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THE CROSSING NUMBER OF THE GENERALIZED PETERSEN GRAPH $P(10,4)$ IS FOUR

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ABSTRACT. The crossing number of a graph is the least number of crossings of edges among all drawings of the graph in the plane. By an *adhoc* argument we establish that the crossing number of the generalized Petersen graph $P(10,4)$ is equal to 4.

Let $n \geq 3$ and $k \in \mathbb{Z}_n \setminus \{0\}$. The *generalized Petersen graph* $P(n,k)$ is defined on the set of vertices $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ with adjacencies $x_i x_{i+1}$, $x_i y_i$ and $y_i y_{i+k}$. For the definition and other details about the *crossing number* of graphs (denoted here by ν), see [1]. Fiorini compiled a table ([1; p. 240]) of small generalized Petersen graphs with known ν, in which the smallest unresolved cases result to be the graphs $P(10,3)$ and $P(10,4)$. For the first one, McQuillan and Richter [2] showed that $\nu(P(10,3))$ is either 5 or 6. For the second, it will be proved in the sequel that $\nu(P(10,4)) = 4$.

The graph $P(10,4)$ is drawn in Figure 1(a) in order to reveal its symmetry, while Figure 1(b) shows that there is a drawing with just 4 crossings. For the rest of this note, the "outer" 10-cycle of $P(10,4)$ (induced by the vertices x_i) will be denoted by K and the two "inner" 5-cycles (induced by the vertices y_i) by C_1, C_2. The edges of the form $x_i y_i$ will be called *spokes.* Figure 1(a) also shows that $P(10,4)$ can be thought of as being consisted of two copies of the Petersen graph $P(5,2)$, amalgamated together along the "outer" 5-cycles after subdividing every edge of them. This observation gives us the idea that $\nu(P(10,4))$ is likely to equal twice the number $\nu(P(5,2)) = 2$.

Now take an arbitrary drawing Δ of $P(10,4)$ and denote by KC_i the subgraph induced by $V(K) \cup V(C_i)$ for $i = 1,2$. Each of the two subgraphs contributes at least 2 to $\nu(P(10,4))$ because both are homeomorphic to $P(5,2)$. If K does not cross itself, then there are at least 4 different crossings in Δ. Therefore, it remains to check the case when K crosses itself. If C_1 (resp. C_2) crosses

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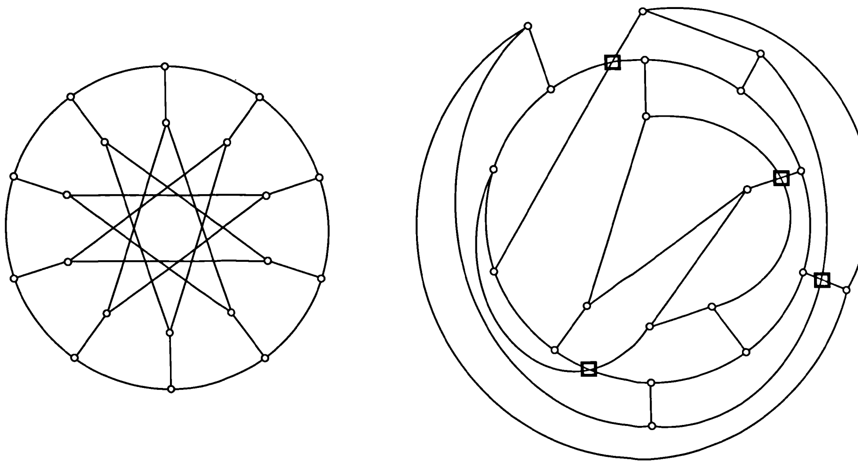
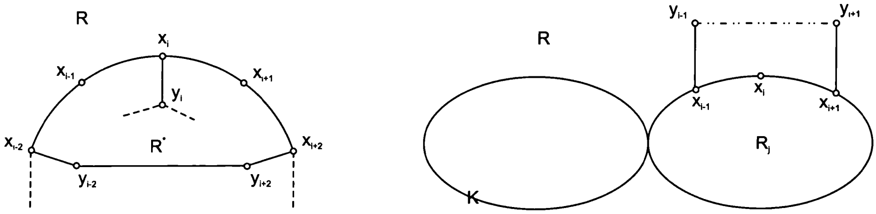
K, then we have at least 3 crossings already, because any two vertex-disjoint cycles must cross each other an even number of times (see also [2; p. 312]). If there are not more than three crossings, then KC_2 (resp. KC_1) is responsible for another crossing different from the previous three. This leads us to assume, from now on, that neither C_1 nor C_2 crosses K. Denote by Δ_K the subdrawing of K in Δ. This subdrawing divides the plane into several regions; if R is anyone of them, ∂R will stand for its boundary. Let p be the number of crossings of Δ_K; thus, $p \geq 1$, and we have two cases to deal with.



(a) (b)

FIGURE 1.

Case 1: $p = 1$. Each of KC_i yields at least one additional crossing. Suppose that there are exactly 3 crossings in Δ. Since $p = 1$, Δ_K is in the form of the number 8 and thus divides the plane into 3 regions: R, R_1 and R_2. One of them (say R) has the property that $\partial R \supset V(K)$. Let us first assume that one of C_i (say C_1) lies in R_j (say R_1). Let q be the number of vertices of K which do not belong to ∂R_1 and are joined to C_1. Clearly, $q \geq 1$ and the spokes joining these vertices to C_1 contribute q to $\nu\big(P(10,4)\big)$. Since 2 crossings are due to KC_2, we have $q + 2 \leq 3$, or $q = 1$, hence ∂R_1 contains 7 or 8 vertices of K. Choose $x_i \in V(K)$ such that $y_i \in V(C_1)$ and that $x_{i-2}, x_i, x_{i+2} \in \partial R_1$, as seen in Figure 2(a). The spoke $x_i y_i$ does not cross any other edge; therefore, y_i must belong to the subregion R^* confined within the cycle $x_i x_{i+1} x_{i+2} y_{i+2} y_{i-2} x_{i-2} x_{i-1}$. Since x_{i-4} and x_{i+4} lie outside R^*, the 2-paths $y_i y_{i-4} x_{i-4}$ and $y_i y_{i+4} x_{i+4}$ are responsible for another 2 crossings. At least one of them is different from the previous three, which is a contradiction. Hence both C_1 and C_2 lie in R. Choose a region R_j and vertices $x_{i-1}, x_{i+1} \in \partial R_j$ such that the spokes $x_{i-1}y_{i-1}$ and $x_{i+1}y_{i+1}$ do not cross K (see Figure 2(b)). This is possible because there are 10 vertices on ∂R, and if no such pair of spokes existed, we would have at least 3 spokes which would cross K. Let $y_{i-1}, y_{i+1} \in V(C_1)$. If C_2 were found inside the subregion bounded by the curve $x_{i-1}x_ix_{i+1}y_{i+1}y_{i+5}y_{i-1}$, the spokes joining C_2 to K would be responsible for 4 crossings. Otherwise, C_2 would lie outside this subregion, but then the spoke x_iy_i would provide another crossing, which is again a contradiction. Therefore, Δ must contain at least 4 crossings in this case.



(a) (b)

FIGURE 2.

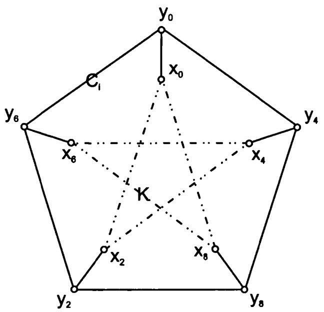


FIGURE 3.

Case 2: $p \geq 2$. Consider the subdrawing Δ_{KC_i} of KC_i and suppose that the only crossings in it are those found in Δ_K. Then the edges of C_i and the spokes joining C_i to K are responsible for no crossing (Figure 3). But the 2-path $x_0x_1x_2$ must cross both 2-paths $x_4x_5x_6$ and $x_6x_7x_8$, which holds for the rest of K, too. This means that there are at least 5 crossings of K. If this is not the case, both Δ_{KC_1} and Δ_{KC_2} contain crossings not in Δ_K, and then we have $p+2 \geq 4$ crossings.

Thus, all the cases are exhausted, and we always end up with at least 4 crossings. This proves that indeed $\nu(P(10,4)) = 4$.

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