Hilda Draškovičová
Modular median algebras generated by some partial modular median algebras


Persistent URL: http://dml.cz/dmlcz/136679

Terms of use:
© Mathematical Institute of the Slovak Academy of Sciences, 1996

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
MODULAR MEDIAN ALGEBRAS GENERATED BY SOME PARTIAL MODULAR MEDIAN ALGEBRAS

HILDA Draškovičová

(Communicated by Tibor Katriňák)

ABSTRACT. Let $M$ denote the variety of algebras with one ternary operation $(abc)$ satisfying the identities $(abb) = b$ and $((abc)dc) = (ac(dcb))$. The subvariety $T$ of the variety $M$ is given by the identity $((abc)de) = ((ade)(bde)(cde))$. It is known that the lattice of subvarieties of the variety $T$ forms a strictly increasing sequence (a chain) of varieties $T_n$, $n = 1, 2, \ldots, \omega$, and $T = T_\omega$. For each $T_n$, $1 < n < \omega$, it is given a finite base of identities. The free algebra $F_M(3)$ on three generators over the variety $M$ belongs to the variety $T$. Since we do not know anything about the free algebra $F_M(4)$ on four generators over $M$, we give results about the algebras in $M$ or in $T$, respectively, which are generated by some partial algebras.

Introduction

Denote by $M$ the variety of algebras $A$ with a single ternary operation $(xyz)$ (notation $A = (A; ()))$ satisfying the identities

1. $(abb) = b,$
2. $((abc)dc) = (ac(dcb)).$

The algebras from $M$ are called modular median algebras (shortly m.m. algebras) as in the papers [6] and [8]. Denote by $D$ the subvariety of $M$ given by the identity

$$(D) \quad (abc) = (bac).$$

AMS Subject Classification (1991): Primary 08B15; Secondary 06C05.
Key words: modular median algebra.

These results were presented at the Conference on General Algebra, Vienna, June 1990. Research supported by VEGA MS SR No. 1/1486/94.
The variety $\mathcal{M}$ was studied by M. Kolibiar and T. Marcisová in [15]. They have shown that the varieties $\mathcal{M}$ and $\mathcal{D}$ are related to the varieties of modular and distributive lattices, respectively: In a modular lattice $L$, the ternary operation

$$ (o) \quad (xyz) = (x \land (y \lor z)) \lor (y \land z) = (x \lor (y \land z)) \land (y \lor z) $$

satisfies the identities (1) and (2). Moreover, if $L$ is distributive, then also (3) is satisfied. Also a partial converse is true (see [15]): Consider an algebra $A \in \mathcal{M}$ which contains two specific elements 0, 1 and satisfies the identity $0xy = x$. Then the algebra $(A; \land, \lor)$, where $x \land y = (x0y)$, $x \lor y = (x1y)$, is a modular lattice in which 0 and 1 are the least and the greatest element, respectively, and the identity (o) holds. This lattice is distributive if $A \in \mathcal{D}$.

The study of ternary algebras related to distributive lattices was initiated by G. Birkhoff and S. A. Kiss [5] and followed by M. Sholander (in [19], [20], [21]) and many other authors (e.g., [1], [15], [13]; a survey can be found in [3]).

The study of ternary algebras related to modular lattices was initiated by J. Hashimoto [10] and followed by other authors (e.g., [15], [11], [12], [13], [6], [8]). More general ternary algebras were investigated by J. R. Isbell [13] and J. Hedlíková [12].

Denote by $\mathcal{T}$ and $\mathcal{U}$ the subvariety of the variety $\mathcal{M}$ satisfying the identity

$$ (T) \quad ((abc)de) = ((ade)(bde)(cde)) $$

and

$$ (U) \quad ((abc)ad) = (ab(cad)) $$

respectively.

E. Fried and A. F. Pixley [9] introduced the notion of a dual discriminator variety. It was shown in [6] that $\mathcal{T}$ is a dual discriminator variety. $\mathcal{T}$ has equationally definable principal congruences, $\mathcal{T}$ has congruence extension property, and any algebra from $\mathcal{T}$ can be embedded in a modular lattice. Independently, the variety $\mathcal{T}$ appeared as a special subvariety of media introduced by J. R. Isbell [13] (he called them isotropic media). The identity (U) appeared in an algebraic description of block graphs (alias Husimi trees) performed by L. Nebeský [18]. Both identities (T) and (U) are used (see [4; Theorem 3]) in a characterization (solely by algebraic identities) of quasi-median algebras, i.e., algebras associated with quasi-median graphs introduced by H. M. Mulder in [17].

It was shown in [8; Theorem 1] that the varieties $\mathcal{T}$ and $\mathcal{U}$ coincide. It holds $\mathcal{D} \subset \mathcal{T}$, $\mathcal{D} \neq \mathcal{T}$ (see, e.g., [8]). Denote by $\mathcal{L}(\mathcal{M})$ the lattice of all subvarieties of the variety $\mathcal{M}$. It was shown in [8; Theorem 2, Theorem 3] that each of the identities (D) and (T) splits the lattice $\mathcal{L}(\mathcal{M})$ into two parts. The free algebra $F_{\mathcal{M}}(3)$ on three generators over the variety $\mathcal{M}$ has six elements and can
be embedded in the free modular lattice on three generators (cf. [13; Corollary to 2.2]). Moreover, $F_{\mathcal{M}}(3)$ belongs to the variety $\mathcal{T}$ (cf. [13; below 5.14]). We do not know anything about the free algebra $F_{\mathcal{M}}(4)$ on four generators from the variety $\mathcal{M}$. We know from [13; 5.14] that the variety $\mathcal{T}$ is locally finite.

In the present paper, some results are given about an algebra $A \in \mathcal{T}$ generated by a partial algebra of order four (Theorem 2 and Theorem 3 below) and $A \in \mathcal{M}$ generated by a partial algebra of order five (Theorem 1), respectively. It is given a finite base of identities for each subvariety of the variety $\mathcal{T}$ (Theorem 4 below).

**Preliminary results**

**Lemma A.** ([15; Lemma]) The following identities and implications hold in each $A \in \mathcal{M}$.

1. $(aba) = a$
2. $(abc) = (acb)$
3. $(aa\bar{a}) = a$
4. $((abc)bc) = (abc)$
5. $((abc)ac) = (ac(abc)) = (abc)$
6. $(ab(cab)) = (abc)$
7. $(ab(cab)) = (abc)$
8. $(abc) = c$ implies $(bac) = c = (cab)$
9. $(bac) = (cab)$ implies $(abc) = (bac)$
10. $(a(dbc)(abc)) = (abc)$

Recall from [6; Remark 1.1] that $\mathcal{M}$ is a congruence distributive variety since (1), (3) and (5) give the majority term.

Let $A \in \mathcal{M}$, $x, y, z \in A$. We say that $y$ is between $x$ and $z$, and write $xyz$, if $(xyz) = y$. By (9) and (4), $xyz$ implies $zyx$.

**Lemma B.** ([6; Lemma 1.2, Lemma 1.3, Lemma 2.1]) The following identities and implications hold in each $A \in \mathcal{M}$.

1. $((abc)(cab)) = (abc)$
2. $((acd)cb) = (ac(dcb)) = (ac(bcd)) = ((acb)cd)$
3. $(ab(cda)) = a(bda)(cda) = (ac(bda))$
4. $a.xb$ and $ayb$ imply $(xay) = (axy) = (yax)$
5. An algebra $A \in \mathcal{T}$ is subdirectly irreducible if and only if for every $x, y, z \in A$ $(xyz) = x$ if $y \neq z$ and $(xyz) = y$ if $y = z$.
6. Let $\theta \in \text{Con} A$, $A \in \mathcal{M}$, $x, y, z, u \in A$. If $(xyz)$, $yzu$ and $x\theta u$, then $y\theta z$. In particular, $xyz$, $yzu$ and $x = z$ imply $y = z$. 

407
Denote by $T_2$ the two element algebra from $\mathcal{M}$. If $A \in \mathcal{M}$, $a$, $b$, $c$ are pairwise different elements of $A$, and $a = (abc)$, $b = (bac)$ and $c = (cab)$ hold, then we say that the elements $a$, $b$, $c$ form a triangle, and we use the notation $T_3$ for it. For each cardinal $n \geq 3$ denote by $T_n$ the algebra of order $n$ in which any three elements form a subalgebra isomorphic to the triangle $T_3$. The algebras $T_n$ are the only subdirectly irreducible algebras in the variety $\mathcal{T}$ (see, e.g., (16)). Let $A \in \mathcal{M}$, $a, b, c, d \in A$. A quadruple $(a, b, c, d)$ is said to be cyclic whenever $abc$, $bcd$, $cda$ and $dab$ hold.

Results

The following Theorem is due to J. Hedlíková (oral communication).

**Theorem 1.** Let $A \in \mathcal{M}$, $x, y, z, u, s \in A$, $y \neq u$, $\{(x,y,z);():\} \cong T_3$ and $(y,z,s,u)$ be a cyclic quadruple. Then the elements $x$, $y$, $z$, $s$, $u$ generate a subalgebra $B$ of $A$, where $B = \{(x,y,z,t = (xsu),s,u);():\}$, which is isomorphic to the direct product $T_3 \times T_2$. Moreover, $B \in \mathcal{T}$.

**Proof.** Note that $y \neq s$ because of (3) in Lemma A. Using (17) of Lemma B, from $y \neq u$, we get $s \neq z$. Similarly, $y \neq z$ implies $u \neq s$. Hence, $y \neq u \neq s \neq z$ hold. We shall prove that the following relations follow from our assumptions:

1. $x = (xys)$ and $x = (xzv)$,
2. $y = (xyu)$ and $z = (zxs)$,
3. $y = (yxz)$ and $z = (zxu)$,
4. $z = (sxy)$ and $y = (uxz)$,
5. $u = (usy)$ and $s = (sux)$.

From the cyclic quadruple $(y, z, s, u)$, we get $yzs$, hence, by (9).

6. $(zys) = z$.

Then $(xys) = ((xyz)ys) = (xy(zys)) = (xys) = x$. Symmetrically, $(xzv) = x$ can be proved and (1.1) holds. $(xyu) = ((xyz)yu) = (xy(yzu)) = (xyy) = y$ (zyu holds since $(y, z, s, u)$ is a cyclic quadruple). Symmetrically, $z = (xzs)$ and (1.2) holds. $(yxz) = ((yxz)xs) = (yx(zxs)) = (yxz) = y$. Symmetrically, $(zxu) = z$ and (1.3) holds. $(sxy) = (sx(yxz)) = (sxz)xy = (zxy) = z$.

Symmetrically, $(uxz) = y$ and (1.4) holds. $(usy) = ((usu)sy) = (ysu) = u$. Symmetrically, $(sux) = s$ and (1.5) holds.

Take $t = (xsu)$. According to (1.5), $(usu) = u \neq s = (sux)$, we get $u \neq t \neq s$ by (10) of Lemma A. In view of (12),

7. $\{(t,u,s);():\} \cong T_3$. 

408
Since \((u, s, z, y)\) is a cyclic quadruple, too, and \(u \neq y \neq z \neq s\) hold, we get that the analogous relations to (1.1)–(1.5) hold:

\[
\begin{align*}
(1.8) \quad & t = (tuz) \text{ and } t = (tsy), \\
(1.9) \quad & u = (tuy) \text{ and } s = (tsz), \\
(1.10) \quad & u = (utz) \text{ and } s = (sty), \\
(1.11) \quad & s = (zut) \text{ and } u = (yts), \\
(1.12) \quad & y = (yzt) \text{ and } z = (zyt).
\end{align*}
\]

Now we shall show that

\[
(1.13) \quad (y, x, t, u) \text{ is a cyclic quadruple.}
\]

According to (4) and (9), we get

\[
(1.14) \quad xtu, \text{ hence, } utx.
\]

With respect to (1.2), (4), and (9), we get

\[
(1.15) \quad xyu, \text{ hence, } uyx.
\]

In view of (15), (1.14), and (1.15), we get

\[
(1.16) \quad (xyt) = (txy) = (ytx).
\]

Then \((xyt) = (xyt) = ((xyz)yt) = (xy(zyt)) = (xyz) = x\). It implies

\[
(1.17) \quad ytx.
\]

Now (1.13) follows from (1.14), (1.17), (1.15) and (1.9). Analogously, it can be proved that

\[
(1.18) \quad (z, x, t, s) \text{ is a cyclic quadruple, in particular, } txz,
\]

hence,

\[
(1.19) \quad (tzx) = x.
\]

\[
(1.20) \quad (tyz) = x:\n\]

\[
(1.21) \quad t \neq y:
\]

In view of (1.13), \(tuy\) and \(uyx\). If \(t = x\), then according to (17), \(y = u\), a contradiction.

\[
(1.22) \quad t \neq y:
\]

Let \(t = y\). Then \(t = (xts) = (xys) = x\), hence, \(y = x\), a contradiction. Analogously, it can be proved

\[
(1.23) \quad t \neq z.
\]

We have proved that all elements from \(B\) are pairwise different. Denote \(\alpha = \theta(x, y)\), \(\beta = \theta(x, t)\). According to (1.13), (1.18), (1.7), and (17), we get \(B/\alpha \cong T_2\) and \(B/\beta \cong T_2\). It is easy to see that \(B \cong B/\alpha \times B/\beta\). Hence, \(B \cong T_2 \times T_2\).

Finally, \(B \in T\) by (16). □
THEOREM 2. Let $A \in T$, $a, b, c, d \in A$, $(\{a, b, c\}; ()) \cong T_3$, $c \neq d \neq a$, and $cda$ hold. Then the subalgebra $B$ of $A$ generated by the elements $a, b, c, d$ is isomorphic to the direct product $T_3 \times T_3$.

**Proof.** Let $B \subseteq \Pi(A_i : i \in I)$ be a subdirect decomposition of subdirectly irreducible algebras $A_i$, $A_i \in T$, $i \in I$. Without loss of generality, we can suppose that for each $i \in I$ the algebra $A_i$ has more than one element, and that all projections $p_i$ from $B$ onto $A_i$ have pairwise different kernels $\ker p_i$.

For arbitrary element $x \in B$ denote by $x_i$ the $i$th component of the element $x$, hence, $x = (x_i : i \in I)$. The elements $a, b, c$ form a triangle, hence, for each $i \in I$ either $a_i = b_i = c_i$ or $a_i \neq b_i \neq c_i \neq a_i$ holds. The element $d_i$ has to be between the elements $a_i$ and $c_i$ in $A_i \cong T_n$, which is possible only if $d_i \in \{a_i, c_i\}$ by (16) of Lemma B. In the case $a_i = b_i = c_i$, the algebra $A_i = p_i(B)$ has only one element. Hence, for each $i \in I$, $a_i \neq b_i \neq c_i \neq a_i$ holds and $A_i \cong \{(a_i, b_i, c_i); ()) \cong T_3$. According to $a \neq d \neq c$, the elements $i, j \in I$ must exist such that $d_i = a_i$ and $d_j = c_j$. We shall show that $I = \{i, j\}$.

Let $k \in I$. Without loss of generality, suppose $d_k = a_k$. Then the mapping $f : A_k \rightarrow A_i$ given by $f(a_k) = a_i$, $f(b_k) = b_i$, $f(c_k) = c_i$ ($f(d_k) = d_i$ holds, too) is an isomorphism, and $p_i = f \circ p_k$ holds (since these homomorphisms coincide on the set $\{a, b, c, d\}$ of generators of the algebra $B$). It implies $\ker p_i = \ker p_k$, hence, $i = k$ (for we have supposed that different projections have different kernels). It was shown that $B \subseteq A_i \times A_j \cong T_3 \times T_3$. It is easy to verify that the elements $a = (a_i, a_j)$, $b = (b_i, b_j)$, $c = (c_i, c_j)$, $d = (a_j, c_i)$ generate the whole algebra $A_i \times A_j$. Really, for the elements $e = (bad)$, $f = (cbe)$, $g = (ae)$, $h = (bag)$, $l = (cbh)$ the following equalities hold: $e = (a_i, b_j)$, $f = (c_i, b_j)$, $g = (c_i, a_j)$, $h = (b_i, a_j)$, $l = (b_i, c_j)$.

**Theorem 3.** Let $A \in T$, $a, b, c, c' \in A$, $c \neq c'$, and $(\{a, b, c\}; ()) \cong T_3 \cong (\{a, b, c\}; ())$. Then the subalgebra $B$ of $A$ generated by the elements $a, b, c, c'$ is isomorphic either to $T_4$ or to the direct product $T_4 \times T_3$.

**Proof.** Similarly as in the proof of Theorem 2, let $B \subseteq \Pi(A_i : i \in I)$ be a subdirect decomposition of subdirectly irreducible algebras $A_i$, $A_i \in T$, $A_i \geq 1$, $i \in I$, and all projections $p_i$ of $B$ onto $A_i$ have pairwise different kernels $\ker p_i$. For each $i \in I$ either $a_i = b_i = c_i$ or $a_i \neq b_i \neq c_i \neq a_i$ holds in the case $a_i = b_i = c_i$, we get $a_i = c'$ and $A_i = 1$. Hence, $a_i \neq b_i \neq c_i \neq a_i$, and analogously, $b_i \neq c' \neq a_i$. According to $c \neq c'$, there exists $i \in I$ such that the elements $a_i, b_i, c_i, c_i'$ are pairwise different, hence, $A_i \cong T_4$. Now we have two possibilities:

a) There does not exist $j \in I$ with the property $c_j = c_j'$. Then for each $k \in I$ the elements $a_k$, $b_k$, $c_k$, $c_k'$ are pairwise different. Similarly as in the proof of Theorem 2, the mapping $f : A_k \rightarrow A_i$ given by $f(a_k) = a_j$, $f(b_k) = b_j$.
MODULAR MEDIAN ALGEBRAS

$f(c_k) = c_i$, $f(c'_k) = c'_i$ is an isomorphism such that $p_i = f \circ p_k$ holds. Then Ker $p_i = \text{Ker} \ p_k$, and $k = i$, $I = \{i\}$, $B = A_i$.

b) There exists $j \in I$ such that $c_j = c'_j$. Then $A_j = \{a_j, b_j, c_j\} \cong T_3$. We shall show that $I = \{i, j\}$. If $k \in I$, then we have either $c_k \neq c'_k$ and then we get Ker $p_k = \text{Ker} \ p_i$ and $k = i$, or $c_k = c'_k$ and then we get Ker $p_k = \text{Ker} \ p_j$ and $k = j$. It implies that $B \subseteq A_i \times A_j \cong T_3 \times T_3$. It is easy to verify that the elements $a = (a_i, a_j)$, $b = (b_i, b_j)$, $c = (c_i, c_j)$, $c' = (c'_i, c'_j)$ generate the whole algebra $A_i \times A_j$. Recall that $T$ is locally finite variety by [13; 5.14]. If $k$ and $m$ are infinite cardinals, then the algebras $T_k$ and $T_m$ generate the same variety $T'$, since they all have the same finitely generated subalgebras. For $n$ finite let $T_n$ be the subvariety of $T$ generated by the subdirectly irreducible algebra $T_n$ (or equivalently, by all subdirectly irreducible algebras $A \in T$ with card $A \leq n$). The varieties $T_n$, $n = 1, 2, \ldots, \omega$, form a strictly increasing sequence (a chain) and $T = T_\omega$ (cf. [13; 5.16]).$

In the paper [9], it was found a finite equational base for a finite algebra in a dual discriminator variety using results of [2] and [16]. Recall from [6] that $M$ (hence, $T$, too) is a congruence distributive variety. The next Theorem will give a different finite base of such identities.

**Theorem 4.** The subvariety $T_n$ of the variety $T$, $1 < n < \omega$, has the following finite base of identities: (1), (2), (T), and

\[(T_n) \quad d_n = d_n^*,\]

where

\[d_2 = (x_0x_1x_2), \quad d_2^* = (x_1x_0x_2),\]

and for $i > 2$ define inductively

\[d_3 = (((d_2x_3x_0)x_3x_1)x_3x_2), \quad d_3^* = (((d_2^*x_3x_0)x_3x_1)x_3x_2),\]

\[\vdots\]

\[d_n = (\ldots (((d_{n-1}x_nx_0)x_nx_1)x_nx_2)\ldots x_nx_{n-1}),\]

\[d_n^* = (\ldots (((d_{n-1}^*x_nx_0)x_nx_1)x_nx_2)\ldots x_nx_{n-1}).\]

**Proof.** According to (16) of Lemma B, it is easy to see that in $T_n$, the identity $(T_n)$ is satisfied whenever at least two of the elements $x_0, x_1, \ldots, x_n$ are equal, but fails whenever all $n + 1$ elements are pairwise different. Hence, it holds in $T$, but fails in $T_{n+1}$. \(\square\)

411
REFERENCES


Received December 27, 1991
Revised May 15, 1996

HILDA DRAŠKOVICOVÁ

Department of Algebra and Number Theory
Faculty of Mathematics and Physics
Comenius University
Mlynská dolina
SK 842 15 Bratislava
SLOVAKIA