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Dedicated to the memory of Professor Milan Kolibiar

WHICH COUNTABLE ORDERED SETS HAVE A DENSE LINEAR EXTENSION?

ALEKSANDER RUTKOWSKI

(*Communicated by Tibor Katriňák*)

ABSTRACT. We try to answer the question: when a partial order can be extended to an order isomorphic to the ordering of rationals? One necessary and a few sufficient conditions for the existence of such an extension are presented.

1. Introduction

This article is located on the crossing of paths leading from the two fundamental results on ordered sets, namely: Marczewski Theorem and Cantor Theorem.

MARCZEWSKI THEOREM. ([13]) *Every partial order can be extended to a linear order (with the same underlying set). Moreover, it is the intersection of such extensions.*

CANTOR THEOREM. ([5]) *Every countable linearly and densely ordered set containing neither the least nor the greatest element is order-isomorphic to the set* $\mathbb{Q}$ *of all rationals (with natural order).*

Marczewski Theorem generates many natural questions on relations between ordered sets and their linear extensions. Which properties are preserved? How many linear extensions of an order determines that order? How many comparabilities should we add to a given order to make it linear? How to obtain a linear order of a given type?

Investigations on the second and third of the above questions have been focused mainly on finite ordered sets and brought the expansion of the dimension theory, the examinations of the jump number and of correlation problems.

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Typical properties to preserve are: a noncontainment of any chain of a given type (e.g., ω, ω^*, η, etc), a local chain-completeness (in the sense that all maximal chains in closed segments are complete).

We will examine the following problem, a special case of the last question:

([6]) *Which order can be extended to a linear order isomorphic to the order on $\mathbb{Q}$, the set of rationals?*

The analogous questions with $\mathbb{Z}$, the set of integers, $\mathbb{N}$, the set of natural numbers, and $\mathbb{R}$, the set of reals, have been considered in [12] and [11]. A countable ordered set has an extension which is order-isomorphic to $\mathbb{Z}$ (resp. $\mathbb{N}$) if and only if each its segment (resp. principal ideal) is finite ([12]). With $\mathbb{R}$ (more generally: with locally chain-complete linearly ordered sets) the situation is not so clear. Each locally chain complete set with no infinite antichains has a linear locally chain complete extension. Moreover, for countable sets the antichain assumption can be removed ([11]). Linear extensions of countable ordered sets are examined in the sequence of papers [8], [9] and [10], in particular: the existence of the least (with respect to the embedding) such an extension, its uniqueness (up to the isomorphism). The problem of density (or conversely: of the dispersity) of ordered sets and their linear extensions is the topic of [2], [3], and [1]. A survey of results and methods obtained up to early eighties can be found in [4].

On the other side, Cantor Theorem had a substantial influence on the development of model theory as an inspiration of such notions as *categorical theory* and *saturated model*.

2. Preliminaries

An *order* (what means here the same as *partial order*) on a set P is denoted by $\leq_P$ and its strict version by $<_P$, although the subscript is usually omitted unless it will lead to a misunderstanding. For $X \subseteq P$, sets $\{p \in P : (\exists x \in P)\ p \leq x\}$, $\{p \in P : (\exists x \in P)\ p \geq x\}$ will be denoted by $X^\wedge$, $X^\vee$, respectively. If $p \in P$, then $p^\wedge$, $p^\vee$ mean $\{p\}^\wedge$, $\{p\}^\vee$, respectively. For $p, q \in P$, q covers p in P ($p \ll_P q$) if $p < q$ and there is no $x \in P$ with $p < x < q$. P is *densely ordered* if $(\forall x, y)\left[x < y \implies (\exists z)\ x < z < y\right]$ (or equivalently: there are no covering pairs in P). *Countability* means here that a given set is of the same cardinality as the set $\mathbb{N}$ of all natural numbers. As usual, η denotes the order type of $\mathbb{Q}$, the set of all rationals. Cantor Theorem establishes that there are only four countable dense ordered types: η, $1+\eta$, $\eta+1$ and $1+\eta+1$.

For an order-type α, an ordered set P has an *α-extension* if its order $\leq_P$ has a linear extension of type α.

Let us recall that, for a given ordered set T and a family $\{P_t : t \in T\}$ of ordered sets, the lexicographic sum $\mathcal{L}_{t\in T}P_t$ is a union $\bigcup_{t\in T} P_t$ ordered by a relation (with no loss of generality, sets P_t can be assumed to be pairwise disjoint):

$$x \le y \iff \quad (\exists t \in T)\,(x, y \in P_t \;\&\; x \le_{P_t} y)$$
$$\text{or} \quad (\exists t, s \in T)\,(x \in P_t \;\&\; y \in P_s \;\&\; t <_T s)\,.$$

For T being a two element chain $0 < 1$ and $P_0 = P$, $P_1 = Q$, the lexicographic sum $\mathcal{L}_{t\in T}P_t$ will be denoted by $P \oplus Q$.

Consider a Cartesian product $P \times Q$ of ordered sets. We recall its two orderings. The first is *componentwise*:

$$(x_1, y_1) \le (x_2, y_2) \iff x_1 \le_P x_2 \;\&\; y_2 \le_Q y_2\,.$$

The second one is called *lexicographical* and it is defined by a formula:

$$(x_1, y_1) \preceq (x_2, y_2) \iff x_1 <_P x_2 \;\text{ or }\; (x_1 = x_2 \;\&\; y_2 \le_Q y_2)\,.$$

Its name is justified as $P\times Q$ ordered by $\preceq$ can be identified with $\mathcal{L}_{x\in P}\big(\{x\}\times Q\big)$. It is easy to see that $\preceq$ extends $\le$ on $P \times Q$, and it is a linear order whenever both $\le_P$ and $\le_Q$ are linear.

There are two trivial examples of ordered sets having an η-extension: $\mathbb{Q}$ itself and a countable antichain. Using lexicographic sums and direct products we can produce from these two examples a great variety of sets with an η-extension.

THEOREM 1.

1. *For every nonempty finite or countable ordered set T and a family $\{P_t : t \in T\}$ of ordered sets with η-extensions, the lexicographical sum $\mathcal{L}_{t\in T}P_t$ has an η-extension.*

2. *The direct product $P \times Q$ (with the componentwise order) of an arbitrary ordered set Q with an η-extension and an arbitrary finite or countable ordered set P has an η-extension.*

Proof. Let $T' = T$, $P'_t = P_t$, let $\le_{T'}$ be an arbitrary linear extension of $\le_T$, and $\le_{P'_t}$ be an η-extension of $\le_{P_t}$. Consider $\mathcal{L}_{t\in T'}P'_t$. Obviously, it is a linear extension of $\mathcal{L}_{t\in T}P_t$ satisfying the assumptions of Cantor Theorem, hence it is order-isomorphic to $\mathbb{Q}$. Point 2 is a consequence of point 1 and of the fact that the lexicographic order of $P\times Q$ is an extension of its componentwise order. □

A chain C in an ordered set P is *saturated in* P if, for every $p \in P - C$, if $C \cup \{p\}$ is a chain, then p is either a lower or an upper bound of C. We call it nontrivial if C contains at least two elements. A subset X of P is *convex* if, for each $x, y \in X$ and $z \in P$, $x < z < y$ implies $z \in X$.

Obviously, a saturated chain in a subset of P can be not saturated in P.

LEMMA 1.

1. *Every chain can be extended to a saturated chain.*
2. *If X is convex in an ordered set P, and C is a saturated chain in X, then C is saturated in P as well.*

P r o o f . To prove point 1, let X be a convex hull of C, i.e., $X = \{p \in P : (\exists x, y \in C)\ x \le p \le y\}$. Obviously, $C \subseteq X$ and C_1, its extension to a maximal chain in X, is saturated in P. Point 2 is evident. □

Observe that a dense chain in an ordered set P need not have a saturated dense extension. An example is $A \cup B$ in $P_0 = A \oplus \{a, b\} \oplus B$, where A (respectively B) is the set of all negative (positive) rationals with the natural order and $\{a, b\}$ is an antichain.

3. The necessary condition

First, let us recall the simplest version of D u s h n i k – M i l l e r theorem ([6]). We prove it for the completeness of our reasoning.

LEMMA 2. ([6]) *Let P be an ordered set with no infinite chains and no infinite antichains. Then P is finite.*

P r o o f . Obviously, for each element $p \in P$ there exists a minimal element q with $q \le p$. Let $P_0 = P = \bigcup_{x \in X_0} x^{\vee}$, where X_0 is the set of all minimal elements of P. Assume P to be infinite. X_0, being an antichain, is finite, hence $x_0^{\vee}$ is infinite for some $x_0 \in X_0$. Let $P_1 = x_0^{\vee} - \{x_0\}$, X_1 be a set of all minimal elements in P_1, and x_1 be such an element of X_1 that $x_1^{\vee}$ is infinite. Let $P_2 = x_1^{\vee} - \{x_1\}$, etc. We obtain a strictly increasing sequence $x_0 < x_1 < \dots$ of elements of P, in contradiction with the finiteness of chains in P. □

THEOREM 2. *Let P be an ordered set with an η-extension. Then P contains an infinite antichain or a nontrivial dense saturated chain.*

P r o o f . Let $\preceq$ be an η-extension of $\le_P$ (we will postpone the subscript P). Assume that no nontrivial saturated chain in P is dense (with respect to $\le$). We define the decreasing sequences of convex subsets of P:

$$P = P_0 \supseteq P_1 \supseteq P_2 \supseteq \dots,$$

and a sequence

$$C_0, C_1, C_2, \dots,$$

of nontrivial saturated chains such that, for all i, C_i is saturated in P_i (hence in all P_j for $j < i$) and disjoint with P_{i+1}.

Moreover, we define two such sequences a_i, b_i of elements of P that

$$a_0 \prec a_1 \prec a_2 \prec \cdots \prec \cdots \prec b_2 \prec b_1 \prec b_0,$$

and a_i is covered in P_i by b_i.

If P is an antichain, then the proof is unnecessary. Otherwise, there exists a nontrivial saturated chain C_0 in $P_0 = P$. As it is a nondense chain, there exist $a_0, b_0 \in C_0$ such that $a_0 \ll_{C_0} b_0$. By the saturation of C_0, this covering relation holds in P_0 as well. Obviously, $a_0 \prec b_0$.

Let $n > 0$, and assume, for $i < n$, that P_i, C_i, a_i, b_i have been defined and they satisfy the above inequalities and inclusions. Define

$$P_n = \{x \in P : \ a_{n-1} \prec x \prec b_{n-1}\}.$$

Obviously, P_n is an infinite convex subset of P_{n-1} with no dense nontrivial saturated chains and $P_n \cap C_{n-1} = \emptyset$. Again, we can assume P_n to be not an antichain. Let C_n be an arbitrary nontrivial saturated chain in P_n, and a_n, b_n be a covering pair (in P_n) of its elements given by the saturation and nondensity of C_n.

Let $X = A \cup B$, where $A = \{a_i : \ i = 0,1,2,\dots\}$ and $B = \{b_i : \ i = 0,1,2,\dots\}$. Now, by Lemma 2, we can assume X to contain an infinite chain C, otherwise the proof is finished. At least one of sets $C \cap A$, $C \cap B$, say the first of them, is infinite. Consider $D = \{b_i : \ a_i \in C \cap A\}$ and a pair b_{i_0}, b_{i_1} of its elements, where $i_0 < i_1$. Observe that a_{i_0} is comparable with a_{i_1} as they are elements of the chain C. Hence $a_{i_0} < a_{i_1}$ because of $a_{i_0} \prec a_{i_1}$. Therefore b_{i_1} is not less than b_{i_0} (otherwise a_{i_0}, a_{i_1} would not be a covering pair). Moreover, b_{i_1} is not over b_{i_0}, because of $b_{i_1} \prec b_{i_0}$. Hence b_{i_1} is noncomparable to b_{i_0}, and D is an infinite antichain. □

Although both the conditions of Theorem 2 can be satisfied, for example: for the rational plane, it seems reasonable to consider them separately in order to look for a sufficient condition. We will make it in the next two sections.

Now, observe that none of the above conditions is sufficient. Consider ordered set $P_0 = A \oplus \{a,b\} \oplus B$ from the previous section. Obviously, in any linear extension of the ordering of P, $\{a,b\}$ is a covering pair, therefore none of those extensions is a dense order. A similar situation occurs when we consider $P_1 = A \oplus \{a,b\} \oplus B$, where both A and B are countable antichains and $\{a,b\}$ is an antichain.

These examples show that we need to strengthen our necessary conditions to make them sufficient.

4. What happens when P contains a chain of type η?

Actually, we assume P to embed $\mathbb{Q}$, the set of all rationals, i.e, P contains a chain C of the type η. The question is what should we do with elements of $P-C$. Loosely speaking, we can treat them as candidates for irrational numbers and to insert them into gaps determined by Dedekind cuts of C. Example P_0 in Section 2 shows that the problem arises when there is more than one candidate to be inserted into the same gap.

For a given subset X of an ordered set P and $p, q \in P$ define $p \sim_X q$ ($\{p, q\}$ is X-autonomous) if and only if for each $x \in X$,

$$x < p \iff x < q \qquad \text{and} \qquad p < x \iff q < x\,.$$

By $X(p)$, we denote the set $\{q \in P : \ p \sim_X q\}$. Obviously, always $p \in X(p)$. For example, in P_0, $(A \cup B)(a) = \{a, b\}$.

Lemma 3. *Let P be a countable ordered set. If P contains a maximal chain C of the type η such that, for each $p \in C$, $C(p) = \{p\}$, then P has an η-extension.*

P r o o f. Let $P - C = \{p_0, p_1, \dots\}$. Define an increasing sequence of extensions of $\leq_P$ by "inserting" elements p_i in turn "within" chain C. Let $\leq_0 = \leq_P$, $C_0 = C$. Assume that $\leq_n$ and C_n have been defined, and C_n is a maximal chain in P (with respect to $\leq_n$), its type is η and $C_n(p) = \{p\}$ for each $p \in P_n$. Let $A = \{x \in C_n : \ x <_n p_n\}$, $B = \{y \in C_n : \ p_n <_n y\}$ and $D = C_n - (A \cup B)$. Observe that $A \neq C_n$, $B \neq C_n$, $D \neq \emptyset$, otherwise, C_n is not a maximal chain. Moreover, D is not a singleton, otherwise if, say, $D = \{p\}$, then $\{p_n, p\}$ would be C_n-autonomous. Therefore D is a segment in C_n. Consider an arbitrary irrational number in D and substitute it by p_n. To be more precise, let (A', B') be a Dedekind cut of C_n such that $A \subseteq A'$, $B \subseteq B'$ and $A' \cap B' = \emptyset$ (i.e., this cut corresponds to an irrational in D). Define now $C_{n+1} = A' \oplus \{p_n\} \oplus B'$, and $\leq_{n+1}$ to be a transitive closure of the sum of the linear order on C_{n+1} and $\leq_n$ on $P - C_{n+1}$ (i.e., $x <_{n+1} y$ for $x \in A'$, $p_n \leq_n y$ and dually). Hence C_{n+1} is maximal in P. By Cantor Theorem, its type is η. It remains to prove that, for each $p \in C_{n+1}$, $C_{n+1}(p) = \{p\}$. Assume that $p \sim_{C_{n+1}} q$ for some $q \neq p$. If $p = p_n$, then q would be comparable with all elements of C_n, despite the maximality of C_n. If $p \neq p_n$, then $p \in C_n$. Hence $p \sim_{C_n} q$ despite the inductive assumption. Obviously, $P = \bigcup_i C_i$ and it is linearly ordered in type η by a relation $\leq = \bigcup_i \leq_i$. □

The sufficient condition in the above Theorem is too strong. Assume a modified example P_0 with an infinite antichain instead of $\{a, b\}$. Extending the disorder of this antichain to an η-order, we obtain a linearly ordered set of the type $\eta + \eta + \eta = \eta$. The following Theorem includes this case.

THEOREM 3. *Let P be a countable ordered set. If P contains a maximal chain C of the type η such that for each $p \in C$, $C(p)$ is either $\{p\}$ or it has an η-extension, then P has an η-extension.*

P r o o f . Let $D = \{p \in C : C(p)$ has an η-extension$\}$, and, for $p \in D$, let $\preceq_p$ be a fixed linear extension of $\leq_{C(p)}$. Substitute in C each $p \in D$ by $C(p)$ (ordered by $\preceq_p$). The obtained chain $C' = (C - D) \cup \bigcup_{p \in D} C(p)$ is maximal in P, and, by Cantor Theorem, is still of the type η. Now, let $\leq'$ be an order of P determined by this new situation, i.e., it is the transitive closure of the sum of that linear order on C' and $\leq_{P-C'}$. It is easy to see that $\leq'$ is an extension of $\leq_P$, and for each $p \in C'$, $C'(p)$ contains p only. Then we use the last Lemma. $\square$

5. What happens if P contains an infinite antichain?

For an ordered set P containing neither the least nor the greatest element, let $\hat{P}$ means $P \cup \{p_0, p_1\}$ with comparabilities $p_0 < x$, $x < p_1$ added, for $x \in P$, to the order of P.

Denote by $(\star)$ the following property of an ordered set P:

$(\star)$ *Let X and Y be finite antichains in P such that*

$$(\forall x \in X)(\forall y \in Y)\ y \nleq_P x\,.$$

Then there exists $p \in P$ such that $p \notin X^\wedge$ and $p \notin Y^\vee$.

Note that if P is linearly ordered, then $(\star)$ means just its density. Any ordered set with the property $(\star)$ has neither the greatest nor the least element. Indeed, apply $(\star)$ to $X = \{x\}$, $Y = \emptyset$, where x is the greatest element of P, and to $X = \emptyset$, $Y = \{y\}$, where y is the least element of P.

The proof of the following theorem applies a "back-and-forth" technique discovered by Georg Cantor for the proof of his Theorem.

THEOREM 4. *If a countable ordered set P satisfies $(\star)$, then it has an η-extension. Moreover, $\leq_P$ is the intersection of all its η-extensions.*

P r o o f . Consider sets $\hat{P}$ and $\hat{\mathbb{Q}}$. With no loss of generality, $\hat{\mathbb{Q}}$ can be assumed as the set of all rationals from the closed segment $[0, 1]$ of the real line. As the matter of fact, we will construct a $(1 + \eta + 1)$-extension of $\hat{P}$ which, obviously, determines the η-extension of P.

Let us enumerate elements of $\hat{P}$ and $\hat{\mathbb{Q}}$:

$$\hat{P} = \{p_0, p_1, p_2, \dots\},$$

where p_0 (p_1) is the least (resp. the greatest) element of $\hat{P}$,

$$\hat{\mathbb{Q}} = \{q_0, q_1, q_2, \dots\},$$

where $q_0 = 0$ and $q_1 = 1$.

We construct a strictly increasing, one-to-one mapping f of $\hat{P}$ onto $\hat{\mathbb{Q}}$ by setting $f(p_{i_k}) = q_{j_k}$, where $p_{i_0}, p_{i_1}, p_{i_2}, \dots$ and $q_{j_0}, q_{j_1}, q_{j_2}, \dots$ are special permutations of $\hat{P}$ and $\hat{\mathbb{Q}}$, respectively. We define p_{i_k} and q_{j_k} by induction, separately on even and odd steps. Let $P_m = \{p_{i_k} : k < m\}$ and $Q_m = \{q_{j_k} : k < m\}$.

Step 0. $p_{i_0} = p_0$,

Step 1. $q_{i_0} = q_0$.

Let $m > 1$. f has been defined on P_m and it is strictly increasing.

Step $m = 2n$. Let $i_{2n} = \min\{i : p_i \notin P_{2n}\}$, $X_{2n} = \{x \in P_{2n} : x < p_{i_{2n}}\}$ and $Y_{2n} = \{x \in P_{2n} : x > p_{i_{2n}}\}$. Obviously, $p_{i_0} \in X_{2n}$ and $p_{i_1} \in Y_{2n}$. Define

$$j_{2n} = \min\Big(\{j : f(x) < q_j < f(y) \text{ for each } x \in X_{2n},\ y \in Y_{2n}\} - Q_{2n}\Big).$$

Step $m = 2n + 1$. Let

$$j_{2n+1} = \min\{j : q_j \notin Q_{2n+1}\}.$$

Define X_{2n+1} (Y_{2n+1}) as the set of all maximal (minimal) elements of the set $\{x \in P_{2n+1} : f(x) < q_{j_{2n+1}}\}$ (respectively, $\{y \in P_{2n+1} : f(y) > q_{j_{2n+1}}\}$). It is easy to see that $y \nleq x$ for each $x \in X_{2n+1}$, $y \in Y_{2n+1}$. Therefore they satisfy the premise of $(\star)$. Thus the set $\{i : p_i \in P - (X^{\wedge}_{2n+1} \cup Y^{\vee}_{2n+1})\}$ is nonempty. Let i_{2n+1} be its first element.

It is easy to see that the mapping $f(p_{i_k}) = q_{j_k}$ is defined on the whole set $\hat{P}$, and all elements of $\hat{\mathbb{Q}}$ are reached as its values. It is also true that, for each k, $p_{i_k} \notin P_k$ and $q_{j_k} \notin Q_k$, hence f is a one-to-one function. It remains to show that f is strictly increasing. Let $p_{i_k} < p_{i_l}$ and $k < l$ (i.e., $p_{i_k} \in P_l$; if $l < k$, the proof is similar). Assume l to be even. Then $p_{i_k} \in X_l$, hence $q_{i_k} < q_{i_l}$. Now, assume l to be odd and, despite our hypothesis, $q_{i_k} > q_{i_l}$. Then, by the definition of Y_l, $p_{i_k} \geq y$ for some Y_l. Therefore $p_{i_l} \in Y_l^{\vee}$, in contradiction to the definition of this element.

Now, we prove that $\leq_P$ is the intersection of its η-extensions. Obviously, it is enough to show that for each noncomparable pair a, b of elements of P there exists an η-extension $\preceq$ such that $a \preceq b$. Denote by $\leq$ an order which is the transitive closure of the relation $\leq_P \cup \{(a, b)\}$. It is easy to see that

$$x \leq y \iff x \leq_P y \vee (x \leq_P a \ \&\ b \leq_P y).$$

In particular, $a \leq b$. We prove that $(P, \leq)$ has the property $(\star)$, what by the just proved part of our theorem gives us a desired $\preceq$.

Let X and Y be $\leq$-antichains (then they are $\leq_P$-antichains, as well) and

$$(\forall x \in X)(\forall y \in Y)\ y \not\leq x\,.$$

It is easy to see that the same holds for $\leq_P$. Applying $(\star)$ to $\leq_P$ we obtain such an element p of P that, for each $x \in X$, $y \in Y$, $p \not\leq_P x$ and $y \not\leq_P p$. If p satisfies the conclusion of $(\star)$ for $\leq$, then we are done. So suppose $p \leq x$ for some $x \in X$ (the dual case, $p \geq y$ for some $y \in Y$, is the same). We will find another element u coming true this conclusion. It holds $p \leq_P a$ and $b \leq_P x$. Obviously, $a \not\leq_P z$ for each $z \in X$. Now, let $x_1 \in X$. We prove that $x_1 \not\leq_P a$. So assume $x_1 \leq_P a$. $x_1 \neq x$, otherwise $b \leq_P a$. Hence $x_1 \leq x$ (because of $x_1 \leq_P a$, $b \leq_P x$), despite that X is an $\leq$-antichain. Therefore, $X \cup \{a\}$ is an $\leq_P$-antichain. Observe, that for any $y \in Y$, $y \not\leq_P a$, otherwise $y \leq x$. Hence, $X \cup \{a\}$ and Y satisfy the premise of $(\star)$ (for $\leq_P$). Therefore, by its conclusion, there is u such that

$$\big(\forall x \in X \cup \{a\}\big)(\forall y \in Y)\ \ u \not\leq_P x\ \ \&\ \ y \not\leq_P u\,.$$

Easily, $u \not\leq x$ for each $x \in X$. Moreover, for any $y \in Y$, $y \not\leq u$, otherwise $y \leq_P a \leq_P x$ $\square$

COROLLARY 1. *Let P be a countable ordered set with the property:*

> *there is an infinite antichain $A \subseteq P$ such that, for each $p \in P$, A_p, the set of all elements of A which are comparable with p, is finite.*

Then P has an η-extension.

Proof. Let A_p denotes the set mentioned above, and X, Y be the finite sets from the assumption of Theorem 2. Obviously, set $C = \bigcap_{p \in X \cup Y} (A - A_p)$ is nonempty, and $C \subseteq P - (X^{\wedge} \cup Y^{\vee})$. $\square$

Consider the set $\mathbb{Z} \times \mathbb{Z}$ with a componentwise order. It is easy to see that each pair (x, y) of that set is comparable with only finitely many elements of the antichain $\{(n, -n) :\ n \in \mathbb{Z}\}$. Hence $\mathbb{Z} \times \mathbb{Z}$ has an η-extension.

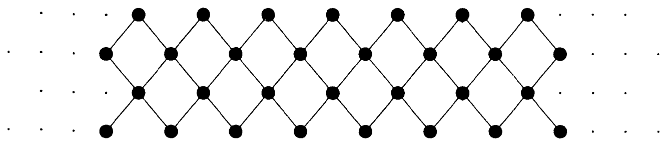
A nice example of the application of that Corollary is the family of finite subsets of a countable set with relation $\subseteq$ as the order. The desired antichain is the family of all singletons. The similar argumentation shows that the family of cofinite subsets (i.e. subsets with a finite complement) of the same set has an η-extension as well. Hence, it is easy to see that the family of all subsets which are finite or cofinite has the $(\eta{+}\eta)$-extension which itself is of type η.

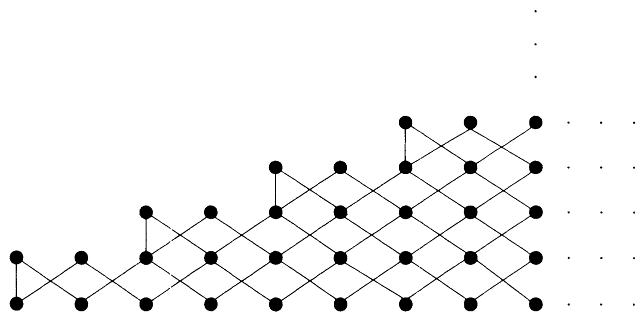
Another example is the family of all words over a countable alphabet ordered by a relation "to be a subword". Then the desired antichain is the alphabet itself.

COROLLARY 2. *Let P be a countable ordered set with no infinite chains. If each element of P is comparable with only finitely many elements of P, then P has an η-extension.*

Proof. By Lemma 1, P contains an infinite antichain A. □

The last result can be applied, among others, to every infinite subset of an infinite fence or of such ordered sets like those with diagrams shown in Figures.





COROLLARY 3. *Let P, Q be countable ordered sets. If P contains a chain with no upper bound, and Q contains a chain with no lower bound, then $P \times Q$ has an η-extension.*

Proof. Let C and D be those chains in P, Q, respectively. Obviously, they are infinite. Let $C = \{c_0, c_1, \dots\}$ and $D = \{d_0, d_1, \dots\}$ (orders of their labelling need not be in any relation to $\leq_P$ and $\leq_Q$). Let $x_0 = c_0$ and, for $k > 0$, $x_k = c_{j_k}$, where $j_k = \min\left\{i : \ c_i > \max_{\leq_P}\{c_0, c_1, \dots, c_{k-1}, x_{k-1}\}\right\}$. It is easy to see that elements x_i form a strictly increasing sequence which is cofinite with a chain C (i.e., $(\forall i)(\exists k)\ c_i < x_k$). Hence it has no upper bound, as well. Dually, define $y_k \in D$, a strictly decreasing sequence which is coinitial with D. Let $A = \{(x_0, y_0), (x_1, y_1), \dots\}$. A is an antichain in $P \times Q$. Observe that each element $(p, q) \in P \times Q$ is comparable to at most finitely many elements of A. Indeed, assume the converse situation. With no loss of generality, we can assume that (p, q) is below infinitely many elements of A. Then q would be a lower bound of the whole sequence y_i and hence – of a chain D. □

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