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Mathematica Slovaca, Vol. 45 (1995), No. 4, 381--393

Persistent URL: http://dml.cz/dmlcz/136655

Mathematica Slovaca
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Slovak Academy of Sciences

Math. Slovaca, **45** (1995), No. 4, 381–393

ABSTRACT REGULARITY OF ADDITIVE AND σ-ADDITIVE GROUP–VALUED SET FUNCTIONS[1]

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(*Communicated by Miloslav Duchoň*)

ABSTRACT. Three types of regularity, namely, $\mathcal{L}$-regularity, $\mathcal{G}$-regularity and $(\mathcal{L},\mathcal{G})$-regularity, are introduced for an abelian Hausdorff group G-valued additive or σ-additive set function defined on a ring of sets $\mathcal{R}$ and some sufficient conditions are given on $\mathcal{L}$, $\mathcal{G}$ and $\mathcal{R}$ to ensure the equivalence of all these three types of regularity. Also [9; Theorem 52.F] of Halmos is generalized to G-valued σ-additive set functions in this abstract set-up.

Fixing two classes of sets $\mathcal{L}$ and $\mathcal{G}$, we introduce the concepts of $\mathcal{L}$-regularity, $\mathcal{G}$-regularity and $(\mathcal{L},\mathcal{G})$-regularity for an abelian Hausdorff group G-valued additive or σ-additive set function defined on a ring of sets $\mathcal{R}$ and study some sufficient conditions on $\mathcal{L}$, $\mathcal{G}$ and $\mathcal{R}$ to ensure the equivalence of these three types of regularity. The main results are *Theorems* 4.3 and 4.6 and their *corollaries* on locally compact spaces and metric spaces. In the abstract set-up, these theorems give generalizations of [9; Theorem 52.F] of Halmos to G-valued measures.

Similar studies in abstract set-up in the study of topological measures have been done by Bachman and Cohen in [1], Bachman and Sultan in [2], [3], and in the study of regularity property of vector lattice-valued measures by Hrachovina in [10]. The advantage of this type of study is that it gives a unified approach to problems which are of topological nature. The strength of our study is brought out well by *Corollaries* 4.9 and 4.10 on locally compact spaces and *Corollary* 4.11 on metric spaces.

AMS Subject Classification (1991): Primary 28B10.
Key words: $\mathcal{L}$-regularity, $\mathcal{G}$-regularity, $(\mathcal{L},\mathcal{G})$-regularity, group-valued set function.
[1] Supported by the C.D.C.H.T. project C-586-93 of the Universidad de los Andes, Mérida, Venezuela.

1. Preliminaries

In this section, we introduce the notation and terminology and state some results from D r e w n o w s k i [7], [8].

G denotes an abelian Hausdorff topological group, whose binary operation is denoted by $+$. Ω is a non-void set, in general. Sometimes Ω is considered as a topological space with special properties. $\mathcal{R}$ is a ring of subsets of Ω, $\lambda\colon \mathcal{R} \to G$ is additive, and $\mu\colon \mathcal{R} \to G$ is σ-additive.

We fix two classes of sets $\mathcal{L}$ and $\mathcal{G}$ in $\mathcal{P}(\Omega)$, with respect to which we introduce the concepts of regularity as in *Definition* 2.1. In general, $\mathcal{L}$ and $\mathcal{G}$ are not assumed to be lattices of sets. However, different properties are assumed for $\mathcal{L}$ and $\mathcal{G}$ explicitly whenever necessary.

DEFINITION 1.1. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be classes of sets in Ω. Then:

(i) $\mathcal{C}_1$ is said to be $\mathcal{C}_2$-*complemented* if $C_2 \setminus C_1 \in \mathcal{C}_2$ for $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$.

(ii) $\mathcal{C}_1$ is said to be $\mathcal{C}_2$-*bounded* if for each $C_1 \in \mathcal{C}_1$ there exists $C_2 \in \mathcal{C}_2$ such that $C_1 \subset C_2$.

(iii) $\mathcal{C}_1$ is said to be $\mathcal{C}_1$-*boundedly* $\mathcal{C}_2$-*dominated* if for each $C_1 \in \mathcal{C}_1$ there exists $C_2 \in \mathcal{C}_2$ and $D_1 \in \mathcal{C}_1$ such that $C_1 \subset C_2 \subset D_1$.

(iv) $\mathcal{C}_1$ is said to be $\sigma\mathcal{C}_2$-*bounded* if for each $C_1 \in \mathcal{C}_1$ there exists a sequence $(C_{2,n})_{n=1}^{\infty} \subset \mathcal{C}_2$ such that $C_1 \subset \bigcup\limits_{n=1}^{\infty} C_{2,n}$.

DEFINITION 1.2. Suppose $\mathcal{L}$ is closed under unions and $\mathcal{G}$ under intersections. Let $\emptyset \in \mathcal{L} \cap \mathcal{G}$. Let $I(K,U) = \{A \subset \Omega : \ K \subset A \subset U\}$ for $K \in \mathcal{L}$ and $U \in \mathcal{G} \cup \{\Omega\}$. As $\{I(K,U) : \ K \in \mathcal{L}, \ U \in \mathcal{G} \cup \{\Omega\}\}$ is closed under intersections and $\mathcal{P}(\Omega) = I(\emptyset, \Omega)$, it follows that this collection is a basis for a unique topology $\tau(\mathcal{L},\mathcal{G})$ on $\mathcal{P}(\Omega)$.

CONVENTION 1.3. Whenever the topology $\tau(\mathcal{L},\mathcal{G})$ is referred to, it is tacitly assumed that $\mathcal{L}$ is closed under unions, $\mathcal{G}$ under intersections and $\emptyset \in \mathcal{L} \cap \mathcal{G}$.

PROPOSITION 1.4. (D r e w n o w s k i [7; p. 271, 1.9]) *Let* $\lambda\colon \mathcal{R} \to G$ *be additive and let* $\mathcal{B}$ *be a local base of symmetric closed neighbourhoods of* 0 *in* G. *For each* $W \in \mathcal{B}$, *let*

$$\mathcal{R}_W(\lambda) = \{E \subset \Omega : \ \lambda(F) \in W \ \text{for each} \ F \subset E, F \in \mathcal{R}\}.$$

Then $\mathcal{R} \cap \mathcal{R}_W(\lambda) = \{E \in \mathcal{R} : \ \lambda(F) \in W \ \text{for each} \ F \subset E, \ F \in \mathcal{R}\}$ *is a local base at* $\emptyset$ *in* $\mathcal{R}$ *for the* FN*-topology* $\Gamma(\lambda)$ *determined by* λ *on* $\mathcal{R}$.

PROPOSITION 1.5. (D r e w n o w s k i [8; p. 440, 8.4]) *If* $\mu\colon \mathcal{R} \to G$ *is* σ*-additive and* $E_n \downarrow \emptyset$ *in* $\mathcal{R}$, *then* $E_n \to \emptyset$ *in* $\Gamma(\mu)$*-topology.*

NOTATION 1.6. If q is a quasi-norm on G and $\varepsilon > 0$, then $B_q(0,\varepsilon) = \{x \in G : q(x) < \varepsilon\}$. Given $W \in \mathcal{B}$, there exists a finite family of continuous quasi-norms $(q_i)_1^k$ on G such that $W_\varepsilon = \bigcap_{i=1}^{k} B_{q_i}(0,\varepsilon) \subset W$. Then $W_{\varepsilon/2^n}$ denotes the set $\bigcap_{i=1}^{k} B_{q_i}(0,\varepsilon/2^n)$. For $n \in \mathbb{N}$, nW denotes $W + \cdots + W$ (n times).

For a class of sets $\mathcal{C}$, $\mathcal{D}(\mathcal{C})$ (resp. $R(\mathcal{C})$, $\mathcal{S}(\mathcal{C})$) denotes the δ-ring (resp. ring, σ-ring) generated by $\mathcal{C}$.

2. $(\mathcal{L},\mathcal{G})$-regularity of G-valued additive set functions

In this section, we introduce the notions of $\mathcal{L}$-regularity, $\mathcal{G}$-regularity and $(\mathcal{L},\mathcal{G})$-regularity for a G-valued additive set function λ on $\mathcal{R}$ and give some sufficient conditions to ensure that λ is $(\mathcal{L},\mathcal{G})$-regular whenever λ is $\mathcal{L}$-regular or $\mathcal{G}$-regular. As a concrete application, we give *Theorem* 2.6 which generalizes the results in [4; §15] of Dinculeanu to G-valued set functions on locally compact spaces. Moreover, the results of this section are needed in the subsequent sections.

DEFINITION 2.1. Let $\lambda\colon \mathcal{R} \to G$ be additive. For $A \in \mathcal{R}$, λ is said to be *$\mathcal{L}$-regular* (resp. *$\mathcal{G}$-regular*) *in* A, if for a given $W \in \mathcal{B}$ there exists $K \in \mathcal{L}$ (resp. $U \in \mathcal{G}$) such that $K \subset A$ and $A \setminus K \in \mathcal{R}_W(\lambda)$ (resp. $A \subset U$ and $U \setminus A \in \mathcal{R}_W(\lambda)$). If λ is $\mathcal{L}$-regular (resp. $\mathcal{G}$-regular) in each $A \in \mathcal{C} \subset \mathcal{R}$, then λ is said to be *$\mathcal{L}$-regular* (resp. *$\mathcal{G}$-regular*) *in* $\mathcal{C}$. Moreover, λ is said to be *$(\mathcal{L},\mathcal{G})$-regular in* $A \in \mathcal{R}$ (resp. *in* $\mathcal{R}$) if λ is both $\mathcal{L}$-regular and $\mathcal{G}$-regular in A (resp. in $\mathcal{R}$).

The following proposition is evident from the above definition and the additivity of λ.

PROPOSITION 2.2. *A G-valued additive set function λ on $\mathcal{R}$ is $(\mathcal{L},\mathcal{G})$-regular in $\mathcal{R}$ if and only if, for each $E \in \mathcal{R}$ and $W \in \mathcal{B}$, there exists $U \in \mathcal{G}$ and $K \in \mathcal{L}$ such that $K \subset E \subset U$ and $U \setminus K \in \mathcal{R}_W(\lambda)$.*

THEOREM 2.3. *Let $\lambda\colon \mathcal{R} \to G$ be additive, $\mathcal{G}$ be $\mathcal{L}$-complemented and $\mathcal{L}$ be $\mathcal{R}$-bounded. If $E \in \mathcal{R}$ is $\mathcal{L}$-bounded and λ is $\mathcal{G}$-regular in $\mathcal{R}$, then λ is $(\mathcal{L},\mathcal{G})$-regular in E. Consequently, if $\mathcal{R}$ is $\mathcal{L}$-bounded, then λ is $(\mathcal{L},\mathcal{G})$-regular in $\mathcal{R}$ if and only if λ is $\mathcal{G}$-regular in $\mathcal{R}$.*

Proof. Suppose λ is $\mathcal{G}$-regular in $\mathcal{R}$. By the hypothesis on E and $\mathcal{L}$, there exist $K \in \mathcal{L}$ and $F \in \mathcal{R}$ such that $E \subset K \subset F$. As $F \setminus E \in \mathcal{R}$, given $W \in \mathcal{B}$, there exists $U \in \mathcal{G}$ such that $F \setminus E \subset U$ and $U \setminus (F \setminus E) \in \mathcal{R}_W(\lambda)$. Since $\mathcal{G}$ is $\mathcal{L}$-complemented, $C = K \setminus U \in \mathcal{L}$ and $C \subset K \cap (F \cap E')' = E$. Moreover, $E \setminus C = E \cap U = U \setminus (U \setminus E) \subset U \setminus (F \setminus E)$, and hence $E \setminus C \in \mathcal{R}_W(\lambda)$. Thus

λ is $(\mathcal{L},\mathcal{G})$-regular in E. If $\mathcal{R}$ is $\mathcal{L}$-bounded, then, from the first part, it follows that λ is $(\mathcal{L},\mathcal{G})$-regular in $\mathcal{R}$. The converse is obvious. □

THEOREM 2.4. *Let $\lambda\colon \mathcal{R} \to G$ be additive and let $\mathcal{L}$ be $\mathcal{G}$-complemented. If $E \in \mathcal{R}$ is such that there exists $U \in \mathcal{G}$ and $F \in \mathcal{R}$ such that $E \subset U \subset F$, and if λ is $\mathcal{L}$-regular in $\mathcal{R}$, then λ is $(\mathcal{L},\mathcal{G})$-regular in E. Consequently, if $\mathcal{R}$ is $\mathcal{G}$-bounded and $\mathcal{G}$ is $\mathcal{R}$-bounded, then λ is $(\mathcal{L},\mathcal{G})$-regular in $\mathcal{R}$ if and only if λ is $\mathcal{L}$-regular in $\mathcal{R}$.*

Proof. The proof is similar to that of Theorem 2.3. □

COROLLARY 2.5. *Suppose $\mathcal{L}$ is $\mathcal{G}$-complemented, $\mathcal{R}$-bounded and $\mathcal{L}$-boundedly $\mathcal{G}$-dominated. If $E \in \mathcal{R}$ is $\mathcal{L}$-bounded, then λ is $(\mathcal{L},\mathcal{G})$-regular in E whenever λ is $\mathcal{L}$-regular in $\mathcal{R}$. Consequently, if $\mathcal{R}$ is $\mathcal{L}$-bounded, then λ is $(\mathcal{L},\mathcal{G})$-regular in $\mathcal{R}$ if and only if λ is $\mathcal{L}$-regular in $\mathcal{R}$.*

Proof. Suppose $E \in \mathcal{R}$ is $\mathcal{L}$-bounded. Then there exists $K \in \mathcal{L}$ such that $E \subset K$. Now, by the hypothesis on $\mathcal{L}$ there exist $U \in \mathcal{G}$, $C \in \mathcal{L}$ and $F \in \mathcal{R}$ such that $K \subset U \subset C \subset F$ so that $E \subset U \subset F$. Thus the hypothesis of Theorem 2.4 is satisfied by E, and the corollary is proved. □

As an application of the above results we give the following theorem on locally compact spaces.

THEOREM 2.6. *Let Ω be a locally compact Hausdorff space. Suppose $\mathcal{K}$ (resp. $\mathcal{K}_0$) is the family of all compact (resp. compact G_δ) subsets of Ω and $\mathcal{U}_\sigma$ (resp. $\mathcal{U}_0$) is that of all open sets in $\mathcal{D}(\mathcal{K})$ (resp. in $\mathcal{D}(\mathcal{K}_0)$). Let the ordered pair $(\mathcal{L},\mathcal{G})$ be either $(\mathcal{K},\mathcal{U}_\sigma)$ or $(\mathcal{K}_0,\mathcal{U}_0)$. Let $\mathcal{R}$ be a ring of relatively compact subsets of Ω and let $\lambda\colon \mathcal{R} \to G$ be additive. Then:*

(i) *$\mathcal{R}$ is $\mathcal{L}$-bounded.*
(ii) *$\mathcal{G}$ is $\mathcal{L}$-complemented and $\mathcal{L}$ is $\mathcal{G}$-complemented.*
(iii) *$\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{G}$-dominated.*
(iv) *$\mathcal{L}$ and $\mathcal{G}$ are lattices of sets.*
(v) *Suppose one of the following conditions is satisfied:*
  (α) *$\mathcal{K}_0$ is $\mathcal{R}$-bounded.*
  (β) *$\mathcal{G} \subset \mathcal{R}$.*
  Then $\mathcal{L}$ is $\mathcal{R}$-bounded.
(vi) *If anyone of conditions (α) or (β) of (v) holds, then the following are equivalent:*
  (a) *λ is $(\mathcal{L},\mathcal{G})$-regular in $\mathcal{R}$.*
  (b) *λ is $\mathcal{G}$-regular in $\mathcal{R}$.*
  (c) *λ is $\mathcal{L}$-regular in $\mathcal{R}$.*
(vii) *If $\mathcal{R}$ is $\tau(\mathcal{L},\mathcal{G})$-dense in $\mathcal{P}(\Omega)$, then $\mathcal{K}_0$ is $\mathcal{R}$-bounded (and hence (vi) holds).*

P r o o f.

(i): By [4; §14, Proposition 11], $\mathcal{R}$ is $\mathcal{K}_0$-bounded, and hence $\mathcal{R}$ is $\mathcal{L}$-bounded.

(ii): Obviously, $\mathcal{U}_\sigma$ is $\mathcal{K}$-complemented. For $U \in \mathcal{U}_0$ and $K \in \mathcal{K}_0$, $K \setminus U$ is compact and $K \setminus U \in \mathcal{D}(\mathcal{K}_0)$. Consequently, by [4; §14, Proposition 13], $K \setminus U \in \mathcal{K}_0$, and hence $\mathcal{U}_0$ is $\mathcal{K}_0$-complemented. Thus $\mathcal{G}$ is $\mathcal{L}$-complemented. Trivially, $\mathcal{L}$ is $\mathcal{G}$-complemented.

(iii): Given $K \in \mathcal{L}$, by [4; §14, Proposition 11], there exist $U \in \mathcal{G}$ and $C \in \mathcal{L}$ such that $K \subset U \subset C \subset \Omega$, and hence (iii) holds.

(iv): Obvious.

(v): If (α) holds, then, by [4; §14, Proposition 11], $\mathcal{L}$ is clearly $\mathcal{R}$-bounded. Suppose (β) holds. Let $K \in \mathcal{L}$. Then, again by the same proposition of [4], there exists $U \in \mathcal{G} \subset \mathcal{R}$ such that $K \subset U$. Hence $\mathcal{L}$ is $\mathcal{R}$-bounded.

(vi): Suppose anyone of (α) or (β) holds. Then, by (v), $\mathcal{L}$ is $\mathcal{R}$-bounded. Therefore, by (i) and (ii) and Theorem 2.3, conditions (a) and (b) are equivalent, while, by (i) – (v) and Corollary 2.5, conditions (a) and (c) are equivalent.

(vii): Suppose $\mathcal{R}$ is $\tau(\mathcal{L},\mathcal{G})$-dense in $\mathcal{P}(\Omega)$. Then, given $K_0 \in \mathcal{K}_0$, there exists $F \in \mathcal{R}$ with $K_0 \subset F \subset \Omega$, since $I(K_0,\Omega)$ is $\tau(\mathcal{L},\mathcal{G})$-open by definition and $\mathcal{K}_0 \subset \mathcal{L}$. Thus $\mathcal{K}_0$ is $\mathcal{R}$-bounded. □

R e m a r k 2.7. If G is a normed space, then Theorem 2.6 clearly subsumes [4; §15, Proposition 6] as a very particular case.

3. $(\mathcal{L},\mathcal{G})$-regularity of G-valued σ-additive set functions

When $\mu\colon \mathcal{R} \to G$ is σ-additive, we give a set of sufficient conditions to extend Theorem 2.3 and Corollary 2.5 to $\sigma\mathcal{L}$-bounded sets in $\mathcal{R}$. As a concrete application of these, we obtain a theorem on locally compact spaces. The results of this section will be used in the next section.

In the sequel, we shall assume $\mathcal{L}$ to be closed under unions.

THEOREM 3.1. *Let $\mu\colon \mathcal{R} \to G$ be σ-additive. Suppose $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{R}$-dominated and $\mathcal{L}$-boundedly $\mathcal{G}$-dominated. Let $\mathcal{G}$ be $\mathcal{L}$-complemented. Then:*

- (i) *If $E \in \mathcal{R}$ is $\sigma\mathcal{L}$-bounded, then μ is $(\mathcal{L},\mathcal{G})$-regular in E whenever μ is $\mathcal{G}$-regular in $\mathcal{R}$.*
- (ii) *If $\mathcal{R}$ is $\sigma\mathcal{L}$-bounded, then μ is $(\mathcal{L},\mathcal{G})$-regular in $\mathcal{R}$ if and only if μ is $\mathcal{G}$-regular in $\mathcal{R}$.*

P r o o f.

(i): Given $W \in \mathcal{B}$, choose $W_0 \in \mathcal{B}$ such that $2W_0 \subset W$. As E is $\sigma\mathcal{L}$-bounded, there exists a sequence $(K_n)_1^\infty \subset \mathcal{L}$ such that $E \subset \bigcup_1^\infty K_n$. Since $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{G}$-dominated, for each n there exist $U_n \in \mathcal{G}$ and $C_n \in \mathcal{L}$ such that $K_n \subset U_n$

$\subset C_n$. As $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{R}$-dominated, for each C_n there exist $F_n \in \mathcal{R}$ and $D_n \in \mathcal{L}$ such that $C_n \subset F_n \subset D_n$. Let $E_n = \bigcup_{k=1}^{n} F_k$. Then $E_n \in \mathcal{R}$, E_n is $\mathcal{L}$-bounded for each n, $E_n \uparrow$ and $E \subset \bigcup_1^\infty E_n$. Taking $B_n = E \cap E_n$, it follows that $B_n \in \mathcal{R}$, $B_n \uparrow E$, and each B_n is $\mathcal{L}$-bounded. As $E \setminus B_n \downarrow \emptyset$ and μ is σ-additive, by Proposition 1.5, there exists n_0 such that $E \setminus B_{n_0} \in \mathcal{R}_{W_0}(\mu)$. Since μ is $\mathcal{G}$-regular and B_{n_0} is $\mathcal{L}$-bounded, by Theorem 2.3, μ is $(\mathcal{L}, \mathcal{G})$-regular in B_{n_0}. Thus there exists $K \in \mathcal{L}$ such that $K \subset B_{n_0}$ and $B_{n_0} \setminus K \in \mathcal{R}_{W_0}(\mu)$. Consequently, $K \subset E$ and $E \setminus K \in \mathcal{R}_W(\mu)$ since μ is additive and $2W_0 \subset W$. Thus μ is $(\mathcal{L}, \mathcal{G})$-regular in E.

(ii): This follows from (i). □

COROLLARY 3.2. *Let $\mu\colon \mathcal{R} \to G$ be σ-additive and let $\mathcal{G}$ be $\mathcal{L}$-complemented. Suppose $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{G}$-dominated. If $\mathcal{G} \subset \mathcal{R}$, or if $\mathcal{R}$ is $\tau(\mathcal{L}, \mathcal{G})$-dense in $\mathcal{P}(\Omega)$, then $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{R}$-dominated. Consequently, if E is $\sigma\mathcal{L}$-bounded (resp. $\mathcal{R}$ is $\sigma\mathcal{L}$-bounded), then μ is $(\mathcal{L}, \mathcal{G})$-regular in E (resp.in $\mathcal{R}$) if (resp. if and only if) μ is $\mathcal{G}$-regular in $\mathcal{R}$.*

Proof. Given $K \in \mathcal{L}$, by the hypothesis on $\mathcal{L}$, there exist $U \in \mathcal{G}$ and $C \in \mathcal{L}$ such that $K \subset U \subset C$. If $\mathcal{G} \subset \mathcal{R}$, take $F = U \in \mathcal{R}$. If $\mathcal{R}$ is $\tau(\mathcal{L}, \mathcal{G})$-dense, then there exists $F \in \mathcal{R}$ such that $K \subset F \subset U$. In both cases, it follows that $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{R}$-dominated. The rest is immediate from Theorem 3.1. □

THEOREM 3.3. *Let $\mu\colon \mathcal{R} \to G$ be σ-additive. Suppose $\mathcal{G}$ is closed under countable unions and $\mathcal{L}$ is $\mathcal{G}$-complemented. Let $\mathcal{L}$ be $\mathcal{L}$-boundedly $\mathcal{G}$-dominated and $\mathcal{G} \subset \mathcal{R}$. Then:*

- (i) *If $E \in \mathcal{R}$ is $\sigma\mathcal{L}$-bounded, then μ is $(\mathcal{L}, \mathcal{G})$-regular in E whenever μ is $\mathcal{L}$-regular in $\mathcal{R}$.*
- (ii) *If $\mathcal{R}$ is $\sigma\mathcal{L}$-bounded, then μ is $(\mathcal{L}, \mathcal{G})$-regular in $\mathcal{R}$ if and only if μ is $\mathcal{L}$-regular in $\mathcal{R}$.*

Proof.

(i): Given $W \in \mathcal{B}$, there exists a finite family of continuous quasi-norms $(q_i)_1^k$ on G and $\varepsilon > 0$ such that $W_\varepsilon = \bigcap_{i=1}^{k} B_{q_i}(0, \varepsilon) \subset W$. By Corollary 3.2, $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{R}$-dominated. Thus, as in the proof of Theorem 3.1, there exists $(B_n)_1^\infty \subset \mathcal{R}$ such that $B_n \uparrow E$ and each B_n is $\mathcal{L}$-bounded. For each $C \in \mathcal{L}$, by hypothesis, there exists $U \in \mathcal{G} \subset \mathcal{R}$ such that $C \subset U$ so that $\mathcal{L}$ is $\mathcal{R}$-bounded. Consequently, by Corollary 2.5, μ is $(\mathcal{L}, \mathcal{G})$-regular in each B_n. Thus, for each n there exists $U_n \in \mathcal{G}$ such that $B_n \subset U_n$ and $U_n \setminus B_n \in \mathcal{R}_{W_{\varepsilon/2^{n+1}}}(\mu)$. Since $\mathcal{G}$ is closed under countable unions, $U = \bigcup_1^\infty U_n \in \mathcal{G}$. Moreover, $E = \bigcup_1^\infty B_n \subset$

$\bigcup\limits_1^{\infty} U_n = U \in \mathcal{R}$. Let $A \in \mathcal{R}$ with $A \subset U \setminus E$. If $H_n = U_n \setminus B_n$, then $A \subset \bigcup\limits_1^{\infty}(U_n \setminus B_n) = \bigcup\limits_{n=1}^{\infty}\Big(H_n \setminus \Big(\bigcup\limits_{i<n} H_i\Big)\Big)$, and hence

$$\mu(A) = \sum_{n=1}^{\infty}\left\{\mu(A \cap H_n) - \mu\left(A \cap H_n \cap \left(\bigcup_{i<n} H_i\right)\right)\right\} \in W$$

since $q_i \circ \mu(A \cap H_n) < \varepsilon/2^{n+1}$ and $q_i \circ \mu\Big(A \cap H_n \cap \Big(\bigcup\limits_{i<n} H_i\Big)\Big) < \varepsilon/2^{n+1}$ for $i = 1, 2, \ldots, k$. Thus μ is $(\mathcal{L}, \mathcal{G})$-regular in E.

(ii): This follows from (i). □

As a consequence of the above results, we can give the following theorem for G-valued σ-additive set functions on locally compact spaces.

THEOREM 3.4. *Let Ω be a locally compact Hausdorff space and let $\mathcal{K}_0$ and $\mathcal{K}$ be as in Theorem* 2.6. *Suppose Σ is a σ-ring such that $\mathcal{S}(\mathcal{K}_0) \subset \Sigma \subset \mathcal{S}(\mathcal{K})$. Let $\mu\colon \Sigma \to G$ be σ-additive. Let $\mathcal{G}$ be the family of all open sets in $\mathcal{S}(\mathcal{K}_0)$ and let $\mathcal{L} = \mathcal{K}_0$. Then the following are equivalent:*

- (i) *μ is $(\mathcal{L}, \mathcal{G})$-regular in Σ.*
- (ii) *μ is $\mathcal{G}$-regular in Σ.*
- (iii) *μ is $\mathcal{L}$-regular in Σ.*

Proof. Clearly, $\mathcal{L}$ is $\mathcal{G}$-complemented. By [4; §14, Proposition 13], $\mathcal{G}$ is $\mathcal{L}$-complemented. By [4; §14, Proposition 11], $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{G}$-dominated. Clearly, $\mathcal{G} \subset \Sigma$ and $\mathcal{G}$ is closed under countable unions. Since Σ is $\sigma\mathcal{L}$-bounded by [4; §14, Proposition 11], the result is now immediate from Corollary 3.2 and Theorem 3.3. □

Remark 3.5. When $\mathcal{R}$ coincides with anyone of $\mathcal{D}(\mathcal{L})$, $\mathcal{S}(\mathcal{L})$, $\mathcal{D}(\mathcal{G})$ or $\mathcal{S}(\mathcal{G})$, under suitable conditions we can strengthen all the theorems in Sections 2 and 3 substantially. (See Theorems 4.3 and 4.6 and their corollaries.)

4. Generalizations of Theorem 52.F of Halmos ([9])

Let $\mathcal{L}$ be closed under unions, $\mathcal{G}$ under intersections and $\emptyset \in \mathcal{L} \cap \mathcal{G}$. Let $\mathcal{R}^{(1)} = \mathcal{D}(\mathcal{L})$ and $\mathcal{R}^{(2)} = \mathcal{S}(\mathcal{L})$. Under additional hypothesis on $\mathcal{L}$, $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$, we shall prove *Theorems* 4.3 and 4.6, which state that a G-valued σ-additive set function μ_i on $\mathcal{R}^{(i)}$ is $(\mathcal{L}, \mathcal{G})$-regular in $\mathcal{R}^{(i)}$ if and only if μ_i is $\mathcal{G}$-regular on $\mathcal{L}$ (resp. if and only if μ_i is $\mathcal{L}$-regular on $\mathcal{L}$-bounded sets in $\mathcal{G}$). Then, in the abstract set-up, the said theorems generalize [9; Theorem 52.F] of Halmos to G-valued σ-additive set functions. As a consequence, the classical results of Dinculeanu and Kluvánek [5], Dinculeanu and

Lewis [6] and of Khurana [11] on the regularity of vector- or group-valued Baire measures on locally compact Hausdorff spaces are obtained as corollaries. Moreover, the classical closed-open regularity of finite positive measures on the Borel sets of a metric space also gets extended naturally to G-valued σ-additive set functions and this generalization is given in *Corollary* 4.11.

Hereafter we shall assume that $\mathcal{L}$ is closed under unions, $\mathcal{G}$ is closed under intersections and $\emptyset \in \mathcal{L} \cap \mathcal{G}$. We shall also assume that $\mathcal{L}$ is $\mathcal{G}$-bounded and $\mathcal{G}$-complemented and that $\mathcal{G}$ is $\mathcal{L}$-complemented. We state the following two conditions $(*)$ and $(**)$.

$(*)$ $\mathcal{G} \subset \mathcal{D}(\mathcal{L})$ and $\bigcup_1^\infty U_n \in \mathcal{G}$ whenever $(U_n)_1^\infty \subset \mathcal{G}$ and $\bigcup_1^\infty U_n \in \mathcal{D}(\mathcal{L})$.

$(**)$ $\mathcal{G} \subset \mathcal{S}(\mathcal{L})$ and $\mathcal{G}$ is closed under countable unions.

LEMMA 4.1.

(a) *Suppose condition $(*)$ holds for $\mathcal{D}(\mathcal{L})$. Then:*
   (i) *$\mathcal{G}$ is a lattice of sets.*
   (ii) *$\mathcal{L}$ is closed under countable intersections and, in particular, $\mathcal{L}$ is a lattice of sets.*
   (iii) *$\mathcal{D}(\mathcal{L}) = \{E \in \mathcal{S}(\mathcal{L}) :\ E$ is $\mathcal{L}$-bounded$\}$.*

(b) *If condition $(**)$ holds for $\mathcal{S}(\mathcal{L})$, then $\mathcal{S}(\mathcal{L})$ is $\mathcal{G}$-bounded. Moreover,* (i) *and* (ii) *of* (a) *are also true.*

Proof.

(a): (i): Obvious.

(ii): Let $(C_n)_1^\infty \subset \mathcal{L}$ and let $C = \bigcap_1^\infty C_n$. By the hypothesis on $\mathcal{L}$ and $\mathcal{G}$, there exists $U \in \mathcal{G}$ such that $C_1 \subset U$ so that $C_1 \cap C_2 = C_1 \setminus (U \setminus C_2) \in \mathcal{L}$. Moreover, $U \setminus C_n \in \mathcal{G}$ for all n, and hence, by condition $(*)$, $\bigcup_1^\infty (U \setminus C_n) \in \mathcal{G}$. Then

$$C = C_1 \setminus (C_1 \setminus C) = C_1 \setminus (U \setminus C) = C_1 \setminus \left(\bigcup_1^\infty (U \setminus C_n) \right) \in \mathcal{L}.$$

(iii): Let $\mathcal{R} = \{E \in \mathcal{S}(\mathcal{L}) :\ E$ is $\mathcal{L}$-bounded$\}$. Clearly, $\mathcal{R}$ is a δ-ring and $\mathcal{R} \supset \mathcal{D}(\mathcal{L})$. On the other hand, if $E \in \mathcal{R}$, then there exist $(E_n)_1^\infty \subset \mathcal{D}(\mathcal{L})$ and $K \in \mathcal{L}$ such that $E = \bigcup_1^\infty E_n \subset K$. Then $E = \bigcup_1^\infty (E_n \cap K) \in \mathcal{D}(\mathcal{L})$.

(b): If $B \in \mathcal{S}(\mathcal{L})$, then $B = \bigcup_1^\infty B_n$, $B_n \uparrow$, $B_n \in \mathcal{D}(\mathcal{L})$ for all n, and for each n there exist $K_n \in \mathcal{L}$ and $U_n \in \mathcal{G}$ such that $B_n \subset K_n \subset U_n$. Then $B \subset \bigcup_1^\infty U_n \in \mathcal{G}$ by condition $(**)$. The last part is evident from the proof of (a).

□

Lemma 4.2. *Let $\mathcal{R}^{(1)} = \mathcal{D}(\mathcal{L})$ and $\mathcal{R}^{(2)} = \mathcal{S}(\mathcal{L})$. Suppose condition $(*)$ holds for $\mathcal{R}^{(1)}$ and $(**)$ for $\mathcal{R}^{(2)}$. Let $\mu_i\colon \mathcal{R}^{(i)} \to G$ be σ-additive and let $\mathcal{M}_i = \{E \in \mathcal{R}^{(i)} : \mu_i$ is $\mathcal{G}$-regular in $E\}$ for $i = 1,2$. Then:*

(i) *If $\mathcal{L} \subset \mathcal{M}_i$, then $R(\mathcal{L}) \subset \mathcal{M}_i$ for $i = 1,2$.*

(ii) *$\mathcal{M}_1$ is a monotone class with respect to $\mathcal{R}^{(1)}$, and $\mathcal{M}_2$ is a monotone class.*

Proof.

(i): Let $C_1, C_2 \in \mathcal{L}$ and let $W \in \mathcal{B}$. As $\mathcal{L} \subset \mathcal{M}_i$, there exists $U \in \mathcal{G}$ such that $C_1 \subset U$ and $U \setminus C_1 \in \mathcal{R}_W^{(i)}(\mu_i)$. Then $V = U \setminus C_2 \in \mathcal{G}$, $C_1 \setminus C_2 \subset V$ and $V \setminus (C_1 \setminus C_2) \subset U \setminus C_1$. Thus $C_1 \setminus C_2 \in \mathcal{M}_i$. Let $E \in \mathcal{R}(\mathcal{L})$. Then E is of the form $E = \bigcup_{j=1}^{n} E_j$, $E_j \cap E_{j'} = \emptyset$ for $j \neq j'$, and $E_j = C_j \setminus D_j$ with $C_j, D_j \in \mathcal{L}$ for $j = 1,2,\ldots,n$. Choose $W_0 \in \mathcal{B}$ such that $2nW_0 \subset W$. Since each $E_j \in \mathcal{M}_i$, there exists $U_j \in \mathcal{G}$ such that $E_j \subset U_j$ and $U_j \setminus E_j \in \mathcal{R}_{W_0}^{(i)}(\mu_i)$. Put $U = \bigcup_{j=1}^{n} U_j$. Then, by condition $(*)$ (resp. by $(**)$), $U \in \mathcal{G}$ and $\mathcal{G} \subset \mathcal{R}^{(1)}$ (resp. $\mathcal{G} \subset \mathcal{R}^{(2)}$) so that $E \subset U \in \mathcal{G} \subset \mathcal{R}^{(i)}$. For $A \in \mathcal{R}^{(i)}$ with $A \subset U \setminus E$, let $H_j = A \cap (U_j \setminus E_j)$. Then

$$\mu_i(A) = \mu_i\left(\bigcup_1^n H_j\right) = \sum_{j=1}^{n}\left\{\mu_i(H_j) - \mu_i\left(H_j \cap \left(\bigcup_{l<j} H_l\right)\right)\right\} \in 2nW_0 \subset W$$

for $i = 1,2$. Hence (i) holds.

(ii): Let $W \in \mathcal{B}$. Choose continuous quasi-norms $(q_j)_1^k$ on G and $\varepsilon > 0$ such that $W_\varepsilon = \bigcap_{j=1}^{k} B_{q_j}(0,\varepsilon) \subset W$. Let $W_0 \in \mathcal{B}$ such that $2W_0 \subset W$. Clearly, $\emptyset \in \mathcal{M}_i$. Let $E_p \uparrow E$, with $(E_p)_1^\infty \subset \mathcal{M}_i$. For $i = 1$, let $E \in \mathcal{R}^{(1)}$. Then for each E_p there exists $U_p \in \mathcal{G}$ such that $E_p \subset U_p$ and $U_p \setminus E_p \in \mathcal{R}^{(i)}_{W_{\varepsilon/2^{p+1}}}(\mu_i)$. For $i = 1$, by Lemma 4.1 (a) (iii), there exists $K \in \mathcal{L}$ such that $E \subset K$. Since $\mathcal{L}$ is $\mathcal{G}$-bounded, there exists $V \in \mathcal{G}$ such that $K \subset V$. Let $V_p = V \cap U_p$. Then by Lemma 4.1 (a) (i), $V_p \in \mathcal{G}$, so that, by condition $(*)$, $V_0 = \bigcup_1^\infty V_p \in \mathcal{G}$ and $E \subset V_0$. For $i = 2$, take $V_p = U_p$ for $p \in \mathbb{N}$. Thus $E \subset V_0 \in \mathcal{G} \subset \mathcal{R}^{(i)}$ for $i = 1,2$. For $A \in \mathcal{R}^{(i)}$ with $A \subset V_0 \setminus E$, we have $A = A \cap \left(\bigcup_1^\infty (V_p \setminus E_p)\right)$. Let $H_p = A \cap (V_p \setminus E_p)$. Then $\mu_i(A) = \sum_{p=1}^{\infty}\left\{\mu_i(H_p) - \mu_i\left(H_p \cap \left(\bigcup_{j<p} H_j\right)\right)\right\}$. Thus $q_j \circ \mu_i(A) < \varepsilon$ for $j = 1,2,\ldots,k$. Therefore, $\mu_i(A) \in W$, and hence $E \in \mathcal{M}_i$ for $i = 1,2$. Let $E_p \downarrow E$, $E_p \in \mathcal{R}^{(i)}$ for $p \in \mathbb{N}$. Then $E_p \setminus E \downarrow \emptyset$ in $\mathcal{R}^{(i)}$, and hence, by Proposition 1.5, there exists p_0 such that $E_{p_0} \setminus E \in \mathcal{R}^{(i)}_{W_0}(\mu_i)$. As $E_{p_0} \in \mathcal{M}_i$, there exists $U \in \mathcal{G}$ such that $E_{p_0} \subset U$ and $U \setminus E_{p_0} \in \mathcal{R}^{(i)}_{W_0}(\mu_i)$. Consequently, $E \subset U \in \mathcal{G}$ and $U \setminus E \in \mathcal{R}^{(i)}_{W}(\mu_i)$. Thus $E \in \mathcal{M}_i$ for $i = 1, 2$. Hence (ii) holds. □

Theorem 4.3. *Let $\mathcal{R}^{(1)} = \mathcal{D}(\mathcal{L})$ and $\mathcal{R}^{(2)} = \mathcal{S}(\mathcal{L})$. Suppose $\mu_i \colon \mathcal{R} \to G$ is σ-additive for $i = 1, 2$. Let $\mathcal{R}^{(1)}$ satisfy condition $(*)$, and $\mathcal{R}^{(2)}$ condition $(**)$. In the case of $\mathcal{R}^{(2)}$, suppose $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{G}$-dominated. Then μ_i is $(\mathcal{L}, \mathcal{G})$-regular in $\mathcal{R}^{(i)}$ if and only if μ_i is $\mathcal{G}$-regular in $\mathcal{L}$ for $i = 1, 2$.*

Proof. Clearly, the condition is necessary. Conversely, let μ_i be $\mathcal{G}$-regular in $\mathcal{L}$. By Lemma 4.2, $R(\mathcal{L}) \subset \mathcal{M}_i$, $\mathcal{M}_1$ is a monotone class with respect to $\mathcal{R}^{(1)}$, and $\mathcal{M}_2$ is a monotone class. Thus by [4; §1, Proposition 1], $\mathcal{M}_1 = \mathcal{R}^{(1)}$, and by [9; Theorem 6.B], $\mathcal{M}_2 = \mathcal{R}^{(2)}$. Thus μ_i is $\mathcal{G}$-regular in $\mathcal{R}^{(i)}$ for $i = 1, 2$.

For $i = 1$, $\mathcal{R}^{(1)} = \mathcal{D}(\mathcal{L}) \supset \mathcal{L}$ so that $\mathcal{L}$ is $\mathcal{R}^{(1)}$-bounded and $\mathcal{R}^{(1)}$ is $\mathcal{L}$-bounded. By hypothesis, $\mathcal{G}$ is $\mathcal{L}$-complemented. Therefore, by Theorem 2.3, μ_1 is $(\mathcal{L}, \mathcal{G})$-regular in $\mathcal{R}^{(1)}$.

For $i = 2$, $\mathcal{R}^{(2)} = \mathcal{S}(\mathcal{L}) \supset \mathcal{L}$ so that $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{R}^{(2)}$-dominated. Moreover, by the additional hypothesis on $\mathcal{L}$, $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{G}$-dominated. As $\mathcal{G}$ is $\mathcal{L}$-complemented and $\mathcal{R}^{(2)}$ is $\sigma\mathcal{L}$-bounded, by Theorem 3.1, we conclude that μ_2 is $(\mathcal{L}, \mathcal{G})$-regular in $\mathcal{R}^{(2)}$. □

Lemma 4.4. *Let $\mathcal{R}^{(1)} = \mathcal{D}(\mathcal{L})$ and $\mathcal{R}^{(2)} = \mathcal{S}(\mathcal{L})$. Suppose $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ satisfy conditions $(*)$ and $(**)$, respectively. Then $\mathcal{L} \subset R(\mathcal{G})$, $\mathcal{R}^{(1)} = \mathcal{D}(\mathcal{G})$ and $\mathcal{R}^{(2)} = \mathcal{S}(\mathcal{G})$.*

Proof. Let $K \in \mathcal{L}$. As, by hypothesis, $\mathcal{L}$ is $\mathcal{G}$-bounded, there exists $U \in \mathcal{G}$ such that $K \subset U$. Since $\mathcal{L}$ is $\mathcal{G}$-complemented and $K = U \setminus (U \setminus K)$, it follows that $K \in R(\mathcal{G})$ so that $\mathcal{L} \subset R(\mathcal{G})$. Consequently, $\mathcal{R}^{(1)} \subset \mathcal{D}(\mathcal{G})$ and $\mathcal{R}^{(2)} \subset \mathcal{S}(\mathcal{G})$. On the other hand, by condition $(*)$, $\mathcal{G} \subset \mathcal{D}(\mathcal{L})$, and by condition $(**)$, $\mathcal{G} \subset S(\mathcal{L})$, whence $\mathcal{D}(\mathcal{G}) = \mathcal{R}^{(1)}$ and $S(\mathcal{G}) = \mathcal{R}^{(2)}$. □

Lemma 4.5. *Let $\mu \colon \mathcal{S}(\mathcal{L}) \to G$ be σ-additive and let condition $(**)$ hold for $\mathcal{S}(\mathcal{L})$. Suppose moreover that $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{G}$-dominated. Then μ is $\mathcal{G}$-regular in $\mathcal{L}$ if and only if μ is $\mathcal{L}$-regular in every $\mathcal{L}$-bounded set $U \in \mathcal{G}$.*

Proof. By Theorem 4.3, the condition is necessary. Conversely, let μ be $\mathcal{L}$-regular in every $\mathcal{L}$-bounded set in $\mathcal{G}$. Let $K \in \mathcal{L}$. Since $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{G}$-dominated, there exists $U \in \mathcal{G}$ and $K_1 \in \mathcal{L}$ such that $K \subset U \subset K_1$. As $\mathcal{L}$ is $\mathcal{G}$-complemented, $V = U \setminus K \in \mathcal{G}$ and V is $\mathcal{L}$-bounded. Consequently, given $W \in \mathcal{B}$, by hypothesis, there exists $C \in \mathcal{L}$ such that $C \subset V$ and $V \setminus C \in \mathcal{S}(\mathcal{L})_W(\mu)$. Then $K \subset (U \setminus C) \in \mathcal{G}$ and $(U \setminus C) \setminus K = V \setminus C \in \mathcal{S}(\mathcal{L})_W(\mu)$. Hence μ is $\mathcal{G}$-regular in $\mathcal{L}$. □

THEOREM 4.6. *Let $\mathcal{R}^{(1)} = \mathcal{D}(\mathcal{L})$ and $\mathcal{R}^{(2)} = \mathcal{S}(\mathcal{L})$. Suppose $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ satisfy conditions $(*)$ and $(**)$, respectively. In the case of $\mathcal{R}^{(2)}$, let $\mathcal{L}$ be $\mathcal{L}$-boundedly $\mathcal{G}$-dominated. Let $\mu_i\colon \mathcal{R}^{(i)} \to G$ be σ-additive. Then:*

(i) $\mathcal{R}^{(1)} = \mathcal{D}(\mathcal{G})$ *and* $\mathcal{R}^{(2)} = \mathcal{S}(\mathcal{G})$.

(ii) μ_i *is $(\mathcal{L},\mathcal{G})$-regular in $\mathcal{R}^{(i)}$ if and only if μ_i is $\mathcal{L}$-regular in $\mathcal{G}$ for* $i = 1, 2$.

(iii) μ_i *is $(\mathcal{L},\mathcal{G})$-regular in $\mathcal{R}^{(i)}$ if and only if μ_i is $\mathcal{L}$-regular in every $\mathcal{L}$-bounded set in $\mathcal{G}$.*

Proof. In the light of Lemmas 4.4 and 4.5, it suffices to prove that $\mathcal{L}$-regularity of μ_i in $\mathcal{G}$ implies $(\mathcal{L},\mathcal{G})$-regularity of μ_i in $\mathcal{R}^{(i)}$. Let $\mathcal{N}_i = \{E \in \mathcal{R}^{(i)} : \mu_i \text{ is } \mathcal{L}\text{-regular in } E\}$ and let μ_i be $\mathcal{L}$-regular in $\mathcal{G}$. Then $\mathcal{G} \subset \mathcal{N}_i$, so that, by Lemma 4.4 and by an argument similar to that in the proof of Lemma 4.2 (i), $R(\mathcal{G}) \subset \mathcal{N}_i$.

Given $W \in \mathcal{B}$, choose continuous quasi-norms $(q_j)_{j=1}^k$ on G, $\varepsilon > O$ and $W_0 \in \mathcal{B}$ such that $W_\varepsilon = \bigcap_{j=1}^k B_{q_j}(0,\varepsilon) \subset W$ and $2W_0 \subset W$. Let $E_n \uparrow E$ and $F_n \downarrow F$ with E_n, F_n in $\mathcal{N}_i$ for all n. For $i = 1$, let us assume that $E \in \mathcal{R}^{(1)}$. Then $E \setminus E_n \downarrow \emptyset$, and hence, by Proposition 1.5, there exists n_0 such that $E \setminus E_{n_0} \in \mathcal{R}^{(i)}_{W_0}(\mu_i)$. As $E_{n_0} \in \mathcal{N}_i$, there exists $C \in \mathcal{L}$ such that $C \subset E_{n_0}$ and $E_{n_0} \setminus C \in \mathcal{R}^{(i)}_{W_0}(\mu_i)$. Consequently, $C \subset E$ and $(E \setminus C) \in \mathcal{R}^{(i)}_{W}(\mu_i)$. Therefore, $E \in \mathcal{N}_i$ for $i = 1, 2$. As $F_n \in \mathcal{N}_i$, there exists $C_n \in \mathcal{L}$ such that $C_n \subset F_n$ and $F_n \setminus C_n \in \mathcal{R}^{(i)}_{W_{\varepsilon/2^{n+1}}}(\mu_i)$ for $n \in \mathbb{N}$. If $C = \bigcap_1^\infty C_j$, then, by Lemma 4.1, $C \in \mathcal{L}$ and $C \subset F$. Since $F \setminus C \subset \bigcup_1^\infty (F_n \setminus C_n)$, for $A \in \mathcal{R}^{(i)}$ with $A \subset F \setminus C$ we have $A = \bigcup_1^\infty \big(A \cap (F_n \setminus C_n)\big)$. Then it follows that $q_j \circ \mu_i(A) < \sum_{n=1}^\infty 2\varepsilon/2^{n+1} = \varepsilon$ for $j = 1, 2, \ldots, k$. Thus $A \in \mathcal{R}^{(i)}_W(\mu_i)$, and hence $F \in \mathcal{N}_i$. Then, as in the proof of Theorem 4.3, it follows that $\mathcal{N}_i = \mathcal{R}^{(i)}$ for $i = 1, 2$. Finally, by Theorem 2.4 (resp. by Theorem 3.3 (ii)), μ_1 is $(\mathcal{L},\mathcal{G})$-regular on $\mathcal{R}^{(1)}$ (resp. μ_2 is $(\mathcal{L},\mathcal{G})$-regular on $\mathcal{R}^{(2)}$). □

DEFINITION 4.7. The lattice of sets $\mathcal{L}$ is said to satisfy the *G_δ-property relative to $\mathcal{G}$* if every $C \in \mathcal{L}$ is of the form $C = \bigcap_1^\infty U_n$ with $(U_n)_1^\infty \subset \mathcal{G}$. Similarly, $\mathcal{G}$ is said to satisfy the *F_σ-property relative to $\mathcal{L}$* if every $\mathcal{L}$-bounded member of $\mathcal{G}$ is a countable union of members of $\mathcal{L}$.

COROLLARY 4.8. *Suppose* $\mu_i\colon \mathcal{R}^{(i)} \to G$ *is* σ*-additive for* $i = 1,2$*, where* $\mathcal{R}^{(1)} = \mathcal{D}(\mathcal{L})$ *and* $\mathcal{R}^{(2)} = \mathcal{S}(\mathcal{L})$*. Let* $\mathcal{R}^{(1)}$ *satisfy condition* $(*)$*, and* $\mathcal{R}^{(2)}$ *condition* $(**)$*. In the case of* $\mathcal{R}^{(2)}$*, let* $\mathcal{L}$ *be* $\mathcal{L}$*-boundedly* $\mathcal{G}$*-dominated. If* $\mathcal{L}$ *has the* G_δ*-property relative to* $\mathcal{G}$ *(resp.* $\mathcal{G}$ *has the* F_σ*-property relative to* $\mathcal{L}$*), then* μ_i *is* $(\mathcal{L},\mathcal{G})$*-regular in* $\mathcal{R}^{(i)}$ *for* $i = 1,2$*.*

Proof. By Proposition 1.5, μ_i is $\mathcal{G}$-regular in $\mathcal{L}$ (resp. $\mathcal{L}$-regular in every $\mathcal{L}$-bounded set in $\mathcal{G}$). Consequently, the result is immediate from Theorem 4.6. □

COROLLARY 4.9. *Let* Ω *be a locally compact Hausdorff space. Let* $\mathcal{K}$*,* $\mathcal{K}_0$ *be as in Theorem* 2.6*. Suppose* $\mathcal{L}$ *is a lattice of sets such that* $\mathcal{K}_0 \subset \mathcal{L} \subset \mathcal{K}$ *and such that* $\mathcal{L}$ *is precisely the collection of all compact sets in* $\mathcal{D}(\mathcal{L})$*. Let* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *be the families of all open sets in* $\mathcal{D}(\mathcal{L})$ *and* $\mathcal{S}(\mathcal{L})$*, respectively. Let* $\mathcal{R}^{(1)} = \mathcal{D}(\mathcal{L})$ *and* $\mathcal{R}^{(2)} = \mathcal{S}(\mathcal{L})$ *and let* $\mu_i\colon \mathcal{R}^{(i)} \to G$ *be* σ*-additive for* $i = 1,2$*. Then the following are equivalent:*

(i) μ_i *is* $(\mathcal{L},\mathcal{G}_i)$*-regular in* $\mathcal{R}^{(i)}$*.*
(ii) μ_i *is* $\mathcal{G}_i$*-regular in* $\mathcal{L}$*.*
(iii) μ_i *is* $\mathcal{L}$*-regular in each* $\mathcal{L}$*-bounded set of* $\mathcal{G}_i$*.*

Proof. By hypothesis, $\mathcal{G}_i$ is $\mathcal{L}$-complemented and, clearly, $\mathcal{L}$ is $\mathcal{G}_i$-complemented. By [4; §14, Proposition 11], $\mathcal{L}$ is $\mathcal{L}$-boundedly $\mathcal{G}_i$-dominated. Now the corollary is immediate from Theorems 4.3 and 4.6. □

COROLLARY 4.10. *Let* Ω *and* $\mathcal{K}_0$ *be as in Corollary* 4.9*. Let* $\mathcal{G}_1 = \{U \in \mathcal{D}(\mathcal{K}_0) : U \text{ open}\}$ *and* $\mathcal{G}_2 = \{U \in \mathcal{S}(\mathcal{K}_0) : U \text{ open}\}$*. Then every* G*-valued* σ*-additive set function on* $\mathcal{D}(\mathcal{K}_0)$ *(resp. on* $\mathcal{S}(\mathcal{K}_0)$*) is* $(\mathcal{K}_0,\mathcal{G}_1)$*-regular (resp.* $(\mathcal{K}_0,\mathcal{G}_2)$*-regular).*

Proof. Use Corollary 4.8 and [4; §14, Proposition 11]. □

COROLLARY 4.11. *Let* Ω *be a metric space with* $\mathcal{L}$ *the family of all closed subsets and* $\mathcal{G}$ *the family of all open subsets of* Ω*. Then every* G*-valued* σ*-additive set function* μ *on* $\mathcal{S}(\mathcal{L})$ $(= \mathcal{B}(\Omega))$ *is* $(\mathcal{L},\mathcal{G})$*-regular.*

Remark 4.12. Corollary 4.9 extends [9; Theorem 52.F] of Halmos to G-valued σ-additive set functions when $\mathcal{L} = \mathcal{K}_0$ or $\mathcal{K}$.

Remark 4.13. Theorem 4 of Dinculeanu and Kluvánek [5] is a particular case of Corollary 4.10. Dinculeanu and Lewis [6] give a direct proof of [5; Theorem 4]. Corollary 4.10 is the same as the first part of [11; Theorem 1 and Corollary 4] of Khurana . The method used here is quite general, elegant and powerful.

R e m a r k 4.14. Corollary 4.11 generalizes the classical result known for finite positive measures on the Borel sets of a metric space. (See [9; Exercise 43.3].)

In the light of Corollaries 4.9, 4.10 and 4.11, our abstract approach has the advantage of unifying results on locally compact spaces and metric spaces.

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Received April 7, 1993
Revised November 30, 1993

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