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*Dedicated to the memory
of Professor Milan Kolibiar*

PRODUCTS OF MODE VARIETIES AND ALGEBRAS OF SUBALGEBRAS

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ABSTRACT. A mode is an idempotent and entropic algebra. The aim of this paper is to describe the structure of subalgebra modes of modes in a product of varieties, in particular varieties such that at least one of them is a variety of affine spaces. We show that certain reducts of such modes may be constructed as Płonka sums. This result is applied to describe subalgebra modes of some binary modes.

1. Introduction

A *mode* is an idempotent, entropic algebra, i.e., with each singleton a subalgebra, and each operation a homomorphism [RS2; p. 145]. The two properties may be expressed algebraically by means of the *idempotent* and *entropic identities*

$$x \dots x\omega = x\,, \tag{I}$$

$$x_{11} \dots x_{1n}\omega \dots x_{m1} \dots x_{mn}\omega\omega' = x_{11} \dots x_{m1}\omega' \dots x_{1n} \dots x_{mn}\omega'\omega \tag{E}$$

that are satisfied in each mode (A, Ω), for any n-ary operation ω and m-ary operation ω' in Ω. Examples of modes are furnished by affine spaces and their reducts, semilattices and convex sets. Modes were studied in detail in [RS2]. Some further information may be found in the list of references at the end of the paper.

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Given a mode (A, Ω) with a set Ω of operations $\omega\colon A^{\omega\tau} \to A$, one may form the set $(A,\Omega)S$ or AS of non-empty subalgebras of (A,Ω). This set AS carries an Ω-algebra structure under the complex products

$$\omega\colon AS^{\omega\tau} \to AS\,; \qquad (X_1,\dots,X_{\omega\tau}) \mapsto \{x_1 \dots x_{\omega\tau}\omega \mid x_i \in X_i\}$$

and it turns out that the algebra (AS,Ω) is again a mode preserving many of the algebraic properties of (A,Ω) [RS2; p. 146]. This self reproducing property plays an important role in the theory of modes, and also in the theory of semilattice ordered modes studied under the name of modals in [RS2]. See also [RS3], [RS4].

One of the most important examples of modes is given by affine spaces (or affine modules) over a ring R. Modes of subspaces of affine spaces over fields were investigated in [RS1]. In that paper, one described affine geometry, projective geometry, and the passage between them purely algebraically, using such modes of subspaces. The results of [RS1] were then generalized in [PRS] to the case of affine spaces over arbitrary commutative rings with unity. It was shown there that certain reducts of such modes may be constructed as P ł o n k a sums of reducts of affine spaces over the corresponding projective space [PRS; Theorem 3.9]. For certain varieties of modes, this result gives a complete characterization of algebras of subalgebras.

This paper is a sequel to [PRS] and continues the study of algebras (AS,Ω). It deals with subalgebra modes of modes in a product of varieties, in particular varieties such that at least one of them is a variety of affine spaces. We refer the reader to Section 3 for the definition of such product we use in this paper and a brief discussion concerning the notion in the case of modes. We describe the structure of subalgebra modes of modes in such products in general, and then focus our attention to products of certain varieties of binary (or groupoid) modes. In Section 2, we recall basic definitions and properties of affine spaces and their algebras of subspaces. Section 3 is devoted to products of mode varieties. In Section 4, we discuss the structure of subalgebra modes in products of mode varieties. Finally, Section 5 is devoted to subalgebra modes in certain binary mode varieties.

The notation and terminology of the paper is similar to that in the book [RS2] and in the paper [PRS]. We use "Polish" notation for words (terms) and operations, e.g., instead of $w(x_1,\dots,x_n)$ we write $x_1 \dots x_n w$. Moreover, the symbol $x_1 \dots x_n w$ means that $x_1,\dots,x_n$ are exactly variables appearing in the word w. The traditional notation is used in the case of groupoid words. For such words we frequently use non-brackets notation, as follows

$$x_1x_2 := x_1 \cdot x_2\,, \qquad x_1 \dots x_n := (x_1 \dots x_{n-1}) \cdot x_n\,.$$

$$xy^0 := x\,,$$

$$xy^n := xy_1 \dots y_n \qquad \text{with} \quad y = y_1 = \dots = y_n\,.$$

$$x^n y := x_n\big(x_{n-1}\big(\dots(x_1 y)\dots\big)\big) \qquad \text{with} \quad x = x_1 = \dots = x_n\,.$$

Two words (terms) of given type are *mode equivalent* if each one can be deduced from the other using only consequences of idempotent and entropic laws. An identity $w_1 = w_2$ is *regular* if the sets of variable symbols on both sides are equal, and it is *linear* if the multiplicities of each argument of w_1 and w_2 are at most 1. In particular, for any mode (A, Ω), the algebra (AS, Ω) satisfies all idempotent and all linear identities true in (A, Ω). (See [RS2].) Algebras and varieties are equivalent if they have the same derived (term) operations. We refer the reader to the book [RS2] for all undefined notions and results.

2. Affine spaces and algebras of subalgebras

Let R be a commutative ring with unity, and let $(E, +, R)$ be a module over R. For each element r of R, define a binary operation

$$\underline{r}\colon E \times E \to E\,; \qquad (x, y) \mapsto xy\underline{r} := x(1-r) + yr\,,$$

and the Mal'cev operation

$$P\colon E \times E \times E \to E\,; \qquad (x, y, z) \mapsto x - y + z\,.$$

The algebra $(E, \underline{R}, P)$ with the ternary operation P and the set $\underline{R}$ of binary operations $\underline{r}$ for r in $\underline{R}$ is equivalent to the full idempotent reduct $\left(E, \left\{x_1 r_1 + \cdots + x_n r_n \mid r_1, \ldots, r_n \in R\,,\ \sum_{i=1}^{n} r_i = 1\right\}\right)$ of the module $(E, +, R)$. Consequently, it can be identified with the affine space (or module) over the ring R. (See, e.g., [RS2].) Carrying out this identification we will refer to the algebra $(E, \underline{R}, P)$ as an affine space over R or an affine R-space. It is well known that the class of affine spaces over the ring R forms a variety. This variety is equivalent to the variety $\underline{\underline{R}}$ of Mal'cev modes $(A, \underline{R}, P)$ with the ternary Mal'cev operation P and one binary operation $\underline{r}$ for each r in R, satisfying certain identities given in [RS2].

The affine subspaces (or affine submodules) of the module $(E, +, R)$ (i.e., cosets of submodules of $(E, +, R)$) are exactly the subalgebras of the algebra $(E, \underline{R}, P)$. Consider the set $(E, \underline{R}, P)S$ or ES of non-empty subalgebras of $(E, \underline{R}, P)$. The set ES forms an algebra under the complex products

$$\underline{r}\colon ES \times ES \to ES\,; \qquad (X, Y) \mapsto \{xy\underline{r} \mid\ x \in X\,,\ y \in Y\}$$

for r in R, and

$$P\colon ES \times ES \times ES \to ES\,; \qquad (X, Y, Z) \mapsto \{xyzP \mid\ x \in X\,,\ y \in Y\,,\ z \in Z\}\,.$$

It turns out that the algebra $(ES, \underline{R}, P)$ is again a mode satisfying each linear identity satisfied by $(E, \underline{R}, P)$. (See [RS1], [RS2].)

Projective space is considered here as the set $L(E) = (E,+,R)S$ of submodules of the R-module $(E,+,R)$, together with the semilattice operation $+$, where for submodules U and V of E, $U+V = \{u+v \mid u \in U,\ v \in V\}$ is the sum of U and V. The inclusion structure is recovered from $(L(E),+)$ via $U \leq V$ if an only if $U+V=V$.

In [PRS], the structure of the algebra $\left(ES, \underline{J}^0_R\right)$, where J^0_R comprises the set of units r of R for which $1-r$ is also invertible, was described using the concept of a Płonka sum ([P1], [RS2; p. 236]). Let (Ω) denote the category of Ω-algebras and homomorphisms between them. Consider the semilattice $(H,+)$ as a small category (H) with a set H of objects and with unique morphism $h \to k$ precisely when $h+k=k$, i.e., $h \leq k$. Let $F\colon (H) \to (\Omega)$ be a functor. Then the Płonka sum of the Ω-algebras (hF,Ω), for h in H, over the semilattice $(H,+)$ by the functor F, is the disjoint union $HF = \bigcup(hF \mid h \in H)$ of the underlying sets hF, equipped with the Ω-algebra structure, given for each n-ary operation ω in Ω and $h_1,\dots,h_n$, $h = h_1+\dots+h_n$ in H, by

$$\omega\colon h_1F \times \ldots \times h_nF \to hF\,; \qquad (x_1,\dots,x_n) \mapsto x_1(h_1 \to h)F \ldots x_n(h_n \to h)F\omega\,.$$

The *canonical projection* of the Płonka sum HF is the homomorphism $\pi\colon (HF,\Omega) \to (H,\Omega)$ with restriction $\pi\colon hF \to \{h\}$. The subalgebras $(hF,\Omega) = \left(\pi^{-1}(h),\Omega\right)$ of (HF,Ω) are the *Płonka fibres*. Recall that for Ω-algebras in an idempotent irregular variety V, the identities satisfied by their Płonka sums are precisely the regular identities holding in the fibres.

THEOREM 2.1. ([PRS]) *For an affine space $(E,\underline{\underline{R}},P)$ in $\underline{\underline{R}}$, each algebra $\left((E,\underline{R},P)S,\Omega\right)$, where $\Omega \subseteq \underline{J}^0_R \cup \{P\}$, is a Płonka sum of Ω-reducts of affine R-spaces $(E/U,\underline{R},P)$ over the projective space $\left(L(E),+\right) = \left((E,+,R)S,+\right)$ by the functor $F\colon \left(L(E)\right) \to (\Omega)$ with $UF = \{x+U \mid x \in E\}$ and $(U \to V)F\colon UF \to VF\,;\ x+U \mapsto x+V$.*

Let V be a variety of Ω-algebras equivalent to a variety $\underline{\underline{R}}$ of affine R-spaces. For each V-algebra (A,Ω), let $V(A)$ be the smallest subvariety of V containing (A,Ω). Then there is a quotient $R(A)$ of the ring R such that the varieties $V(A)$ and $\underline{\underline{R(A)}}$ are equivalent. The algebra (A,Ω) is equivalent to the faithful affine space $\left(A,\underline{R(A)},P\right)$. (The affine space $(E,\underline{R},P)$ is said to be *faithful* if the module $(E,+,R)$ is faithful.)

PROPOSITION 2.2. ([PRS]) *Let V be a variety of Ω-algebras equivalent to a variety $\underline{\underline{R}}$ of affine R-spaces. Let (A,Ω) be in V. If $\Omega \subseteq \underline{J}^0_{R(A)} \cup \{P\}$, then the algebra $\left((A,\Omega)S,\Omega\right)$ is a Płonka sum of $V(A)$-algebras, equivalent to affine $R(A)$-spaces, over the semilattice $\left((A,+,R)S,+\right) = \left((A,+,R(A))S,+\right)$.*

3. Products of mode varieties

Let $V_1, \dots, V_n$ be varieties of Ω-algebras of the same fixed type. The varieties $V_1, \dots, V_n$ are *independent* if there is an n-ary Ω-word $x_1 \dots x_n d$ such that the identity $x_1 \dots x_n d = x_i$ holds in V_i for each $i = 1, \dots, n$. It is well known that whenever the varieties $V_1, \dots, V_n$ are independent, each algebra (A, Ω) in their join $V = V_1 \vee \dots \vee V_n$ is isomorphic to a product $(A_1, \Omega) \times \dots \times (A_n, \Omega)$ with (A_i, Ω) in V_i for each $i = 1, \dots, n$, and algebras (A_i, Ω) are determined up to isomorphism. In this case, we denote the join V of V_i by $V_1 \times \dots \times V_n$ and say that V is the product of its subvarieties $V_1, \dots, V_n$. (See [GLP].) It is easy to see that, in this case, the product $V_1 \times \dots \times V_n$ satisfies the diagonal identity

$$x_{11} \dots x_{1n} d x_{21} \dots x_{2n} d \dots x_{n1} \dots x_{nn} dd = x_{11} x_{22} \dots x_{nn} d . \tag{3.1}$$

Moreover, if $V_1, \dots, V_n$ are varieties of modes, then so is $V_1 \times \dots \times V_n$. (Cf., e.g., [RS2; 2.3], note, however, that in [RS2], the product of varieties is called a "direct sum".) On the other hand, if $x_1 \dots x_n d$ is a word of a variety V of Ω-modes, and V satisfies the identity (3.1), then V is the product $V_1 \times \dots \times V_n$ of its subvarieties $V_1, \dots, V_n$ with each V_i defined by the identity $x_1 \dots x_n d = x_i$. This is a consequence of a more general theorem (cf., e.g., [MMT; 4.4], [FMMT]) saying that a variety V of Ω-algebras is the product of its subvarieties $V_1, \dots, V_n$, whenever $x_1 \dots x_n d$ satisfies (3.1), $x \dots x d = x$, and for each ω in Ω, $x_{11} \dots x_{1\omega\tau} \omega \dots x_{n1} \dots x_{n\omega\tau} \omega d = x_{11} \dots x_{n1} d \dots x_{1\omega\tau} \dots x_{n\omega\tau} d\omega$. Obviously, last identities are always satisfied by Ω-modes. So, in the case of Ω-modes, these identities reduce to (3.1). The word d is called a *decomposition* word and is uniquely defined modulo equational theory of V. As was shown in [AK], in the case the independent varieties $V_1, \dots, V_n$, have finite bases for their identities, their product $V_1 \times \dots \times V_n$ is finitely based, too. In the case of varieties of modes, it is very easy to find its basis.

PROPOSITION 3.2. *Let $V_1, \dots, V_n$ be independent varieties of Ω-modes, with each V_i satisfying the identity $x_1 \dots x_n d = x_i$. Let each V_i be defined by identities $t^i_j = w^i_j$ for $j = 1, 2, \dots, k_i$. Then the product $V_1 \times \dots \times V_n$ is the variety of Ω-modes defined by the identities*

$$x_{11} \dots x_{1n} d x_{21} \dots x_{2n} d \dots x_{n1} \dots x_{nn} dd = x_{11} x_{22} \dots x_{nn} d , \tag{3.3}$$

$$x_1 \dots t^i_j \dots x_n d = x_1 \dots w^i_j \dots x_n d \tag{3.4}$$

for each $i = 1, \dots, n$ and $j = 1, \dots, k_i$.

Proof. Let V be the variety of Ω-modes defined by the identities (3.3) and (3.4). It is easy to see that each variety V_i satisfies the identities (3.3) and (3.4), whence $V_1 \vee \dots \vee V_n \subseteq V$. On the other hand, since the variety V has a decomposition word, each algebra (A, Ω) in V is isomorphic to a product

$(A_1, \Omega) \times \ldots \times (A_n, \Omega)$ with each (A_i, Ω) satisfying $x_1 \ldots x_n d = x_i$, and hence also each $t^i_j = w^i_j$. It follows that (A, Ω) is in $V_1 \times \ldots \times V_n$, and hence $V \subseteq V_1 \times \ldots \times V_n = V_1 \vee \ldots \vee V_n$. □

Let us recall that for any mode (A, Ω) the subalgebra mode (AS, Ω) satisfies all idempotent and all linear identities true in (A, Ω). If (A, Ω) is in the variety $V = V_1 \times \ldots \times V_n$ as in Proposition 3.2, and the identities (3.3) and (3.4) are linear, then the mode (AS, Ω) of subalgebras is again in V and decomposes into product of V_i-algebras.

4. Products of mode varieties and algebras of subalgebras

At first we give some basic properties of subalgebras of a product of nontrivial modes.

If $V_1, \ldots, V_n$ are independent varieties of Ω-algebras, an algebra (A, Ω) is in the variety $V_1 \times \ldots \times V_n$, and (A, Ω) is isomorphic to $(A_1, \Omega) \times \ldots \times (A_n, \Omega)$ with (A_i, Ω) in V_i, then we say that $(A_1, \Omega) \times \ldots \times (A_n, \Omega)$ is a *factorization* of (A, Ω).

LEMMA 4.1. ([FMMT]) *Let $V_1, \ldots, V_n$ be independent varieties of Ω-algebras. Let $(A_1, \Omega) \times \ldots \times (A_n, \Omega)$ be a factorization of an algebra (A, Ω) in the variety $V = V_1 \times \ldots \times V_n$. If (B, Ω) is a subalgebra of $(A_1, \Omega) \times \ldots \times (A_n, \Omega)$, then for each $i = 1, \ldots, n$, there is a subalgebra (B_i, Ω) of (A_i, Ω) such that (B, Ω) is isomorphic to $(B_1, \Omega) \times \ldots \times (B_n, \Omega)$.*

LEMMA 4.2. *For each i in a set I, let (A_i, Ω) be an Ω-algebra. For a fixed j in I, let (A_j, Ω) be equivalent to an affine R-space. If all subalgebras of $\prod_{i \in I} (A_i, \Omega)$ are of the form $\prod_{i \in I} (B_i, \Omega)$, with (B_i, Ω) a subalgebra of (A_i, Ω), then*

(i) *the mapping*

$$\pi \colon \Big(\prod_{i \in I} A_i\Big) S \to L(A_j); \qquad \Big(\prod_{i \in I} B_i\Big) \mapsto U_j,$$

where $B_j = x + U_j$, is an Ω-homomorphism;

(ii) *the mapping*

$$\varphi \colon \pi^{-1}(U_j) \to \pi^{-1}(V_j); \qquad \prod_{i \in I} B_i \mapsto \prod_{i \in I} C_i,$$

where $B_j = x + U_j$, $C_j = x + V_j$, $U_j \subseteq V_j$ and for $i \neq j$, $C_i = B_i$, is an Ω-homomorphism.

P r o o f .

(i) Let $\prod_{i\in I}(B_{ik},\Omega)$ for $k=1,\dots,n$ be subalgebras of $\prod_{i\in I}(A_i,\Omega)$ with $B_{jk} = x_k + U_k$, and ω be n-ary operation in Ω. Then

$$\begin{aligned}\prod_{i\in I} B_{i1}\dots\prod_{i\in I} B_{in}\omega &= \{(b_{i1})_{i\in I}\dots(b_{in})_{i\in I}\omega \mid b_{ik}\in B_{ik}\}\\ &= \{(b_{i1}\dots b_{in}\omega)_{i\in I} \mid b_{ik}\in B_{ik}\} = \prod_{i\in I} B_{i1}\dots B_{in}\omega\,.\end{aligned}$$

Moreover,

$$B_{j1}\dots B_{jn}\omega = (x_1+U_1)\dots(x_n+U_n)\omega = x_1\dots x_n\omega + U_1\dots U_n\omega\,.$$

Hence

$$\begin{aligned}\prod_{i\in I} B_{i1}\dots\prod_{i\in I} B_{in}\omega\pi &= \prod_{i\in I} B_{i1}\dots B_{in}\omega\pi = U_1\dots U_n\omega\\ &= \Big(\prod_{i\in I} B_{i1}\Big)\pi\dots\Big(\prod_{i\in I} B_{in}\Big)\pi\omega\,.\end{aligned}$$

(ii) Let $\prod_{i\in I}(B_{ik},\Omega)$, for $k=1,\dots,n$, be subalgebras of $\prod_{i\in I}(A_i,\Omega)$ with $B_{jk} = x_k + U_j$, and ω be n-ary operation in Ω. Then $\Big(\prod_{i\in I} B_{i1}\dots\prod_{i\in I} B_{in}\omega\Big)\varphi = \Big(\prod_{i\in I} B_{i1}\dots B_{in}\omega\Big)\varphi$, where $B_{j1}\dots B_{jn}\omega = x_1\dots x_n\omega + U_j$.

Let $\Big(\prod_{i\in I} B_{ik}\Big)\varphi = \prod_{i\in I} C_{ik}$; where $B_{jk} = x_k + U_j$, $C_{jk} = x_k + V_j$, and for $i\neq j$, $C_{ik} = B_{ik}$. Then

$$\begin{aligned}\Big(\prod_{i\in I} B_{i1}\Big)\varphi\dots\Big(\prod_{i\in I} B_{in}\Big)\varphi\omega &= \prod_{i\in I} C_{i1}\dots\prod_{i\in I} C_{in}\omega\\ &= \prod_{i\in I} C_{i1}\dots C_{in}\omega = \Big(\prod_{i\in I} B_{i1}\dots\prod_{i\in I} B_{in}\omega\Big)\varphi\,,\end{aligned}$$

since $C_{j1}\dots C_{jn}\omega = x_1\dots x_n\omega + V_j$. □

The next theorem follows directly from Lemma 4.1 and Lemma 4.2.

THEOREM 4.3. *Let $V_1, \ldots, V_n$ be independent varieties of Ω-modes. For a fixed j in $I = \{1, \ldots, n\}$, let V_j be equivalent to a variety $\underline{\underline{R}}$ of affine R-spaces. Let $(A_1, \Omega) \times \ldots \times (A_n, \Omega)$ be a factorization of an algebra (A, Ω) in the variety $V = V_1 \times \ldots \times V_n$. If $B_j = x + U_j$ is a subalgebra of (A_j, Ω) term equivalent to $\left(A_j, \underline{R(A_j)}, P\right)$, then define*

$$\pi \colon \left(\prod_{i \in I} A_i\right) S \to L(A_j)\,; \qquad \prod_{i \in I} B_i \mapsto U_j\,.$$

If $\Omega \subseteq \underline{J}^0_{R(A_j)} \cup \{P\}$, then the algebra $\big((A,\Omega)S, \Omega\big)$ is a Płonka sum of the algebras

$$(\pi^{-1}(U_j), \Omega) = \left\{ \prod_{i \in I} (B_i, \Omega) \mid \ (B_i, \Omega) \leq (A_i, \Omega) \ \text{for } i = 1, \ldots, n \right. \\ \left. \text{and } B_j = x + U_j \ \text{for } x \ \text{in } A_j \right\}$$

over the projective space $\big((A_j, +, R(A_j))S, +\big)$ by the functor $F \colon \big((A_j, +, R(A_j))S, \big) \to (\Omega)$, with $U_j F = \pi^{-1}(U_j)$ and $(U_j \to V_j)F \colon \pi^{-1}(U_j) \to \pi^{-1}(V_j)$; $\prod_{i \in I} B_i \mapsto \prod_{i \in I} C_i$, where $B_j = x + U_j$, $C_j = x + V_j$, and for $i \neq j$, $C_i = B_i$.

P r o o f. The proof is similar to the proof of 2.2 (see [PRS]). Lemma 4.2 implies that π is an Ω-homomorphism onto the semilattice $\big((A_j, +, R(A_j))S, +\big)$, and F is a functor. The Płonka fibres have the form described in the theorem. For $k = 1, \ldots, n$, let $\prod_{i \in I} (B_{ik}, \Omega)$ be subalgebras of $\prod_{i \in I} (A_i, \Omega)$ with $B_{jk} = x_k + U_k$, and let ω be n-ary operation in Ω. As in Lemma 4.2, $\prod_{i \in I} B_{i1} \ldots \prod_{i \in I} B_{in}\omega = \prod_{i \in I} B_{i1} \ldots B_{in}\omega$, where $B_{j1} \ldots B_{jn}\omega = x_1 \ldots x_n\omega + (U_1 + \cdots + U_n)$, since ω is derived from $\underline{J}^0_{R(A_j)} \cup \{P\}$. Hence $\prod_{i \in I} B_{i1} \ldots \prod_{i \in I} B_{in}\omega = \prod_{i \in I} B_{i1}(U_1 \to U_1 + \ldots + U_n)F \ldots \prod_{i \in I} B_{in}(U_n \to U_1 + \cdots + U_n)F\omega$ showing that $\big((A,\Omega)S, \Omega\big)$ is a Płonka sum as claimed. In particular, if for $k = 1, \ldots, n$, $B_{jk} = x_k + U_j$, we obtain that $\prod_{i \in I} B_{i1} \ldots \prod_{i \in I} B_{in}\omega = \prod_{i \in I} B_{i1} \ldots B_{in}\omega$, with $B_{j1} \ldots B_{jn}\omega = x_1 \ldots x_n\omega + U_j$. □

Certain identities on two variables are easily seen to be satisfied in algebras of subalgebras of $V_1 \times \ldots \times V_n$-algebras. To describe them, we need the following lemma.

LEMMA 4.4. *For $1 \leq i \leq n$, let V_i be idempotent varieties of Ω-algebras. If the identity $xyv_i = x$ is satisfied in the variety V_i, then the identities*

$$x \dots xxyv_1v_2 \dots v_n = x = x \dots xxyv_{\sigma(1)}v_{\sigma(2)} \cdots v_{\sigma(n)} \tag{4.5}$$

are satisfied in the variety $V_1 \vee \dots \vee V_n$ for each permutation σ of the set $\{1, 2, \dots, n\}$.

P r o o f. First, let us note that each variety V_i satisfies the identity (4.5). Indeed, since $xyv_i = x$ is satisfied in V_i, it follows that V_i also satisfies

$$\begin{aligned} xx \dots xxyv_1v_2 \dots v_n &= x \dots x\big(x(x \dots xxyv_1v_2 \dots v_{i-1})v_i\big)v_{i+1} \cdots v_n \\ &= x \dots xxv_{i+1} \dots v_n = x \,. \end{aligned}$$

The same holds for any order of $V_1, \dots, V_n$. Consequently, (4.5) holds in the join $V_1 \vee \dots \vee V_n$. □

As a consequence of Lemma 4.4, Proposition 2.2 and Theorem 4.3, one has the following.

PROPOSITION 4.6. *For $1 \leq i \leq n$, let V_i be independent varieties of Ω-modes. If an identity $xyv_i = x$ is satisfied in the variety V_i and is mode equivalent to a linear identity, then the identities*

$$x \dots xxyv_1v_2 \dots v_n = x = x \dots xxyv_{\sigma(1)}v_{\sigma(2)} \cdots v_{\sigma(n)} \tag{4.7}$$

are true in modes of subalgebras of $V_1 \times \dots \times V_n$-modes, for each permutation σ of the set $\{1, 2, \dots, n\}$.

P r o o f. The proof goes by induction on n. Let $n = 2$. Let (A_1, Ω), (B_1, Ω) be in V_1 and (A_2, Ω), (B_2, Ω) be in V_2. Then $(A_1 \times A_2)(A_1 \times A_2)(B_1 \times B_2)v_1v_2 = A_1A_1B_1v_1v_2 \times A_2A_2B_2v_1v_2 = A_1A_1v_2 \times A_2 = A_1 \times A_2$, because modes of subalgebras of V_1-modes satisfy $xyv_1 = x$, and modes of submodes of V_2-modes satisfy $xyv_2 = x$. Similar argument shows that the identity (4.7) implies similar identity for $n+1$, and hence Proposition 4.6 holds. □

COROLLARY 4.8. *For $1 \leq i \leq n$, let V_i be independent varieties of Ω-modes. For a fixed j in I, let V_j be a variety of Ω-algebras equivalent to a variety $\underline{\underline{R}}_j$ of affine R_j-spaces. Moreover, let an identity $xyv_j = x$ be satisfied in the variety $\underline{\underline{R}}_j$, and for $i \neq j$ let algebras of subalgebras of V_i-algebras satisfy the identity $xyv_i = x$. Then the Płonka fibres $\pi^{-1}(U_j)$ satisfy the identities*

$$x \dots xxyv_1v_2 \dots v_n = x = x \dots xxyv_{\sigma(1)}v_{\sigma(2)} \cdots v_{\sigma(n)}$$

for each permutation σ of the set $\{1, 2, \dots, n\}$.

5. Certain binary mode varieties and algebras of subalgebras

In this section, we investigate the structure of algebras in a certain binary (or groupoid) mode variety, and the structure of modes of their subalgebras. The variety in question is the join of three varieties. The first one is the variety $V_{s,t}$ of binary modes defined by the identities

$$y^s x = x = xy^t \,. \tag{5.1}$$

The variety is very well known. (See, e.g., [PRS].) It is a Mal'cev variety with the Mal'cev operation given by

$$xyzP := xy^{t-1} \cdot y^{s-1} z \,.$$

So $V_{s,t}$ is equivalent to $\underline{\underline{R}}_{s,t}$ for some commutative ring $R_{s,t}$ generated by one element, say r. The groupoid multiplication can be identified with the operation $\underline{r}$. In fact, the identities (5.1) hold in a groupoid $(G, \underline{r})$ precisely if $r^s = 1$ and $(1-r)^t = 1$. The ring $R_{s,t}$ is isomorphic to the ring $Z[X]/\big(X^s - 1, (X-1)^t - 1\big)$.

The varieties $V_{s,t}$ contain many well-known varieties of Mal'cev binary modes. Among them are the varieties $G(n,k)$ of groupoids studied by M i t s c h k e, W e r n e r [MW], equivalent to affine spaces over the rings $R(n,k) = Z[X]/(X^n - 1, X^k + X - 1)$. Each $G(n,k)$ is a subvariety of the variety $V_{n,n/[n,k]}$, where $[n,k]$ is the greatest common divisor of n and k. To show it, let us first note that the generator r of the ring $R(n,k)$ satisfies the conditions $r^n = 1$ and $r^k = 1 - r$. Hence $(1-r)^{n/[n,k]} = r^{n(k/[n,k])} = 1$, which implies that $G(n,k)$-groupoids satisfy the identity $xy^{n/[n,k]} = x$, and hence are members of $V_{n,n/[n,k]}$. The varieties $G(q)$ of groupoids equivalent to affine spaces over finite fields $GF(q)$, described by G a n t e r, W e r n e r [GW], are subvarieties of $G(q-1,k)$, where $r + r^k = 1$ and r is a primitive element of $GF(q)$. Any irregular variety $\underline{\underline{2m+1}}$ of commutative binary modes (cp. [JK] and [RS6]) is equivalent to the variety $\underline{\underline{Z}}_{2m+1}$ of affine spaces over the ring Z_{2m+1}. The groupoid multiplication is given by $\underline{r} = \underline{m+1}$. Here $1 - r = r$. For each variety $\underline{\underline{2m+1}}$ there is an n such that $\underline{\underline{2m+1}}$ is contained in the variety $V_{n,n}$. Indeed, since $2m+1$ and $m+1$ are relatively prime, Euler's Theorem shows that there is $n = \varphi(2m+1)$ such that $(m+1)^n = 1 \pmod{(2m+1)}$. Hence each $\underline{\underline{2m+1}}$-groupoid satisfies the identity $xy^n = x$. There is another interesting series of varieties of binary modes equivalent to varieties $\underline{\underline{Z}}_{2m+1}$. These are subvarieties S_{2m+1} of the variety S of *symmetric* binary modes satisfying the identity $xy^2 = x$, defined by the additional identity

$$xys_{2m+1} := (\ldots(y_1 x_2 \cdot y_3)x_4 \ldots)x_{2m} \cdot y_{2m+1} = x \,, \tag{S_{2m+1}}$$

where $y_1 = y_3 = \cdots = y_{2m+1} = y$ and $x_2 = x_4 = \cdots = x_{2m} = x$.

The variety S was thoroughly investigated by B. Roszkowska. See, e.g., [Rs1] and [Rs2]. From results of [Rs2], one can easily deduce that each variety S_{2m+1} is equivalent to the variety $\underline{\underline{Z}}_{2m+1}$. The groupoid multiplication is given by $\underline{r} = \underline{2}$. Since 2 and $2m+1$ are relatively prime, Euler's Theorem again shows that there is $s = \varphi(2m+1)$ such that $2^s = 1 \pmod{(2m+1)}$. It follows that each S_{2m+1}-groupoid satisfies the identity $y^s x = x$ and, consequently, is in the variety $V_{s,2}$.

The other two varieties we will consider in this section, $D_{m,n}$ and $D^*_{k,l}$ defined below, are of interest to us, because they also have interesting models, and because of their connection to idempotent abelian algebras.

First recall that a groupoid $(G,\cdot)$ is called *abelian* if it satisfies the so called *term condition*.

(TC) If $xy_1 \dots y_n w$ is a groupoid word (term), a, b are in G and $(c_1,\dots,c_n)$, $(d_1,\dots,d_n)$ are in G^n, then $ac_1 \dots c_n w = ad_1 \dots d_n w$ implies $bc_1 \dots c_n w = bd_1 \dots d_n w$.

(See, e.g., [MMT].) As was observed by K. Kearnes [K], idempotent abelian groupoids are modes. In particular, Kearnes can show that each finite idempotent abelian groupoid $(G,\cdot)$ decomposes as the product $A \times L \times R$, where $(A,\cdot)$ is equivalent to an affine space, and $(L,\cdot)$ and $(R,*)$, with $x*y = yx$, are in the variety $D_{m,n}$ of groupoid modes defined by the m-reduction law

$$x_1\big(x_2(\dots(x_{m-1}\cdot x_m y)\dots)\big) = x_1\big(x_2(\dots(x_{m-1}\cdot x_m)\dots)\big) \tag{mR}$$

and the n-cyclic law

$$xy^n = x\,. \tag{nC}$$

Moreover, the variety $V(G)$ generated by $(G,\cdot)$ decomposes as the product $V(A)\times V(L)\times V(R)$ of varieties $V(A)$, $V(L)$ and $V(R)$ generated by the groupoids $(A,\cdot)$, $(L,\cdot)$ and $(R,\cdot)$ respectively. Some of varieties $D_{m,n}$ are very well known. The variety defined by (2R) is the variety L of *differential* or *LIR-groupoids*, see, e.g., [RS5]. It contains as subvarieties the variety $D_{2,n}$ of *n-cyclic* groupoids, see [RR2]. The variety of *kei-modes* ([RS2; Chapter 4]) is defined by the identity $x^2y = y$, dual to (2C). In its dual form, i.e., defined by (2C), this variety is the variety of symmetric binary modes. It contains as subvarieties the varieties $D_{n,2}$.

There is a very easy way to show that any two of the three varieties $V_{s,t}$, $D_{m,n}$ and $D^*_{k,l}$, defined dually to $D_{k,l}$, are independent. Similarly, all these three varieties are independent. To show this, we will need the following lemma.

LEMMA 5.2. *Let V be the variety of all binary modes. Let n and i be positive integers with $n \ge i$. The following identities are equivalent in the variety V.*

(i) $x_1\big(x_2(\dots(x_{i-1}(x_i^{n-i+1}y))\dots)\big) = x_1\big(x_2(\dots(x_{i-1}x_i)\dots)\big)$,

(ii) $x_1^n x_2 = x_1$,
(iii) $x_1(x_2(\dots(x_i(x_{i+1}^{n-i}y))\dots)) = x_1(x_2(\dots(x_{i-1}(x_i x_{i+1}))\dots))$.
(iv) $x_1(x_2^{n-1}x_3) = x_1 x_2$,
(v) $x_1(x_2(\dots(x_n y)\dots)) = x_1(x_2(\dots(x_{n-1}x_n)\dots))$,
(vi) $x_1(x_2(\dots(x_n y)\dots)) = x_1(x_2(\dots(x_n z)\dots))$.

Proof.

(i) $\Longrightarrow$ (ii): It follows by substituting x_1 for $x_2, \dots, x_i$ in (i).

(i) $\Longrightarrow$ (iii): As a consequence of the first implication one gets $x_1 = x_1^n x_{i+1}$. Then entropicity and (i) imply the following:

$$\begin{aligned} x_1(x_2(\dots(x_{i+1}^{n-i}y)\dots)) &= (x_1^n x_{i+1})(x_2(x_3(\dots x_i(x_{i+1}^{n-i}y)\dots))) \\ &= (x_1x_2)((x_1^{n-1}x_{i+1})(x_3(\dots(x_{i+1}^{n-i}y)))) \\ &= (x_1x_2)((x_1x_3)((x_1^{n-2}x_{i+1})(x_4(\dots(x_{i+1}^{n-i}y))))) \\ &= \dots \\ &= (x_1x_2)((x_1x_3)(\dots(x_1x_i)((x_1x_{i+1})^{n-i+1}y)\dots)) \\ &= (x_1x_2)((x_1x_3)(\dots((x_1x_i)\cdot(x_1x_{i+1}))\dots)) \\ &= x_1(x_2(\dots(x_ix_{i+1})\dots)). \end{aligned}$$

(iii) $\Longrightarrow$ (i): It follows by substituting x_i for x_{i+1} in (iii).

(ii) $\Longrightarrow$ (iv): It follows by the equivalence of (i) and (iii) for $i = 1$.

(iv) $\Longrightarrow$ (v): Applying successively the equivalence of (i) and (iii) one obtains the following identities true in V.

$$\begin{aligned} x_1(x_2(x_3^{n-2}x_4)) &= x_1(x_2x_3), \\ x_1(x_2(x_3(x_4^{n-3}x_5))) &= x_1(x_2(x_3x_4)), \\ &\vdots \\ x_1(x_2(\dots(x_ny)\dots)) &= x_1(x_2(\dots(x_{n-1}x_n)\dots)). \end{aligned}$$

(v) $\Longrightarrow$ (iv): It is obvious.

(vi) $\Longrightarrow$ (i): It follows by substituting x_i for $x_{i+1}, \dots, x_n, z$ in (vi). □

Let us note that by Lemma 5.2, the variety of $D_{m,n}$-modes can be equivalently defined by the identities

$$x^m y = x \qquad (R_m)$$

and

$$xy^n = x. \qquad (C_n = nC)$$

The dual variety $D^*_{m,n}$ is defined by the dual identities

$$xy^m = y \qquad (R^*_m)$$

and

$$x^n y = y. \qquad (C^*_n)$$

LEMMA 5.3. *For natural k, l, m, n, s and t, any two of the varieties $D^*_{k,l}$, $D_{m,n}$ and $V_{s,t}$ are independent.*

P r o o f. Let (i,j) denote the least common multiple of natural numbers i and j. It is easy to see that the following implications hold

$$\begin{aligned} &(R_m) \implies (R_{(m,l)}),\ (R_{(m,s)}); && (C^*_l) \implies (C^*_{(m,l)}),\ (C^*_{(l,s)});\\ &(C_n) \implies (C_{(n,k)}),\ (C_{(n,t)}); && (R^*_k) \implies (R^*_{(n,k)}),\ (R^*_{(k,t)});\\ &(C_t) \implies (C_{(k,t)}),\ (C_{(t,n)}); && (C^*_s) \implies (C^*_{(l,s)}),\ (C^*_{(m,s)}). \end{aligned}$$

It follows that one can take as a decomposition words:

$$\begin{aligned} &xy^{(n,k)} \text{ or } x^{m,l}y && \text{for } D_{m,n} \text{ and } D^*_{k,l},\\ &x^{(m,s)}y && \text{for } D_{m,n} \text{ and } V_{s,t},\\ &xy^{(k,t)} && \text{for } V_{s,t} \text{ and } D^*_{k,l}. \end{aligned}$$

Consequently, each pair of varieties above is independent. □

LEMMA 5.4. *For natural numbers k, l, m, n, s and t, the three varieties $D^*_{k,l}$, $D_{m,n}$ and $V_{s,t}$ are independent.*

P r o o f. Let (i,j,k) be the least common multiple of natural numbers i, j and k. Let

$$xyzw := \big(x^{(m,s,l)}y\big)\big(yz^{(k,t,n)}\big)^{(k,t,n)}.$$

Then it is easy to check that the identity $w = x$ is satisfied in the variety $D_{m,n}$, the identity $w = y$ is satisfied in the variety $V_{s,t}$, and finally, the identity $w = z$ is satisfied in the variety $D^*_{k,l}$. Hence the varieties $D_{m,n}$, $D^*_{k,l}$ and $V_{s,t}$ are independent. □

PROPOSITION 5.5. *The following hold for any natural numbers k, l, m, n, s and t*

$$D_{m,n} \vee D^*_{k,l} = D_{m,n} \times D^*_{k,l}, \tag{5.6}$$

$$D_{m,n} \vee V_{s,t} = D_{m,n} \times V_{s,t}, \tag{5.7}$$

$$D^*_{k,l} \vee V_{s,t} = D^*_{k,l} \times V_{s,t}, \tag{5.8}$$

$$D_{m,n} \vee D^*_{k,l} \vee V_{s,t} = D_{m,n} \times D^*_{k,l} \times V_{s,t}. \tag{5.9}$$

P r o o f. It follows directly by Lemma 5.3 and Lemma 5.4. □

Note that bases for the identities satisfied in each of the four varieties above can be easily deduced using Proposition 3.2. In the first three cases, the bases can be simplified a little using the following observation.

For any binary operation $x \circ y$ the diagonal identity $(x \circ y) \circ (z \circ t) = x \circ t$ is equivalent to the conjunction of

$$(x \circ y) \circ z = x \circ z\,, \tag{5.10}$$

and

$$x \circ (y \circ z) = x \circ z\,. \tag{5.11}$$

Obviously, the conjunction of (5.10) and (5.11) implies the diagonal identity. Conversely, the diagonal identity applied in different ways to $\big[(x \circ y) \circ (p \circ q)\big] \circ (r \circ z)$ yields (5.10), and a symmetric argument shows that the diagonal identity implies (5.11).

In fact, the identities (5.10) and (5.11) are mode equivalent. Indeed, if (5.10) holds, then $(x \circ y) \circ z = (x \circ z) \circ (y \circ z) = x \circ (y \circ z)$. The proof in the opposite direction is similar.

PROPOSITION 5.12. *Let k, l, m, n, s, t be natural numbers.*

(i) *The variety of $D_{m,n} \times D^*_{k,l}$-modes is defined by the identities*

$$(x^m y)z^{(k,n)} = xz^{(k,n)} = (xy^n)z^{(k,n)}\,,$$
$$z(xy^k)^{(k,n)} = zy^{(k,n)} = z(x^l y)^{(k,n)}\,,$$

or by the identities

$$(x^m y)^{(m,l)}z = x^{(m,l)}z = (xy^n)^{(m,l)}z\,,$$
$$z^{(m,l)}(xy^k) = z^{(m,l)}y = z^{(m,l)}(x^l y)\,.$$

(ii) *The variety of $D_{m,n} \times V_{s,t}$-modes is defined by the identities*

$$(x^m y)^{(m,s)}z = x^{(m,s)}z = (xy^n)^{(m,s)}z\,,$$
$$z^{(m,s)}(xy^t) = z^{(m,s)}x = z^{(m,s)}(x^s y)\,.$$

(iii) *The variety of $D^*_{k,l} \times V_{s,t}$-modes is defined by the identities*

$$(xy^k)z^{(k,t)} = yz^{(k,t)} = (x^l y)z^{(k,t)}\,,$$
$$z(xy^t)^{(k,t)} = zx^{(k,t)} = z(x^s y)^{(k,t)}\,.$$

Proof. We prove only (i). Proofs of (ii) and (iii) can be done in a similar way. First note that if we take the word $xyd = x \circ y = xy^{(n,k)}$ as a decomposition word for $D_{m,n}$ and $D^*_{k,l}$, then the identities (3.4) of Proposition 3.2 take the form of the first two identities of (i). Then $z \circ (xy^k) = z(xy^k)^{(n,k)} = zy^{(k,n)} = z \circ y$ implies $z \circ (x \circ y) = z(xy^{(k,n)})^{(k,n)} = zy^{(k,n)} = z \circ y$. But, by the remark before 5.12, the last identity is equivalent to the diagonal identity $(x\circ y)\circ(z\circ t) = x\circ t$. Then 5.12 (i) follows by Proposition 3.2. □

The decomposition of $D_{m,n}\vee D^*_{k,l}\vee V_{s,t}$-modes given in Proposition 5.5 together with results of Section 4 allows one to give a description of subgroupoid modes for groupoids in this variety.

PROPOSITION 5.13. *Let k, l, m and n be natural numbers. For each $D_{m,n}\times D^*_{k,l}$-mode $(G,\cdot)$, the mode $(GS,\cdot)$ of submodes of $(G,\cdot)$ satisfies the identities*

$$x^m(yx^k) = x = (x^m y)x^k\,.$$

P r o o f. Since each identity (mR) is linear and equivalent to (R_m), and similarly, (mR*) is linear and equivalent to (R^*_m), 5.13 follows by Lemma 4.1, Lemma 4.4 and Proposition 4.6. □

THEOREM 5.14. *Let k, l, m, n, s and t be natural numbers. Let $(A_1,\cdot)\times(A_2,\cdot)\times(A_3,\cdot)$ be a factorization of a $D_{m,n}\vee D^*_{k,l}\vee V_{s,t}$-mode $(A,\cdot)$. Then the mode $(AS,\cdot)$ of submodes of $(A,\cdot)$ is a Płonka sum of binary modes satisfying the identities*

$$x\big(x^m(yx^k)\big)^t = x = \big(x^m(yx^k)\big)^s x\,,$$

over the semilattice $\big((A_3,+,R(A_3))S,+\big)$. Moreover, if $(A,\cdot) = (A_1,\cdot)\times(A_3,\cdot)$ is in the variety $D_{m,n}\vee V_{s,t}$, then the corresponding Płonka fibres satisfy the identities

$$x(x^m y)^t = x = (x^m y)^s x\,.$$

And if $(A,\cdot) = (A_2,\cdot)\times(A_3,\cdot)$ is in the variety $D^*_{k,l}\vee V_{s,t}$, then the corresponding Płonka fibres satisfy the identities

$$x(xy^k)^t = x = (xy^k)^s x\,.$$

P r o o f. It follows by Theorem 4.3, Proposition 4.6, Corollary 4.8 and Proposition 5.5. □

EXAMPLE 5.15. The lattice of subvarieties of the variety S of symmetric binary modes was described in [Rs1]. It is isomorphic to the lattice $\tilde{\mathbb{N}} = \mathbb{N}\cup\infty$ of natural numbers with divisibility relation and with the greatest element added. Each subvariety S_{2n+1} is defined by one additional identity (S_{2n+1}). Each variety S_{2m} coincides with $D_{m,2}$. By Lemma 5.3, the varieties S_{2m} and S_{2n+1} are independent, and by Proposition 5.5, $S_{2m}\vee S_{2n+1} = S_{2m}\times S_{2n+1}$. This was first proved in [Rs2]. Moreover, it was shown there that in fact $S_{2m}\vee S_{2n+1} = S_{2m(2n+1)}$. By Lemma 4.4, each variety $S_{2m(2n+1)}$ satisfies the identity

$$xx^m ys_{2n+1} = x\,. \tag{5.16}$$

In fact, as was shown in [Rs2], this identity defines $S_{2^m(2n+1)}$. This is obviously simpler than the axiomatization that follows from Proposition 5.12. An argument similar to that for Theorem 5.14 shows that the mode $(AS,\cdot)$ of submodes of $(A,\cdot)$ in the variety $S_{2^m(2n+1)}$ is a Płonka sum of binary modes satisfying (5.16).

Let us note, that for a symmetric binary mode $(G,\cdot)$, the mode $(GS,\cdot)$ does not necessarily satisfy the symmetric identity. Indeed, consider the groupoid $(Z_4,\cdot) = (Z_4,\underline{2})$. In $(Z_4S,\cdot)$, one has $(\{0\}\cdot\{0,1,2,3\})\cdot\{0,1,2,3\} = \{0,2\}\cdot\{0,1,2,3\} = \{0,2\} \neq \{0\}$.

EXAMPLE 5.17. The variety $S_4 = D_{2,2}$ of symmetric binary modes is also contained in the variety L of differential groupoids. The lattice of subvarieties of the variety L is described in [RR1]. It is isomorphic to the lattice $\underline{\mathbb{N}}^0 \times \mathbb{N}$ with the greatest and the smallest elements added, where $\underline{\mathbb{N}}^0$ is the lattice of non-negative integers with the usual ordering as the lattice ordering. For (i,j) in $\underline{\mathbb{N}}^0 \times \mathbb{N}$, the subvariety $L_{i,j}$ is defined by one additional identity

$$xy^{i+j} = xy^i\,. \tag{5.18}$$

Let us note that $S_4 = D_{2,2} = L_{0,2}$. Subgroupoid modes of S_4-modes do not inherit the symmetric identity, but they satisfy the identity (R_2).

The natural question arises. Do the subgroupoid modes of S_4-groupoids satisfy any of the identities (5.18)?

To answer this question, let us note that any finite groupoid satisfies an identity of the form (5.18). Indeed, if $(G,\cdot)$ has cardinality n, then for each y in G, the mappings $R_y\colon G\to G;\ g\mapsto gy$, form a finite cyclic monoid. Hence there are an index i and a period p such that $R_y^{i+p} = R_y^i$. It follows that for each x in G, $xy^{i+p} = xy^i$. Consequently, any x and y in $(G,\cdot)$ satisfy the identity $xy^{m+l} = xy^m$, where m is maximal among all indexes, and l is the least common multiple of all periods. Since the subgroupoid mode of a finite S_4-groupoids is finite, it necessarily satisfies an identity of the form (5.18). However, this is no longer true if, instead of a single groupoid, we consider the class S_4S of all subgroupoid modes of all S_4-groupoids.

THEOREM 5.19. *The variety L of differential groupoids is generated by the class S_4S of subgroupoid modes of S_4-groupoid.*

Proof. We will find a sequence $(F_2,\cdot),(F_3,\cdot),\dots$ of S_4-groupoids, such that for each $(i,j)\in\underline{\mathbb{N}}^0\times\mathbb{N}$, there is a groupoid $(F_k,\cdot)$ in this sequence such that $(F_kS,\cdot)$ does not satisfy the identity (5.18). For each natural number n, we define $(F_{n+1},\cdot)$ to be the free S_4-groupoid on $n+1$ free generators $x,y_1,\dots,y_n$. For each $(F_{n+1},\cdot)$, let $A_{n+1} = \{x\}$, one element subalgebra of $(F_{n+1},\cdot)$, and let $B_{n+1} = y_1F_{n+1}\cup\dots\cup y_nF_{n+1}$ be the union of the orbits $y_1F_{n+1},\dots,y_nF_{n+1}$.

Then it is easy to check the following

$$
\begin{aligned}
A_{n+1}B_{n+1} &= \{xy_i \mid i = 1, \dots, n\}, \\
A_{n+1}B_{n+1}^2 &= \{x\} \cup \{xy_iy_j \mid i, j = 1, \dots, n \text{ and } i \neq j\}, \\
A_{n+1}B_{n+1}^3 &= \{xy_i \mid i = 1, \dots, n\} \cup \{xy_iy_jy_k \mid i, j, k = 1, \dots, n \text{ and} \\
&\qquad\qquad i, j, k \text{ are pairwise different}\}
\end{aligned}
$$

and so on. It is easy to see that $A_{n+1}B_{n+1}^{n+1} = A_{n+1}B_{n+1}^{n-1}$, and that all $A_{n+1}B_{n+1}$, $A_{n+1}B_{n+1}^2, \dots, A_{n+1}B_{n+1}^n$ are pairwise different. It follows that in $(F_{n+1}S, \cdot)$, xy^n is different from all x, xy, ..., xy^{n-1}. Consequently, 5.19 holds. □

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