Miloslav Duchoň
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GENERALIZED HERGLOTZ THEOREM
IN VECTOR LATTICES

MILOSLAV DUCHOŇ

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ABSTRACT. We present a Herglotz theorem in the context of vector lattices.

Introduction

It is well known that Fourier-Stieltjes coefficients of positive measures can be characterized as positive definite sequences. Recall that a numerical sequence $(a_n)_{n=-\infty}^{\infty}$ is said to be positive definite if for any (complex) sequence $(z_n)$ having only a finite number of terms different from zero we have

$$\sum_{n,m} a_{n-m} z_n \overline{z}_m \geq 0.$$ 

Now, according to the Herglotz theorem [5; Theorem I.7.6], a numerical sequence $(a_n)_{n=-\infty}^{\infty}$ is positive definite if and only if there exists a positive Borel measure $\mu$ on $[-\pi, \pi]$ with $\mu(\{-\pi\}) = \mu(\{\pi\})$, such that

$$a_n = \int_{[-\pi, \pi]} e^{-ins} \, d\mu(s)$$

for all $n = 0, \pm 1, \ldots$ (cf. also [1] and [4]).

In this paper, we give a generalization of the Herglotz theorem for $a_n$ being elements of a vector lattice. As for terminology and some results from vector lattices we shall use as reference the book [2].
1. Preliminaries

Let $Y$ be a (Dedekind) complete vector lattice. Denote by $L^\omega(X,Y)$ the vector space of all $\omega$-bounded operators on the normed space $X$ into $Y$, that is, if $U \in L^\omega(X,Y)$, then $\{U(x); \|x\| \leq 1\}$ is an $\omega$-bounded subset of $Y$. For $U \in L^\omega(X,Y)$ we put

$$\|U\| = \sup\{|U(x)|; \|x\| \leq 1\}.$$

In the following, let $T$ denote the quotient group $\mathbb{R}/2\pi\mathbb{Z}$ ($\mathbb{R}$ and $\mathbb{Z}$ denoting the additive group of reals, integers, respectively), as a model we may think of the interval $[0, 2\pi)$, and let $C(T)$ denote the space of all scalar continuous functions on $T$ with the usual sup norm. If $U \in L^\omega(C(T), Y)$, then an element of $Y$ of the form

$$\hat{U}(n) = U(e^{i nt})$$

is called the $n$th Fourier coefficient of $U$. The (formal) series

$$\sum_{n \in \mathbb{Z}} \hat{U}(n) e^{inx}$$

is called the Fourier series of $U$. It is clear that there exists an element $0 \leq C \in Y$ such that

$$|\hat{U}(n)| \leq C, \quad n \in \mathbb{Z}.$$

We shall investigate some properties of such Fourier series.

A trigonometric polynomial on $T$ is a function $a = a(t)$ defined on $T$ by $a(t) = \sum_{n} a_n e^{i j n t}$. Denote by $p(T)$ the set of all trigonometric polynomials on $T$.

We shall need the following theorem ([5; Theorem 2.12]) asserting that trigonometric polynomials are dense in $C(T)$.

**Theorem A.** For every $f \in C(T)$ we have $\sigma_n(f) \to f, \ n \to \infty$, in the $C(T)$ norm.

Recall that

$$\sigma_n(f, t) = \sum_{-n}^{n} \left( 1 - \frac{|j|}{n + 1} \right) \hat{f}(j) e^{i j t}.$$

where $\hat{f}(j)$ is the $j$th Fourier-Lebesgue coefficient of $f$ defined by

$$\hat{f}(j) = \frac{1}{2\pi} \int f(t) e^{-ijt} \, dt.$$

(The integration is taken over $T$.)

The following simple lemma will be useful for us.
**Lemma.** Let $U : C(T) \to Y$ be an $(o)$-bounded linear operator. For every $a = \sum_{-n}^{n} a_j e^{ijt}$ we have $U(a) = \sum_{-n}^{n} a_j \hat{U}(-j)$ and $|U(a)| \leq \|a\| \|U\|$, where

$$
\|a\| = \sup_{t} |a(t)|.
$$

We have the following result.

**Theorem 1. (Parseval’s Formula)** Let $f \in C(T)$ and $U \in L^o(C(T), Y)$. Then

$$
U(f) = \lim_{N \to \infty} \sum_{-N}^{N} \left(1 - \frac{|j|}{N + 1}\right) \hat{f}(j) \hat{U}(-j).
$$

**Proof.** Since $f = \lim_{n \to \infty} \sigma_n(f)$ in the $C(T)$ norm, it follows from lemma and the fact that $U$ is $(o)$-bounded (hence $(o)$-continuous) that

$$
U(f) = U \left( \lim_{n \to \infty} \sigma_n(f) \right) = \lim_{n \to \infty} U(\sigma_n(f))
$$

$$
= \lim_{n \to \infty} \sum_{-n}^{n} \left(1 - \frac{|j|}{n + 1}\right) \hat{f}(j) \hat{U}(-j).
$$

Remark. The fact that the preceding limit exists is an implicit part of the theorem. It is equivalent to the $C$-1 (Cesàro) summability of the series $\sum \hat{f}(j) \hat{U}(-j)$, the members of which are elements of the space $Y$. If this last series converges, then clearly,

$$
U(f) = \sum_{-\infty}^{\infty} \hat{f}(j) \hat{U}(-j).
$$

**Corollary. (Uniqueness Theorem)** If $\hat{U}(j) = 0$ for all $j \in \mathbb{Z}$, then $U = 0$.

Parseval’s formula enables us to characterize sequences of Fourier coefficients of $(o)$-bounded linear operators on $C(T)$ similarly as in the case of linear functionals ([5; 7.3])

**Theorem 2.** Let $(y_j)$ be a two-way sequence of elements of $Y$. Then the following two conditions are equivalent:

(a) There is an operator $U \in L^o(C(T), Y)$ with $\|U\| \leq C \in Y$ such that $\hat{U}(j) = y_j$ for all $j \in \mathbb{Z}$.
(b) For all trigonometric polynomials \( a = \sum_{-l}^{l} a_{j} e^{ijt} \) there holds
\[
\left| \sum_{-l}^{l} a_{-j} y_{j} \right| \leq \| a \| C \quad \text{with} \quad 0 \leq C \in Y.
\]

**Proof.** Clearly, (a) implies (b) since
\[
\left| \sum_{-l}^{l} a_{-j} y_{j} \right| = \left| \sum_{-l}^{l} a_{-j} \hat{U}(j) \right|
= \left| \sum_{-l}^{l} a_{-j} U(e^{-jt}) \right| \leq \| U \| \cdot \sup_{t} \left| \sum_{-l}^{l} a_{-j} e^{-ijt} \right| \leq C \| a \|.
\]

Conversely, let for \( \{ y_{j} \} \subset Y \) and for some \( C \in Y \)
\[
\left| \sum_{-l}^{l} a_{-j} y_{j} \right| \leq C \sup_{t} \left| \sum_{-l}^{l} a_{-j} e^{-ijt} \right|.
\]

Put
\[
U \left( \sum_{-l}^{l} a_{-j} e^{ijt} \right) = \sum_{-l}^{l} a_{-j} y_{j}.
\]

Then
\[
\left| U \left( \sum_{-l}^{l} a_{-j} e^{-ijt} \right) \right| \leq C \sup_{t} \left| \sum_{-l}^{l} a_{-j} e^{-ijt} \right|.
\]

It follows that \( U \) is an \( o \)-bounded operator on trigonometric polynomials, these are dense in \( C(T) \), hence \( U \) has an \( o \)-bounded extension to \( C(T) \). Also we obtain \( \hat{U}(j) = y_{j} \). \( \square \)

Let \( (y_{j}) \) be a two-way sequence of elements of \( Y \). Put
\[
\sigma_{N}(Y, t) = \sum_{-N}^{N} \left( 1 - \frac{|j|}{N + 1} \right) y_{-j} e^{-ijt}, \quad N = 1, 2, \ldots.
\]

and denote by \( S_{N}(Y) \) the \( (o) \)-bounded linear operator on \( C(T) \) defined by
\[
S_{N}(Y)(f) = \frac{1}{2\pi} \int_{T} f(t) \sigma_{N}(Y, t) \, dt, \quad f \in C(T), \quad N = 1, 2, \ldots.
\]

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If $U \in L^o(C(\mathbf{T}), Y)$ and if $y_j = \hat{U}(j)$, we shall write

$$\sigma_N(Y,t) = \sigma_N(U,t) \quad \text{and} \quad S_N(Y) = S_N(U).$$

We have

$$S_N(Y)(f) = \frac{1}{2\pi} \int_T f(t)\sigma_N(Y,t) \, dt = \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) \hat{f}(j)y_{-j},$$

$$N = 1, 2, \ldots, \quad f \in C(\mathbf{T}).$$

We may now prove the following.

**Theorem 3.** The members of a two-way sequence $(y_j)$ in $Y$ are the Fourier coefficients of some $U \in L^o(C(\mathbf{T}), Y)$ with $\|U\| \leq C \leq Y$ if and only if $\|S_N(Y)\| \leq C$, $N = 1, 2, \ldots$.

**Proof.**

The necessity: Let $y_j = \hat{U}(j)$ for some $U \in L^o(C(\mathbf{T}), Y)$ with $\|U\| \leq C$. Then $S_N(Y) = S_N(U)$, $N = 1, 2, \ldots$. Recall that $\|\sigma_N(f)\| \leq \|f\|$ for all $f \in C(\mathbf{T})$. Since, for $f \in C(\mathbf{T})$, $S_N(U)(f) = U(\sigma_N(f))$, we have

$$\|S_N(Y)\| = \|S_N(U)\| = \sup\{|S_N(U)(f)| : f \in C(\mathbf{T}), \|f\| \leq 1\},$$

$$= \sup\{|U(\sigma_N(f))| : f \in C(\mathbf{T}), \|f\| \leq 1\},$$

$$\leq \sup\{|U(f)| : f \in C(\mathbf{T}), \|f\| \leq 1\},$$

$$= \|U\| \leq C$$

for $N = 1, 2, \ldots$.

The sufficiency: Take $a = \sum_{-l}^{l} a_j e^{ijt}$. Then we have

$$\sum_{-l}^{l} y_{-j}a_j = \lim_{N \to \infty} \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) y_{-j}a_j = \lim_{N \to \infty} S_N(Y)(a).$$

Thus

$$\left|\sum_{-l}^{l} y_{-j}a_j\right| = \lim_{N \to \infty} |S_N(Y)(a)| \leq \|a\| \sup_N \|S_N(Y)\| \leq \|a\|C.$$  

According to the preceding theorem, there exists $U \in L^o(C(\mathbf{T}), Y)$ such that $y_j = \hat{U}(j)$ and $\|U\| \leq C$. 

\[\square\]
2. Fourier-Stieltjes coefficients of vector measures of \((o)\)-bounded variation

Let \(Y\) be a complete vector lattice. Recall that \(T\) is a compact Hausdorff space, and let \(B(T)\) be the sigma algebra of Borelian subsets of \(T\). Let \(m : B(T) \rightarrow Y\) be an additive set function which satisfies the condition that for any \(E \in B(T)\) the set

\[
G(E) = \left\{ \sum_{i=1}^{k} |m(A_i)| ; (A_1, \ldots, A_k) \text{ is } B(T)-\text{partition of } E \right\}
\]

is \((o)\)-bounded. We shall say that \(m\) is a vector measure of the \((br)\)-type or of \((o)\)-bounded variation, and we shall denote

\[
v_m(E) = \sup G(E).
\]

If \(f\) is a \(B(T)\)-simple function, \(f(t) = \sum_{i=1}^{k} c_i \chi_{A_i}(t)\), we define

\[
\int f(t) \, dm(t) = \sum_{i=1}^{k} c_i m(A_i),
\]

and then we extend this integral for bounded Borel functions on \(T\) ([3]).

Denote by \(BV^\circ(T, Y)\) the vector space of all measures on \(T\) with values in \(Y\) of \(o\)-bounded variation.

Further, if \(m \in BV^\circ(T, Y)\), then an element of \(Y\) of the form

\[
\hat{m}(n) = \frac{1}{2\pi} \int_T e^{-int} \, dm(t)
\]

is called the \(n\)th Fourier-Stieltjes coefficient of \(m\).

We shall make use of the following result.

The general form of the \((o)\)-bounded linear operator \(U : C(T) \rightarrow Y\) is given by the formula

\[
U(f) = \int f(t) \, dm(t),
\]

where \(m : B(T) \rightarrow Y\) is a measure of \((o)\)-bounded variation ([3]).

Now we can prove the following.

**Theorem 4.** Let \(Y\) be a complete vector lattice. Let \((y_k)\) be a two-way sequence of elements of \(Y\). Then the following two conditions are equivalent:

(a) There is a measure \(m : B(T) \rightarrow Y\) of \((o)\)-bounded variation with \(v_m(T) \leq C \in Y\) such that \(y_j\) are Fourier-Stieltjes coefficients of \(m\).
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\[ y_j = \hat{m}(j) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ijt} \, dm(t) \quad \text{for all } j \in \mathbb{Z}. \]

(b) For all trigonometric polynomials \( a = \sum_{-l}^{l} a_j e^{ijt} \in p(\mathbb{T}) \) there holds

\[ \left| \sum_{-l}^{l} a_{-j} y_j \right| \leq \|a\| C \]

for some \( C \in Y \).

Proof. Clearly, (a) implies (b) since

\[ \left| \sum_{-l}^{l} a_{-j} y_j \right| = \left| \sum_{-l}^{l} a_{-j} \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ijt} \, dm(t) \right| \]

\[ = \left| \frac{1}{2\pi} \int \left( \sum_{-l}^{l} a_{-j} e^{-ijt} \right) dm(t) \right| \leq \|a\| v_m(\mathbb{T}) \]

by using ([3; p. 407]).

If we assume (b), then the linear operator \( U : p(\mathbb{T}) \to Y \) from the proof of Theorem 2 is an \((o)\)-bounded linear operator that admits an extension that is an \((o)\)-bounded linear operator on \( C(\mathbb{T}) \) with \( \|U\| \leq C \). But according to ([3; Corollary]), there exists a measure \( m \) of \((o)\)-bounded variation such that

\[ U(f) = \int f(t) \, dm(t), \quad f \in C(\mathbb{T}). \]

Clearly, \( \hat{U}(j) = \hat{m}(j) = y_j \). \( \square \)

If \( m \in BV^{o}(\mathbb{T}, Y) \), then the (formal) series

\[ \sum_{n \in \mathbb{Z}} \hat{m}(n) e^{inx} \]

is called the Fourier-Stieltjes series of \( m \).

If the measure \( m \) is of the \((o)\)-bounded variation, and \( y_j = \hat{m}(j), j \in \mathbb{Z}, \) we shall write

\[ \sigma_N(Y, t) = \sigma_N(m, t) \quad \text{and} \quad S_N(Y) = S_N(m). \]

We can now prove the following.
**Theorem 5.** Let $Y$ be a complete vector lattice. The trigonometric series

$$
\sum_{n \in \mathbb{Z}} y_n e^{inx}, \quad y_n \in Y,
$$

is the Fourier-Stieltjes series of the measure $m: B(T) \to Y$ of the $(o)$-bounded variation, i.e., $y_n = \hat{m}(j)$, $j \in \mathbb{Z}$, if and only if there exists an element $0 \leq C \in Y$ such that

$$
\|S_N(Y)\| \leq C, \quad N = 1, 2, \ldots.
$$

**Proof.** If there exists a measure $m$ of $(o)$-bounded variation, $m \in BV^o(T, Y)$ such that $y_n = \hat{m}(j)$, $j \in \mathbb{Z}$, then, as we know, the equation

$$
U(f) = \int f(t) \, dm(t), \quad f \in C(T),
$$

defines an $(o)$-bounded linear operator $U: C(T) \to Y$ with $\|U\| \leq C$ for some $0 \leq C \in Y$. Hence, according to Theorem 3, we have

$$
\|S_N(Y)\| = \|S_N(U)\| = \|S_N(m)\| \leq C, \quad N = 1, 2, \ldots.
$$

Conversely, if $\|S_N(Y)\| \leq C$, $N = 1, 2, \ldots$, for some $0 \leq C \in Y$, then, according to Theorem 3, there exists an $(o)$-bounded linear operator $U: C(T) \to Y$ such that $\hat{U}(j) = y_j$. But then there exists a measure $m$ of $(o)$-bounded variation such that

$$
U(f) = \int f(t) \, dm(t), \quad f \in C(T).
$$

But $\|U\| = v_m(T) \leq C$. Clearly, $\hat{U}(j) = \hat{m}(j) = y_j$, $j \in \mathbb{Z}$. \hfill \Box

It is useful to establish the Parseval formula explicitly also for the Fourier-Stieltjes series of the measure $m$ of $(o)$-bounded variation.

**Theorem 6.** Let $Y$ be a complete vector lattice, and let $f \in C(T)$. Then we have

$$
\int f(t) \, dm(t) = \lim_{N \to \infty} \sum_{-N}^N \left(1 - \frac{|y|}{N + 1}\right) \hat{f}(j) \hat{m}(-j).
$$

**Proof.** By the Parseval formula from Theorem 1, the last equality holds for $f \in C(T)$. \hfill \Box

It is a very important fact that we have established not only a characterization of the Fourier-Stieltjes series of the measure of $(o)$-bounded variation but also a method how to recapture the measure by means of its Fourier-Stieltjes series. Theorem 6 gives a recipe how to recover the measure $m$. In this sense, we may by abuse of notation, write

$$
\, dm(t) \sim \sum_{j \in \mathbb{Z}} \hat{m}(j) e^{ijx}.
$$
for \( m \in BV^o(T, Y) \).

It is easy to see that if the measure \( m: B(T) \to Y \) is positive, then \( m \) is of the \((a)\)-bounded variation. Hence we may establish the following.

**Theorem 7.** Let \( Y \) be a complete vector lattice. The necessary and sufficient condition for
\[
\sum_{k \in \mathbb{Z}} y_k e^{ikx}
\]
to be the Fourier-Stieltjes series of a positive measure \( m \) with values in \( Y \) is that \( \sigma_N(Y, t) \geq 0 \) for all \( N \) on \( T \).

**Proof.**

The necessity: If \( y_k = \hat{m}(k) \) for a positive measure \( m \), we have
\[
\sigma_N(Y, t) = \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) y_j e^{-ijt} = \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) \hat{m}(-j) e^{-ijt}
\]
\[
= \frac{1}{2\pi} \int \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) e^{-ij(t-s)} \text{d}m(t) = \int K_N(s-t) \text{d}m(t) \geq 0
\]
since \( m \) is positive, and Féjer’s kernel \( K_n \) is nonnegative. So we have \( \sigma_N(Y, t) \geq 0 \) on \( T \).

Assuming \( \sigma_N(Y, t) \geq 0 \) we obtain
\[
\|S_N(Y)\| = \sup_{\|f\| \leq 1} \left| \int f(t) \sigma_N(Y, t) \text{d}t \right| = \frac{1}{2\pi} \int \sigma_N(Y, t) \text{d}t = y_0,
\]
and by Theorem 5,
\[
\sum_{j \in \mathbb{Z}} y_j e^{ijx}
\]
is the Fourier-Stieltjes series for some \( m \in BV^o(T, Y) \). For arbitrary nonnegative \( f \in C(T) \)
\[
\int f(t) \text{d}m(t) = \lim_{N \to \infty} \frac{1}{2\pi} \int f(t) \sigma_N(Y, t) \text{d}t \geq 0,
\]
hence
\[
U: f \to \int f(t) \text{d}m(t)
\]
defines a positive linear operator on \( C(T) \) into \( Y \) which can be extended ([2; 5.1.2, Theorem]) to the positive linear operator (denoted again by) \( U \) defined on the complete vector lattice containing characteristic functions \( c_A \) of Borel sets \( A \) in \( T \). From the definition ([3; Theorem]), \( m(A) = U(c_A) \), and it follows that \( m \) is positive. \( \square \)
It is not unexpected that Theorem 7 gives rise to a representation of positive definite functions defined in a suitable sense, analogous to those known for complex-valued positive definite functions.

Suppose that \( (y_n) \), \( n = 0, \pm 1, \pm 2, \ldots \), is a two-way sequence of elements in a vector lattice \( Y \). Then it is called positive definite if for any sequence \( (c_n) \) of complex numbers having only a finite number of terms different from zero we have
\[
\sum_{m,n} c_n \overline{c_m} y_{n-m} \geq 0.
\]

**Theorem 8.** Let \( Y \) be a complete vector lattice. A necessary and sufficient condition for a sequence \( (y_n)_n \subset Y \) to be positive definite is that there exists a positive measure \( m : B(T) \rightarrow Y \) such that \( y_n = m(n) \) for all \( n \).

**Proof.** Assume \( y_j = \hat{m}(j) \) with \( m : B(T) \rightarrow Y \) positive, then
\[
\sum_{m,n} c_n \overline{c_m} y_{n-m} = \int \left( \sum_{m,n} c_n \overline{c_m} e^{i(n-m)t} \right) \ dm(t) = \int \left| \sum_n c_n e^{int} \right|^2 dm(t) \geq 0.
\]

Conversely, if the sequence \( y_j \) is positive definite, and we take \( c_l = e^{ilt} \), then
\[
\sum_{m,n} c_n \overline{c_m} y_{n-m} = (N + 1) \sigma_N(Y, t) \geq 0,
\]
and it is enough to apply Theorem 7.

**References**


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