \colon C_b(X,E) \to K$, $\mu(g) = \lim \mu_n(g)$. By the principle of uniform boundedness, $\mu \in (C_b(X,E), \|\,\|)'$, and so we have only to prove that $|\mu| \in M_\sigma(X)$ ([7]). Take a zero set $Z \subset \tilde{X} \setminus X$ and take an increasing sequence $\{V_n\}$ of open subsets of $\tilde{X}$ such that $\tilde{X} \setminus Z = \bigcup_{n=1}^{\infty} V_n$. Using the fact that $\tilde{X} \setminus Z$ is para-compact locally compact, we get a partition of unity $\{\dot{h}_n\} \subset C_b(\tilde{X} \setminus Z)$ such that $\sum \dot{h}_n = 1$ on $\tilde{X} \setminus Z$, and $\operatorname{supp}(\dot{h}_n) \subset V_n$, $\forall n$. Let $h_n = \dot{h}_n|_X$. We first prove that $|\mu_k|\Big(\sum_{i=n}^{\infty} h_i\Big) \to 0$, as $n \to \infty$, uniformly in k. Suppose this is not true. This means, taking a subsequence of $\{\mu_n\}$, if necessary, $\exists \eta > 0$, a strictly increasing sequence $\varrho(n) \subset \mathbb{N}$, and a sequence $\{f_n\} \subset C_b(X,E)$ such that $\|f_n\| \le \sum_{i=\varrho(n)}^{\varrho(n+1)-1} h_i$ and

$\mu_n(f_n) > \eta$, $\forall n$. For a subset $M \subset \mathbb{N}$, $f_M = \sum_{i \in M} f_i$ is in $C_b(X,E)$, and $\|f_M\| \leq 1$. Define $\lambda_n \colon 2^{\mathbb{N}} \to K$, $\lambda_n(M) = \mu_n(f_M)$; the conditions of Lemma 1 are satisfied, hence $\mu_n(f_n) \to 0$, which is a contradiction. Fix an $\varepsilon > 0$ and take $p \in \mathbb{N}$ such that $|\mu_n|\Bigl(\sum_{i=p}^{\infty} h_i\Bigr) < \varepsilon/2$, $\forall n$. Let $\varphi^{\sim} \in C(X^{\sim})$, $0 \leq \varphi^{\sim} \leq 1$, $\varphi^{\sim}(Z) = 1$, $\varphi^{\sim}(V_{p+1}) = 0$, and put $\varphi = \varphi^{\sim}\big|_X$. Let $g \in C_b(X,E)$, $\|g\| \leq \varphi$, and $|\mu|(\varphi) \leq |\mu(g)| + \varepsilon/2$.

Now, for every n,

$$|\mu_n|(\varphi) = |\mu_n|\Bigl(\sum_{i=1}^{\infty} \varphi h_i\Bigr) = |\mu_n|\Bigl(\sum_{i=p+1}^{\infty} \varphi h_i\Bigr) \leq |\mu_n|\Bigl(\sum_{i=p+1}^{\infty} h_i\Bigr) \leq \varepsilon/2,$$

hence $|\mu_n(g)| \leq \varepsilon/2$, $\forall n$. Thus $|\mu(g)| \leq \varepsilon/2$, and so $|\mu|(\varphi) \leq \varepsilon$. This gives $|\mu|^{\sim}(Z) \leq \varepsilon$ and, consequently, $|\mu|^{\sim}(Z) = 0$. This proves $|\mu| \in M_\sigma$.

Case of $z = g$.

Using the result proved above for $z = \sigma$, we get $\mu_n \to \mu$, pointwise on $C_b(X,E)$, and $|\mu| \in M_\sigma$. By [8; Theorem 6.5. (v)], it is enough to prove that $|\mu| \in M_g$. Suppose this is not true. Let $\lambda = |\mu| + \sum_{n=1}^{\infty} \frac{1}{2^n}|\mu_n|$. Since $\bigl(M_g(X), \tau\bigl(M_g(X), C_b(X)\bigr)\bigr)$ ([13]) is complete, by Grothendieck's completeness theorem ([10; Theorem 6.2]), there exists an absolutely convex and pointwise compact $H \subset C_b(X)$, H consisting of real-valued functions, such that $|\mu|$ is not continuous on $\bigl(H, \sigma\bigl(C_b(X), M_g(X)\bigr)\bigr)$ at 0. We assume that $|\mu_n|(1) \leq 1$, $\forall n$, and so $|\mu|(1) \leq 1$. There exists an $\eta > 0$ such that for any finite subset $A \subset M_g(X)$ and $\varepsilon > 0$, $H(A,\varepsilon) = \bigl\{f \in H : \ |\langle f, \nu\rangle| \leq \varepsilon, \ \forall \nu \in A, \text{ and } |\mu|(f) > \eta\bigr\} \neq \emptyset$ ([8]). As $H(A,\varepsilon)$ is convex and decreases as A increases and ε decreases, $\bigcap_{A,\varepsilon} \overline{H(A,\varepsilon)} \neq \emptyset$, closure taken in $L_1(X, Ba, \lambda)$ with weak topology (Ba denotes all Baire subsets of X). Take an $f \in \bigcap_{A,\varepsilon} \overline{H(A,\varepsilon)}$. Fix A and ε, and take a sequence $\{f_n\} \subset H(A,\varepsilon)$ such that $f_n \to f$ a.e. $[\lambda]$. Since H is compact, $\exists f_0 \in H$ such that $f = f_0$ a.e. $[\lambda]$. Hence we may assume that $f \in H$. Let $K_1 = \bigl\{x \in X^{\sim} : \ f^{\sim}(x) \leq 0\bigr\}$ and $K = \bigl\{x \in X^{\sim} : \ f^{\sim}(x) \geq \eta/3\bigr\}$, then $K \cap \operatorname{supp}(\mu^{\sim}) \neq \emptyset$. Define $g^{\sim} \in C(X^{\sim})$, $0 \leq g^{\sim} \leq 1$, $g^{\sim}(K) = 1$, $g^{\sim}(K_1) = 0$. This means $|\mu|^{\sim}(f^{\sim}g^{\sim}) > 0$. Put $g = g^{\sim}\big|_X$ and take, for every n, $A_n = \bigl\{g|\mu_i| : \ 1 \leq i \leq n\bigr\} \subset M_g(X)$, and $\varepsilon = \dfrac{1}{n}$. Then $|\mu_i|(gf) \leq 1/n$, $1 \leq i \leq n$, $\forall n$, and so $|\mu_i|(gf) = 0$, $\forall n$. Take an $h \in C_b(X,E)$, $\|h\| \leq fg$, and $|\mu(h)| > 0$. Now $\mu_i(h) = 0$, $\forall i$, implies that $\mu(h) = 0$, which is a contradiction.

Case of $z = \infty$.

The proof is very similar to that of the case of β_g. We only have to note that $\bigl(M_\infty(X), H^\infty\bigr)$ is complete, where H^∞ is the topology of uniform convergence on all subsets of $C_b(X)$ which are uniformly bounded and equicontinuous ([5]), from which it easily follows that $\big(M_\infty(X), \tau\big(M_\infty(X), C_b(X)\big)\big)$ is complete, and then take H to be uniformly bounded, equicontinuous, and pointwise compact of real-valued functions in $C_b(X)$.

Case of $z = \tau$.

From the case $z = \sigma$, we get $|\mu| \in M_\sigma(X)$. Let $C = \operatorname{supp}(\lambda)$, where $\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n}|\mu_n|$. Since $\lambda \in M_\tau(X)$ and X is meta-compact, C is Lindelöf ([10]). Fix $\varepsilon > 0$, take a zero-set $Z \subset X \setminus C$; using the normality of X, get an $f \in C_b(X)$ such that, $0 \le f \le 1$, $f(C) = 0$ and $f(Z) = 1$. Let $g \in C_b(X,E)$ with $\|g\| \le f$ and $|\mu(g)| + \varepsilon > |\mu|(f)$.

Now $|\mu(g)| = \lim |\mu_n(g)| \le \limsup \int_C \|g\| \, \mathrm{d}|\mu_n|$. Thus $|\mu|(f) < \varepsilon$, and so $|\mu|(Z) = 0$. Let $\{U_\alpha : \alpha \in I\}$ be a covering of X by cozero sets ([10]). Since C is Lindelöf, there exists a countable subcovering $\{U_{\alpha(n)} : n \in \mathbb{N}\}$ of C. Since the zero-set $X \setminus \Big(\bigcup_1^\infty U_{\alpha(n)}\Big)$ has $|\mu|$-measure 0, it follows from [14; Part 1, Theorem 25, Corollary 4] that $|\mu| \in M_\tau(X)$. □

THEOREM 4. *Let E be a Banach space, $F_z = \big(C_b(X,E), \beta_z\big)$, and $\{\mu_k\}$ be a sequence in F_z' such that $\mu_k \to \mu$ in $\big(F_z', \sigma(F_z', F_z)\big)$. Then*

(i) *for $z = \sigma$, if $\{f_n\}$ is a sequence in $C_b(X)$, $0 \le f_n \le 1$, $f_n \downarrow 0$, then $|\mu_k|(f_n) \to 0$, uniformly in k;*

(ii) *for $z = \infty$, if $\{f_\alpha\}$ is a net of uniformly bounded and equicontinuous functions in $C_b(X)$ and $f_\alpha \to 0$, pointwise, then $|\mu_k|(f_\alpha) \to 0$, uniformly in k;*

(iii) *when X is meta-compact and normal and $z = \tau$, if $\{f_\alpha\}$ is a net in $C_b(X)$, $f_\alpha \downarrow 0$, then $|\mu_k|(f_\alpha) \to 0$, uniformly in k.*

Proof.

(i) Since $\big(C_b(X,E), \beta_\sigma\big)$ is strongly Mackey ([7; Corollary 6]), $\{\mu_n\}$ is equicontinuous. By Lemma 2, $\big\{|\mu_n|\big\}$ is β_σ-equicontinuous. Also $f_n \downarrow 0$ implies $f_n \to 0$ in $\big(C_b(X), \beta_\sigma\big)$ ([13]). From this the result follows.

(ii) Exactly same argument applies in the case of β_σ.

(iii) As in Theorem 3, there exists a closed Lindelöf subset $C \subset X$, such that $\operatorname{supp}\big(|\mu_n|\big) \subset C$, $\forall n$. We claim that $\big\{|\mu_n|\big\}$ is relatively compact in $\big(M_\tau(X), \sigma\big(M_\tau(X), C_b(X)\big)\big)$. By (i), this is relatively compact in $\big(M_\sigma(X), \sigma\big(M_\sigma(X), C_b(X)\big)\big)$. Let $\nu \in M_\sigma(X)$ be a cluster point of $\big\{|\mu_n|\big\}$. To prove that $\nu \in M_\tau(X)$, by using the techniques of Theorem 3 (case of $z = \tau$), we need only to prove that for any zero-set $Z \subset X \setminus C$, $\nu(Z) = 0$. Using the normality of X, get an $f \in C_b(X)$ such that, $0 \le f \le 1$, $f(C) = 0$, and $f(Z) = 1$. Since $|\mu_n|(f) = 0$, $\forall n$, we get $\nu(f) = 0$, thus $\nu(Z) = 0$. Hence $\big\{|\mu_n|\big\}$ is relatively compact. Since $\{|\mu_n|\} \subset M_\tau^+$, it is β_τ-equicontinuous ([13]). Now $f_\alpha \downarrow 0$ implies $f_\alpha \to 0$ in β_τ, and so the result follows. □

THEOREM 5. *Let E be a Banach space, and $F_z = \big(C_b(X,E), \beta_z\big)$. Then $\big(F_z', \tau(F_z', F_z)\big)$ is complete for $z = \sigma, \infty$, or g. For $z = \tau$ or t, this space is complete if and only if $M_z(X) = M_g(X)$.*

Proof. We will use Grothendieck's completeness theorem. Let $\mu\colon C_b(X,E) \to K$ be a linear mapping such that μ is continuous on every absolutely convex, $\sigma(F_z, F_z')$-compact subset H of $C_b(X,E)$, with $\sigma(F_z, F_z')$ topology.

Case of $z = \sigma$:
Here μ is continuous on every absolutely convex, $\sigma(F_\sigma, F_\sigma')$-compact subset H of $C_b(X,E)$. From this, it easily follows $\mu \in \big(C_b(X,E), \|\ \|\big)'$. So it is enough to prove that $|\mu|$ is in $M_\sigma(X)$ ([7]). Suppose there exists a sequence $\{f_n\} \subset C_b(X)$, $f_n \downarrow 0$, but $|\mu|(f_n) > \eta$, $\forall n$, for some $\eta > 0$. Thus, there is a sequence $\{g_n\} \subset C_b(X,E)$, $\|g_n\| \le f_n$, and $|\mu(g_n)| > \eta$, $\forall n$. This implies that $\{g_n\}$ is equicontinuous, uniformly bounded and pointwise compact, and H, the absolutely convex, pointwise closed hull of $\{g_n\}$, is pointwise compact, uniformly bounded and equicontinuous. We claim H is a $\sigma(F_\sigma, F_\sigma')$-compact subset H of $C_b(X,E)$. Take $\lambda \in F_\sigma'$, fix $\varepsilon > 0$, and select a Baire set $C \subset X$ such that $|\lambda|(X \setminus C) \le \varepsilon$ and $f_n \downarrow 0$, uniformly on C (Egoroff's theorem). This makes $H\big|_C$ a compact subset of $\big(C_b(C,E), \|\ \|\big)$. If, in H, $h_\alpha \to h$, pointwise on X, then, using

$$|\lambda(h_\alpha - h)| \le |\lambda|\big(\|h_\alpha - h\|\big) = \int_C \big(\|h_\alpha - h\|\big)\, \mathrm{d}|\lambda| + \int_{X\setminus C} \big(\|h_\alpha - h\|\big)\, \mathrm{d}|\lambda|,$$

we get $\lambda(h_\alpha) \to \lambda(h)$, and so the claim is proved. Thus $g_n \to 0$ in $\big(H, \sigma(F_\sigma, F_\sigma')\big)$. Since μ is continuous on H, $\mu(g_n) \to 0$, which is a contradiction. This proves $|\mu| \in M_\sigma(X)$.

Case of $z = \infty$:
Here μ is continuous on every absolutely convex, $\sigma(F_\infty, F_\infty')$-compact subset H of $C_b(X,E)$. From this it easily follows $\mu \in \big(C_b(X,E), \|\ \|\big)'$. So it is enough to prove that $|\mu| \in M_\infty$. Take P to be an absolutely convex, pointwise compact, equicontinuous, and uniformly bounded (by 1, in absolute values), subset of real-valued functions in $C_b(X)$. Fix $h \in C_b(X,E)$. The mapping $g \mapsto gh$ $\big(\big(C_b(X), \beta_\infty\big) \to \big(C_b(X,E), \beta_\infty\big)\big)$ is continuous. Suppose $f_\alpha \to f$, pointwise on P. We get $2+f_\alpha \to 2+f$ in $\big(3P, \sigma(F_\infty, F_\infty')\big)$. Fix $\varepsilon > 0$ and take $g \in C_b(X,E)$ such that $\|g\| \le 2+f$ and $|\mu(g)| > |\mu|(f+2) - \varepsilon/2$. Since the mapping $g \mapsto gh$ $\big(\big(C_b(X), \beta_\infty\big) \to \big(C_b(X,E), \beta_\infty\big)\big)$ is continuous, $(2+P)\dfrac{g}{f+2}$ is weakly compact convex in $\big(C_b(X,E), \beta_\infty\big)$, and so its closed absolutely convex hull, H, is

also weakly compact. Since $(2+f_\alpha)\frac{g}{f+2} \to g$ in $\big(3H, \sigma(F_\infty, F'_\infty)\big)$,

$$|\mu(g)| \le \Big|\mu\Big((2+f_\alpha)\frac{g}{f+2}\Big)\Big| + \varepsilon/2, \qquad \forall\alpha \ge \text{some } \alpha_0 .$$

This means $|\mu(g)| \le |\mu|(2+f_\alpha) + \varepsilon/2$, $\forall\alpha \ge \alpha_0$ (note $\Big\|\frac{g}{f+2}\Big\| \le 1$). So $|\mu|(2+f) \le |\mu|(2+f_\alpha) + \varepsilon$, $\forall\alpha \ge \alpha_0$. Thus $|\mu|(f) \le \underline{\lim}\,|\mu|(f_\alpha)$. Similarly, starting with $2 - f_\alpha \to 2 - f$, we will get $|\mu|(-f) \le \underline{\lim}\,|\mu|(-f_\alpha)$. This proves that $|\mu|(f_\alpha) \to |\mu|(f)$, and so $|\mu| \in M_\infty(X)$.

Case of $z = g$:
This case is identical with $z = \infty$.

Case of $z = \tau$:
Suppose $M_g(X) = M_\tau(X)$. This means $\mu \in F'_g$ implies $|\mu| \in M_g(X) = M_\tau(X)$, and so the result follows. Conversely, suppose $\big(F'_\tau, \tau(F'_\tau, F_\tau)\big)$ is complete. This easily implies that $\big(M_\tau(X), \tau\big(M_\tau(X), C_b(X)\big)\big)$ is complete. Take $\mu \in M_g$ and H an absolutely convex compact subset of $\big(C_b(X), \sigma\big(C_b(X), M_\tau(X)\big)\big)$. This means the pointwise topology and $\sigma\big(C_b(X), M_\tau(X)\big)$-topology coincide on H, and so μ is continuous on H, By Grothendieck's completeness theorem, $\mu \in M_\tau(X)$.

Case of $z = t$:
This case is identical with $z = t$. □

Now we consider the measure space $M_{\infty C}(X)$. This is studied in [1], [5], and [9] (in [9], it is denoted by $M(X)$).

THEOREM 6. *Let E be a Banach space and $F = \big(C(X,E), \beta_{\infty C}\big)$. Then $\big(F', \sigma(F', F)\big)$ is sequentially complete and $\big(F', \tau(F', F)\big)$ is complete. If $\{\mu_k\}$ is a sequence in F' such that $\mu_k \to \mu$ in $\big(F', \sigma(F', F)\big)$, and if $\{f_\alpha\}$ is a net of pointwise bounded and equicontinuous functions in $C_b(X)$ and $f_\alpha \to 0$, pointwise, then $|\mu_k|(f_\alpha) \to 0$, uniformly in k.*

Proof. Take a sequence $\{\mu_n\} \subset F'$ such that $\lim \mu_n(g) = \mu(g)$ exists for every $g \in C(X,E)$. This means that $\mu \in \big(C_b(X,E), \|\,\|\big)'$. Suppose $\exists f \ge 0$ in $C(X)$ such that $|\mu|(f) = \infty$. We get a sequence $\{g_n\} \subset C(X,E)$, $\|g_n\| \le f$ and $|\mu(g_n)| \ge 4^n$, $\forall n$. Put $h_n = 1/2^n g_n$. Then $\{h_n\}$ is equicontinuous, pointwise bounded and $h_n \to 0$, pointwise. Define $\lambda_n\colon 2^{\mathrm{N}} \to K$, $\lambda_n(M) = \mu_n\Big(\sum_{i\in M} 1/2^i h_i\Big)$ (note $\sum_{i\in M} 1/2^i h_i \in C(X,E)$). The conditions of Lemma 1 are satisfied, and so $\mu_n\big(1/2^n h_n\big) \to 0$, which is a contradiction. Also proceeding as in Theorem 2, $|\mu| \in M_\infty(X)$. This proves $\mu \in F'$.

Now we consider the completeness of $\big(F', \tau(F', F)\big)$. By Grothendieck's completeness theorem, we only need to prove that any linear $\mu\colon C(X,E) \to K$

such that for every absolutely convex, $\sigma(F, F')$-compact subset $H \subset C(X, E)$, $\mu|_H$ is continuous for $\sigma(F, F')$-topology, is in F'. As in Theorem 5, $\mu \in (C_b(X, E), \| \|)'$. Suppose there exists $f \geq 0$ in $C(X)$ such that $|\mu|(f) = \infty$. We get a sequence $\{g_n\} \subset C(X, E)$, $\|g_n\| \leq f$ and $|\mu(g_n)| \geq 2^n$, $\forall n$. Put $h_n = 1/2^n g_n$. Then $\{h_n\}$ is equicontinuous, pointwise bounded, and $h_n \to 0$, pointwise. Let H be the pointwise closed, absolutely convex hull of $\{h_n\}$ in $C(X, E)$; it is equicontinuous and pointwise compact, and so it is $\sigma(F, F')$-compact. By the continuity of μ on H, $\mu(h_n) \to 0$, which is a contradiction. Also proceeding as in Theorem 3, $|\mu| \in M_\infty(X)$. This proves completeness.

The proof of uniform convergence of $\{f_\alpha\}$ is identical to the case $z = \infty$ in Theorem 4. □

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* *Department of Mathematics*
University of Iowa
Iowa City
Iowa 52242
U. S. A.

** *Department of Mathematics*
King Saud University
Riyadh
Saudi Arabia