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A SADDLE POINT APPROACH TO NONLINEAR EIGENVALUE PROBLEMS

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ABSTRACT. The paper deals with the existence, location and behaviour of solutions of a general nonlinear eigenvalue problem. The applications concern semilinear elliptic eigenvalue problems.

1. Introduction

This paper is devoted to the study of the following general nonlinear eigenvalue problem in a real Hilbert space X

$$I'(u) = \lambda u \qquad \text{for} \quad u \in X \setminus \{0\} \text{ and } \lambda \in \mathbb{R}, \ \lambda > 0. \tag{1}$$

Here I stands for a continuously differentiable functional on X with gradient $I'(u) \in X$ ($\cong X^*$) at a point $u \in X$. By a solution to the problem (1) we mean a pair $(u, \lambda) \in X \times \mathbb{R}$ satisfying (1).

Problem (1) is usually treated by means of critical point theory on the spheres in X. Indeed, denoting by $(\cdot\,,\cdot)$ the scalar product on X with the associated norm $\|\cdot\|$, if $u \in X$ is a critical point of the restriction $I|_{S_r}$ of I to the sphere S_r in X of radius $r > 0$ and centered at 0, that is,

$$S_r = \{v \in X : \ \|v\| = r\}, \tag{2}$$

then (1) holds with $\lambda = (1/r^2)\big(I'(u), u\big)$. Thus the problem amounts to seeking the critical points of the constrained functional $I|_{S_r} : S_r \to \mathbb{R}$ on some sphere S_r in (2). In this connection, a basic tool is provided by the Ljusternik-Schnirelman theory under the assumption that I is an even functional (see Ambrosetti and Rabinowitz [1], Kavian [9], Palais [14], Palais and Terng [15], Rabinowitz [16], [17], Szulkin [21], [22], Zeidler [24] and the references therein).

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A theory for eigenvalue problems involving locally Lipschitz functionals has been developed by Chang [2] (see also Lefter and Motreanu [11] and, for nonlinear eigenvalue problems for variational inequalities, Degiovanni [3], Kubrulski [10], Motreanu and Panagiotopoulos [13]). A different approach to solving (1) is provided by Schechter and Tintarev [19], [20] under the assumption that the functional I has no local maximum on $X\setminus\{0\}$, or under a related property concerning the function $\gamma(r)=\sup\{I(v):\ v\in S_{r^{1/2}}\}$, $r>0$.

In the present paper, we propose a new method of investigating the eigenvalue problem (1) by making use of a variant of the Mountain Pass Theorem of Ambrosetti and Rabinowitz [1], [16], [17], in which the space X and a sphere S_ρ in X are replaced by the product space $X\times\mathbb{R}$ and a hyperplane $X\times\{\rho\}$ in $X\times\mathbb{R}$, respectively. The main point consists in considering a suitable functional F on $X\times\mathbb{R}$ that depends on a rather general coercive function $\beta\in C^1(\mathbb{R},\mathbb{R})$ allowing the Palais-Smale condition for F. By choosing β appropriately one obtains the existence of critical points $(u,s)\in X\times\mathbb{R}$ of F which determine eigensolutions (u,λ) of (1). The advantage is that the graph of β' enables the eigenpair (u,λ) to be located. In addition to the solvability, various qualitative properties dealing with the multiplicity and location of eigensolutions to problem (1) are then deduced. For abstract results regarding the location of critical points of the minimax type we refer to Du [4], Ghoussoub [5], Ghoussoub and Preiss [6], Wang [23].

The applications concern the semilinear eigenvalue Dirichlet problem

$$\begin{aligned} -\Delta u &= \mu p(x,u)\,, && x\in\Omega \\ u &= 0\,, && x\in\partial\Omega\,, \end{aligned} \tag{3}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^N$, $\mu>0$ is a constant, and the function $p\in C(\bar\Omega\times\mathbb{R},\mathbb{R})$ satisfies the growth condition with constants $a_1,a_2\geq 0$

$$\begin{aligned} &|p(x,t)|\leq a_1+a_2|t|^{q-1} && \text{with } 0\leq q<2N/(N-2) && \text{if } N>2\,,\\ &|p(x,t)|\leq a_1\exp\varphi(t) && \text{with } \varphi(t)t^{-2}\to 0 \text{ as } |t|\to\infty && \text{if } N=2\,,\\ &p(x,t) \text{ arbitrary} && && \text{if } N=1\,. \end{aligned} \tag{4}$$

The eigenvalue problem (3) and some extensions have been studied by many authors (see Ambrosetti and Rabinowitz [1], Chang [2], Kavian [9], Lefter and Motreanu [11], Palais and Terng [15], Rabinowitz [16], [17], Schechter and Tintarev [19], [20], Szulkin [21]).

Concerning problem (3) one introduces the functional $I\colon H_0^1(\Omega)\to\mathbb{R}$ by

$$I(v)=\int_\Omega P\big(x,v(x)\big)\,\mathrm{d}x\,,\qquad v\in H_0^1(\Omega)\,, \tag{5}$$

where $P\colon \Omega \times \mathbb{R} \to \mathbb{R}$ denotes the primitive of p,

$$P(x,t) = \int_0^t p(x,\tau)\, d\tau\,, \qquad (x,t) \in \Omega \times \mathbb{R}. \tag{6}$$

Under hypotheses that are weaker than the usual ones, we deduce from the abstract results regarding problem (1) (with the functional I of (5), (6) and $\lambda = 1/\mu$) the existence, multiplicity and location properties of eigensolutions (u,μ) to (3). The same approach works for other eigenvalue boundary value problems, including nonlinear elliptic systems.

The rest of the paper is organized as follows. Section 2 contains a version of the Mountain Pass Theorem. Section 3 deals with the eigenvalue problem (1). Section 4 presents some applications to eigenvalue elliptic problems with Dirichlet boundary conditions.

2. A version of the Mountain Pass Theorem

Throughout this section, X denotes a real Banach space with norm $\|\cdot\|$. The result below represents a variant of the celebrated Mountain Pass Theorem of Ambrosetti and Rabinowitz [1], [16], [17], in which, as a separating surface, a hyperplane is chosen in place of a sphere. In some sense, this can be viewed also as a variant of Saddle Point Theorem (see Rabinowitz [17]).

THEOREM 1. *Let $F\colon X \times \mathbb{R} \to \mathbb{R}$ be a continuously differentiable functional satisfying $F(0,0) = 0$, the Palais-Smale condition, and the further requirements*

(i) *there are positive constants ρ and α such that*

$$F(v,\rho) \ge \alpha \qquad \text{for all} \quad v \in X\,;$$

(ii) *there exists a number $r > \rho$ with $F(0,r) = 0$.*

Then the number

$$c = \inf_{g\in\Gamma} \max_{0\le t\le 1} F\big(g(t)\big)\,, \tag{7}$$

where

$$\Gamma = \big\{g \in C([0,1], X \times \mathbb{R}) :\ g(0) = (0,0)\,,\ g(1) = (0,r)\big\} \tag{8}$$

is a critical value of F, and

$$c \ge \inf_{v\in X} F(v,\rho) \ge \alpha > 0\,. \tag{9}$$

We recall that the functional F on $X \times \mathbb{R}$ is said to satisfy the *Palais-Smale condition* if every sequence (v_n, t_n) in $X \times \mathbb{R}$ with $F(v_n, t_n)$ bounded and

$F'(v_n, t_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence in $X \times \mathbb{R}$. The fact that $c \in \mathbb{R}$ is a critical value of F means that there exists $(u, s) \in X \times \mathbb{R}$ such that

$$F'(u,s) = 0 \qquad \text{and} \qquad F(u,s) = c\,. \tag{10}$$

Proof. The argument is the same as in the classical Mountain Pass Theorem. For the sake of clarity we give it. Relation (9) is clear from (i) because each path $g \in \Gamma$ intersects the hyperplane $X \times \{\rho\}$. Arguing by contradiction we assume now that c in (7) is not a critical value of F. In view of the Deformation Theorem for C^1 functionals, one finds a number ε with $0 < \varepsilon < \alpha \leq c$ and a homeomorphism η of $X \times \mathbb{R}$ such that

$$\eta = \mathrm{id} \qquad \text{on} \quad X \times \mathbb{R} \setminus F^{-1}\big([c-\alpha, c+\alpha]\big) \tag{11}$$

and

$$\eta\big(\{(v,t) \in X \times \mathbb{R} :\ F(v,t) \leq c+\varepsilon\}\big) \subset \big\{(v,t) \in X \times \mathbb{R} :\ F(v,t) \leq c-\varepsilon\big\}\,. \tag{12}$$

Let $g \in \Gamma$ be fixed and such that

$$I\big(g(t)\big) \leq c + \varepsilon \qquad \text{for all} \quad t \in [0,1]\,.$$

Then, by (11) and (ii), it follows that $\eta \circ g \in \Gamma$. Taking into account (12) this contradicts (7), (8). □

From the minimax characterization of c in (7), (8), one can derive various qualitative information for the critical point (u, s) of F in (10). For instance, in view of Hofer [7], one obtains the following property, which is useful in studying the multiplicity of solutions.

COROLLARY 2. *Assume that the conditions of Theorem* 1 *hold. If the critical level set $F^{-1}(c)$ contains only isolated critical points of F, then the point (u, s) in* (10) *can be chosen to be of the mountain pass-type in the sense of Hofer* [7], *i.e., for each open neighborhood W of (u, s) in $X \times \mathbb{R}$, $W \cap F^{-1}(-\infty, c)$ is nonempty and not path-connected.*

As a first application, Theorem 1 can be used in optimization leading to the existence of local or global minimizers. We illustrate this aspect with two existence results for general nonconvex minimization problems.

COROLLARY 3. (Minimization on a hyperplane) *In addition to the assumptions of Theorem* 1, *we suppose that there exists a sequence (w_n) in X such that*

$$\lim_{n\to\infty} \max\Big(\max_{0\leq t\leq 1} F(tw_n, t\rho), \max_{0\leq t\leq 1} F\big(tw_n, t\rho + (1-t)r\big)\Big) = \inf_{v\in X} F(v,\rho)\,. \tag{13}$$

Then $\inf_{v\in X} F(v,\rho)$ *is a critical value of* F *on* $X\times\mathbb{R}$, *so there exists* $(u,s)\in X\times\mathbb{R}$ *such that*

$$F'(u,s)=0 \quad \textit{and} \quad F(u,s)=\inf_{v\in X} F(v,\rho)\,. \tag{14}$$

P r o o f. We claim that the point $(u,s)\in X\times\mathbb{R}$ satisfying (10), as given by Theorem 1, satisfies (14). It suffices to check the second equality in (14). Setting

$$g_n(t)=\begin{cases} (2tw_n, 2t\rho)\,, & 0\le t\le \frac{1}{2}\,,\\ ((2-2t)w_n,(2-2t)\rho+(2t-1)r)\,, & \frac{1}{2}\le t\le 1\,,\end{cases}$$

the paths g_n, $n\ge 1$, belong to the class Γ in (8). Relations (7)-(10) imply

$$\inf_{v\in X} F(v,\rho)\le c=F(u,s)\le \max_{0\le t\le 1} F\big(g_n(t)\big)\,.$$

Passing to the limit for $n\to\infty$ the assertion follows from (13). □

Corollary 4. (Palais Minimization Theorem [14], [15]) *Let* I *be a continuously differentiable functional on the Banach space* X *which is bounded from below and satisfies the Palais-Smale condition. Then* I *attains* $\inf_X I$ *at a (critical) value.*

P r o o f. Fix a point $w\in X$ and a function $f\in C^1(\mathbb{R},\mathbb{R})$ such that for some positive constants $\rho<r$ it satisfies

$$f(t)=t \quad \text{for } t\le 0\,, \qquad f(t)=-t+r \quad \text{for } t\ge r\,,$$
$$\max_{0\le t\le r} f(t)=f(\rho)>I(w)-\inf_X I\,,$$
$$\rho \text{ is the unique critical point of } f\,.$$

We introduce the functional $F\in C^1(X\times\mathbb{R},\mathbb{R})$ by

$$F(v,t)=I(v+w)-I(w)+f(t)\,,\qquad (v,t)\in X\times\mathbb{R}\,.$$

It is straightforward to check that F verifies the hypotheses of Theorem 1. The Palais-Smale condition for F is a direct consequence of the similar property for I. Then Theorem 1 produces a point $u_w\in X$ such that $v_w=u_w+w$ is a critical point of I satisfying

$$f(\rho)-I(w)+\inf_X I\le F(u_w,\rho)=I(v_w)-I(w)+f(\rho)\le \max_{0\le t\le r} F(0,tr)=f(\rho)\,.$$

The first inequality is obtained from (9) with $\alpha=f(\rho)-I(w)+\inf_X I$, and the second one is obtained by taking the path $g(t)=(0,tr)$, $0\le t\le 1$, in (7). Choosing now a minimizing sequence (w_n) for I one finds a corresponding sequence (v_n) of critical points of I such that

$$\inf_X I\le I(v_n)\le I(w_n)\,.$$

Letting $n \to \infty$ and using the Palais-Smale condition for I we establish that the functional I attains its infimum on X at a limit point v of (v_n). $\square$

Remark. The results of this section are valid for locally Lipschitz functionals employing the critical point theory for this type of functions as developed by Chang [2]. Related results and applications of the Palais Minimization Theorem in Corollary 4 are given in [2], [9], [12], [14], [15], [17].

3. A general nonlinear eigenvalue problem

Let X be a real Hilbert space with the scalar product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$, let q and a be fixed numbers with $q \geq 2$ and $a > 0$, and let $I \in C^1(X,\mathbb{R})$ be a given functional satisfying $I(0) = 0$ and $I'(0) \neq 0$. We require that the following conditions hold:

(I_1) there exist constants $a_1, a_2 \geq 0$ such that I satisfies the following one-sided growth condition

$$I(v) \leq a_1 + a_2\|v\|^q \qquad \text{for all} \quad v \in X;$$

(I_2) the gradient $I'\colon X \to X$ of I is a compact mapping in the sense that it maps bounded sets onto relatively compact ones;

(I_3) every sequence (v_n) in X such that $I(v_n) - (a/2)\|v_n\|^2$ is bounded and $I'(v_n) - av_n \to 0$ in X as $n \to \infty$ contains a bounded subsequence in X.

We consider also an additional function $\beta \in C^1(\mathbb{R},\mathbb{R})$ and positive numbers $\rho < r$ verifying the hypotheses below:

(β_1) $\beta(0) = \beta(r) = 0$;

(β_2) $\rho^{q+1} \geq qa_2$ and $\beta(\rho) = \dfrac{q}{q+1}(a_1 + \alpha)$ with some $\alpha > 0$;

(β_3) $\lim\limits_{|t|\to\infty} \beta(t) = +\infty$;

(β_4) $\beta'(t) < 0 \iff t < 0$ or $\rho < t < r$.

Notice that by (β_3), it is always possible to have $\rho > 0$ so that (β_2) is valid. The existence of a function β satisfying (β_1)–(β_4) is clear.

Our aim is to locate the eigensolutions of problem (1) by means of the function β. The argument relies on the minimax principle of Theorem 1 applied to the functional $F \in C^1(X \times \mathbb{R},\mathbb{R})$ given by

$$F(v,t) = \frac{1}{q}|t|^{q+1}\|v\|^q + \frac{q+1}{q}\beta(t) - I(v) + \frac{a}{2}\|v\|^2, \qquad (v,t) \in X \times \mathbb{R}. \quad (15)$$

To utilize Theorem 1, we need the property below.

LEMMA 5. *Assume that conditions* (I_1)-(I_3) *and* (β_1)-(β_4) *are verified. Then the functional* $F\colon X\times\mathbb{R}\to\mathbb{R}$ *in* (15) *satisfies the Palais-Smale condition.*

Proof. Let (v_n, t_n) be a sequence in $X\times\mathbb{R}$ such that

$$|F(v_n,t_n)|\le M\,, \tag{16}$$

$$F_v(v_n,t_n)\to 0 \quad \text{as} \quad n\to\infty\,, \tag{17}$$

$$F_t(v_n,t_n)\to 0 \quad \text{as} \quad n\to\infty\,, \tag{18}$$

where M is a constant and the subscripts v, t denote the variables with respect to which we differentiate F. From (15), (16) and (I_1), we infer that

$$M\ge F(v_n,t_n)\ge\Big(\frac{1}{q}|t_n|^{q+1}-a_2\Big)\|v_n\|^q+\frac{q+1}{q}\beta(t_n)-a_1\,.$$

Taking into account (β_3) this yields the boundedness of the sequence (t_n). Two cases can arise. Suppose first that one has $t_n\to 0$ along a subsequence. Then, by (β_4), one gets $\beta'(t_n)\to\beta'(0)=0$. Since (18) reads as

$$|t_n|^{q-1}t_n\|v_n\|^q+\beta'(t_n)\to 0\,, \tag{19}$$

it follows that

$$t_nv_n\to 0 \quad \text{in} \quad X \text{ as } n\to\infty\,. \tag{20}$$

Then from (15), (16) and (20), under the assumption that $t_n\to 0$, one derives

$$I(v_n)-\frac{a}{2}\|v_n\|^2 \quad \text{is bounded.} \tag{21}$$

On the other hand, property (17) shows that

$$|t_n|^{q+1}\|v_n\|^{q-2}v_n-I'(v_n)+av_n\to 0 \quad \text{as} \quad n\to\infty\,. \tag{22}$$

Combining (20), (22) and the fact that $t_n\to 0$ it follows that

$$I'(v_n)-av_n\to 0 \quad \text{in} \quad X \text{ as } n\to\infty\,. \tag{23}$$

Hypothesis (I_3) and (21), (23) allow us to find a renamed sequence (v_n) which is bounded in X. Then (I_2) and (23) ensure that a subsequence of (v_n) converges in X. This completes the proof of the Palais-Smale condition in the case where $t_n\to 0$. Finally, assume that (t_n) is bounded away from zero. Then (19) shows that (v_n) is bounded in X, thus by (I_2), we may admit that $(I'(v_n))$ converges in X. Consequently, relation (22) enables us to deduce that $(a+|t_n|^{q+1}\|v_n\|^{q-2})v_n$ converges in X as $n\to\infty$. Since $a>0$, it is clear that (v_n) has a convergent subsequence. This establishes the stated result. □

The main result of the Section is formulated below. It provides accurate topological information for eigensolutions (u,λ) in (1).

THEOREM 6. *Assume that the hypotheses* (I_1)–(I_3) *and* (β_1)–(β_4) *are satisfied for a functional* $I \in C^1(X,\mathbb{R})$ *with* $I(0) = 0$ *and* $I'(0) \neq 0$. *Then the following alternative holds:*
either

(i) *a is an eigenvalue of* I', *so*

$$I'(u) = au \qquad \textit{for some} \quad u \in X \setminus \{0\} \tag{24}$$

with

$$\alpha \le -I(u) + \frac{a}{2}\|u\|^2 \le a_1 + \alpha\,; \tag{25}$$

or

(ii) *there exists a number* $s > 0$ *and an element* $u \neq 0$ *in* X *such that*

$$\rho < s < r\,, \tag{26}$$

$$\alpha - \frac{q+1}{q}\beta(s) \le \frac{s^{q+1}}{q}\|u\|^q + \frac{a}{2}\|u\|^2 - I(u) \le a_1 + \alpha - \frac{q+1}{q}\beta(s)\,, \tag{27}$$

$$\|u\| = \big(-\beta'(s)\big)^{1/q} s^{-1}\,, \tag{28}$$

$$(u,\lambda) \in X \times \mathbb{R} \quad \textit{is a solution of } (1) \tag{29}$$

with the eigenvalue

$$\lambda = a + s^3\big(-\beta'(s)\big)^{(q-2)/q}\,. \tag{30}$$

P r o o f. For the function $F \in C^1(X \times \mathbb{R},\mathbb{R})$ defined in (15) we check that the requirements of Theorem 1 are fulfilled. Lemma 5 ensures that F satisfies the Palais-Smale condition. From the assumptions $I(0) = 0$ and (β_1) one finds that $F(0,0) = F(0,r) = 0$, so (ii) of Theorem 1 holds. The assumption (I_1) implies the estimate

$$F(v,\rho) \ge \Big(\frac{1}{q}\rho^{q+1} - a_2\Big)\|v\|^q + \frac{q+1}{q}\beta(\rho) - a_1\,, \qquad v \in X\,.$$

Hence (β_2) shows that (i) of Theorem 1 holds. Therefore, we are in a position to apply Theorem 1. It provides a pair $(u,s) \in X \times \mathbb{R}$ with the properties

$$|s|^{q+1}\|u\|^{q-2}u - I'(u) + au = 0\,, \tag{31}$$

$$|s|^{q-1}s\|u\|^q + \beta'(s) = 0\,, \tag{32}$$

$$F(u,s) = \inf_{g\in\Gamma}\max_{0\le t\le 1} F\big(g(t)\big) \ge \inf_{v\in X} F(v,\rho) \ge \alpha > 0\,, \tag{33}$$

where Γ is given by (8).

Equation (32) ensures that

$$s\beta'(s) \le 0\,. \tag{34}$$

If $s = 0$, then by (33), we know that $u \neq 0$. Formula (31) reveals then that u is an eigensolution of I' whose corresponding eigenvalue is a. Thus equality (24) holds. The first inequality in (25) follows from (33) with $s = 0$. The last inequality of (25) is obtained from (33) by taking the path $g(t) = (0, tr)$, $0 \leq t \leq 1$. We have thus checked that the assertion (i) in the alternative of Theorem 6 is valid. Assume now $s \neq 0$. If $s < 0$, then (β_4) implies $\beta'(s) < 0$, which contradicts (34). So, one has necessarily that $s > 0$. By (β_4) and (34), we conclude that

$$\rho \leq s \leq r. \tag{35}$$

According to (32) and (β_4), the situations $s = \rho$ and $s = r$ lead to $u = 0$. Then (31) is not compatible with the assumption $I'(0) \neq 0$. Thus we have proved that (35) reduces to (26). Using once again (32) we get that $u \neq 0$, and so (28) holds. Inequality (33) gives rise to (27), while (29) follows from (31) by denoting $\lambda = a + s^{q+1}\|u\|^{q-2}$. This last formula coincides with (30) by replacing $\|u\|$ as determined in (28). The proof of Theorem 1 is thus complete. □

COROLLARY 7. *Under the hypotheses of Theorem 6, if $a > 0$ is not an eigenvalue of I' in the sense of (24), then there exists an eigensolution $(u, \lambda) \in X \times \mathbb{R}$ of (1) satisfying a priori estimates in terms of the function β*

$$0 < \lambda - a \leq \max_{\rho \leq t \leq r} t^3 \big(-\beta'(t)\big)^{(q-2)/q},$$

$$0 < \|u\| \leq \max_{\rho \leq t \leq r} t^{-1} \big(-\beta'(t)\big)^{1/q}.$$

Proof. Apply (26), (28), (30). □

COROLLARY 8. *Let $a > 0$ be a fixed number for which the hypotheses of Theorem 6 are verified and which is not an eigenvalue of I'. Then for every $\varepsilon > 0$ there exists an eigensolution $(u_\varepsilon, \lambda_\varepsilon) \in X \times \mathbb{R}$ of problem (1) with $\lambda_\varepsilon > a$ such that $\|u_\varepsilon\| \leq \varepsilon$, $\lambda_\varepsilon \|u_\varepsilon\|^3 \to 0$ and $\lambda_\varepsilon \to +\infty$ and $\varepsilon \to 0$. In particular, there exists a sequence (u_n, λ_n) of eigensolutions of (1) verifying $u_n \to 0$, $\lambda_n \|u_n\|^3 \to 0$ and $\lambda_n \to +\infty$.*

Proof. For each $\varepsilon > 0$ one can take a function $\beta_\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$ having the properties (β_1)–(β_4), and additionally,

$$|\beta'_\varepsilon(t)| \leq \varepsilon^q \qquad \text{for all} \quad t \in \mathbb{R},$$

and accordingly (s_ε) in Theorem 6 with $s_\varepsilon \to +\infty$ as $\varepsilon \to 0$. We give a hint of how to construct of β_ε using a function β satisfying (β_1)–(β_4). We can take $\beta_\varepsilon(t) := \beta(\varepsilon^q t/c)$, where $c = \sup|\beta'| < \infty$, then $\rho_\varepsilon = c\rho/\varepsilon^q$, $r_\varepsilon = cr/\varepsilon^q$. Then relation (28) yields $\|u_\varepsilon\| \leq \varepsilon$ and $s_\varepsilon \|u_\varepsilon\| \leq \varepsilon$. This allows us to conclude that

$$\lambda_\varepsilon \|u_\varepsilon\|^3 = s_\varepsilon^3 \|u_\varepsilon\|^3 \big(-\beta'(s_\varepsilon)\big)^{(q-2)/q} + a\|u_\varepsilon\|^3$$

converges to 0 when $\varepsilon \to 0$. Finally, since $u_\varepsilon \to 0$ as $\varepsilon \to 0$ and $I'(0) \neq 0$, the equality $I'(u_\varepsilon) = \lambda_\varepsilon u_\varepsilon$ necessarily implies $\lambda_\varepsilon \to +\infty$ as $\varepsilon \to 0$. □

COROLLARY 9. *Assume that the hypotheses of Corollary 8 regarding $I \in C^1(X,\mathbb{R})$ and $a > 0$ hold. Then problem (1) admits uncountably many solutions (u,λ) in $(X \setminus \{0\}) \times (0,+\infty)$.*

Proof. Fix positive numbers $\rho < r$ and choose two functions $\beta_1, \beta_2 \in C^1(\mathbb{R},\mathbb{R})$ satisfying (β_1)–(β_4) and $\beta_1'(t) \neq \beta_2'(t)$ for $\rho < t < r$. We can take for example $\beta_2 = b\beta_1$ on $[\rho, r]$ with $b \geq 1$ and β_1 satisfying (β_1)–(β_4). Theorem 6 provides numbers $s_1, s_2 \in (\rho, r)$ and eigensolutions $(u_1,\lambda_1), (u_2,\lambda_2)$ of (1), satisfying the equalities

$$\lambda_i - a = s_i^3\big(-\beta'(s_i)\big)^{(q-2)/q}, \qquad i = 1,2,$$

$$\|u_i\| = s_i^{-1}\big(-\beta_i'(s_i)\big)^{1/q}, \qquad i = 1,2.$$

We infer that

$$\lambda_i - a = s_i^{q+1}\|u_i\|^{q-2}, \qquad i = 1,2.$$

If (u_1,λ_1), (u_2,λ_2) were equal, then we would have $s_1 = s_2$, so by (32), $\beta_1'(s_1) = \beta_2'(s_2)$ contradicting the choice of functions β_1, β_2. This contradiction establishes the conclusion stated. □

4. Applications to nonlinear boundary value problems

The purpose of this Section is to show that the abstract results of Section 3 apply to the eigenvalue elliptic problem (3) with the Dirichlet boundary condition on a bounded and smooth domain Ω in $\mathbb{R}^N$. We recall that p in (3) is a continuous function on $\bar\Omega \times \mathbb{R}$ satisfying the growth condition (4). We assume further that $p(x,0)$ is not identically zero for $x \in \Omega$. This represents a complementary situation to that treated in Ambrosetti and Rabinowitz [1] and Rabinowitz [16], [17], where it is supposed that $p(x,t) = o(t)$ as $t \to 0$. In particular, we consider the case when $p(x,t)$ is not an odd function with respect to the variable $t \in \mathbb{R}$. With problem (3) we associate the functional $I\colon H_0^1(\Omega) \to \mathbb{R}$ given by (5) with P denoting the primitive of p as in (6). It is known that $I \in C^1\big(H_0^1(\Omega),\mathbb{R}\big)$, and its gradient is equal to

$$\big(I'(u),v\big) = \int_\Omega p\big(x,u(x)\big)v(x)\,\mathrm{d}x, \qquad v \in H_0^1(\Omega).$$

The abstract eigenvalue problem (1) reads for the functional I of (4) as

$$\int_\Omega p\big(x,u(x)\big)v(x)\,\mathrm{d}x = \lambda\int_\Omega \nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x, \qquad v \in H_0^1(\Omega),$$

the space $H_0^1(\Omega)$ being endowed with the scalar product

$$(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx\,, \qquad u,v \in H_0^1(\Omega)\,.$$

This is exactly the definition of a weak solution $u \in H_0^1(\Omega)$ to the Dirichlet problem (3) with $\mu = 1/\lambda$. Under additional regularity assumptions on p (e.g., that it is locally Lipschitz) one finds that u is a classical solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of (3). Consequently, problems (1) for I given by (5) on $H_0^1(\Omega)$ and (3) with $\mu = 1/\lambda$ are equivalent.

In order that (I_3) hold, it is clear that we need to impose a further condition on the function p. As a model, we suppose that there exist constants c_1, $c_2 \in \mathbb{R}$ and $\gamma > 2$, $1 \le \sigma < 2$ such that

$$p(x,t)t - \gamma P(x,t) \ge -c_1 - c_2|t|^\sigma \qquad \text{for} \quad (x,t) \in \Omega \times \mathbb{R}\,. \tag{36}$$

Notice that (36) is weaker than the famous hypothesis (p_4) of Ambrosetti and Rabinowitz [1], [16], [17]. This is due to the terms in the right-hand side and to the lack of positive sign condition for P in (36). Applications of the abstract results in Section 3 to nonlinear boundary value problems (3) are achieved through the following result.

THEOREM 10. *Assume that $p \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ with $p(\cdot\,,0)$ not identically zero on Ω satisfies* (4) *and* (36). *Then for every function $\beta \in C^1(\mathbb{R},\mathbb{R})$ satisfying (β_1)–(β_4) for some numbers $0 < \rho < r$ corresponding to the growth condition of type* (4) (*expressed by* (38)), *the conclusion of Theorem* 6 *and its corollaries hold with respect to the functional I in* (5), (6) *for every number $a > 0$. In other words, problem* (3) *possesses eigensolutions $(u,\mu) \in H_0^1(\Omega) \times \mathbb{R}$ such that the pairs $(u, \lambda{=}1/\mu)$ have the existence, multiplicity and location properties of Theorem* 6 *and its corollaries.*

Proof. The proof consists in checking the conditions (I_1)–(I_3) for the functional I on $H_0^1(\Omega)$ constructed in (5), (6). Assume $N > 2$. By integrating (4), one obtains

$$|P(x,t)| \le C_1 + C_2|t|^q\,, \qquad (x,t) \in \Omega \times \mathbb{R}\,, \tag{37}$$

for new constants $C_1, C_2 \ge 0$. Without loss of generality we may suppose $q \ge 2$. Then (5), (6) and (37) yield

$$I(v) \le C_1|\Omega| + C_2\|v\|_{L^q}^q \le C_1|\Omega| + C_2 C\|v\|^q\,, \qquad v \in H_0^1(\Omega)\,, \tag{38}$$

where the constant $C > 0$ comes from the Sobolev imbedding theorem for the continuous inclusion $H_0^1(\Omega) \subset L^q(\Omega)$ with $q < 2N/(N-2)$. Hence (I_1) is true. Under hypothesis (4) for $p \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ it is known that the gradient $I'\colon H_0^1(\Omega) \to H_0^1(\Omega)$ is a compact mapping (see, e.g., Rabinowitz [17;

p. 91]). Hence condition (I_2) holds for the functional I in (5). It remains only to verify (I_3). By (36), we can write

$$\begin{aligned}(I'(v),v)-\gamma I(v) &= \int_\Omega \big(p(x,v)v-\gamma P(x,v)\big)\,\mathrm{d}x \\ &\geq -c_1|\Omega| - c_2\|v\|^\sigma_{L^\sigma} \geq -c_1|\Omega| - c_2 C\|v\|^\sigma \\ &\text{for all} \quad v\in H^1_0(\Omega)\,,\end{aligned} \tag{39}$$

where $C>0$ is a constant determined by the continuous inclusion $H^1_0(\Omega)\subset L^\sigma(\Omega)$. Estimate (39) enables us to derive property (I_3). Indeed, for a fixed $a>0$, let a sequence (v_n) in $H^1_0(\Omega)$ satisfy

$$\Big|I(v_n)-\frac{a}{2}\|v_n\|^2\Big|\leq M\,,\qquad n\geq 1\,,$$

where M is a constant, and

$$I'(v_n)-av_n\to 0 \qquad \text{as} \quad n\to\infty\,.$$

Then, for n large enough, one has

$$\big|\big(I'(v_n)-av_n,v_n\big)\big|\leq\|v_n\|\,.$$

The estimate (39) enables us then to write

$$\begin{aligned}M+\frac{1}{\gamma}\|v_n\| &\geq \frac{a}{2}\|v_n\|^2 - I(v_n)+\frac{1}{\gamma}\big((I'(v_n),v_n)-a\|v_n\|^2\big) \\ &\geq a\Big(\frac{1}{2}-\frac{1}{\gamma}\Big)\|v_n\|^2-\frac{1}{\gamma}\big(b_1+b_2\|v_n\|^\sigma\big)\end{aligned}$$

for appropriate constants b_1, b_2. This clearly implies the boundedness of (v_n) in $H^1_0(\Omega)$ because $a>0$, $\gamma>2$ and $\sigma<2$, thus (I_3) holds. Then Theorem 6 and its corollaries apply. □

The method can be extended to the vector-valued eigenvalue problem (3), that is, for $u\in H^1_0(\Omega;\mathbb{R}^m)$ and $p\in C(\bar\Omega\times\mathbb{R},\mathbb{R}^m)$ with $m\geq 1$. In fact, then (3) becomes a system of equations with a common eigenvalue $\mu>0$. Elliptic systems with a variational structure are encountered frequently. For instance, a system of two coupled semilinear Poisson equations is studied by Hulshoff and van der Vorst [8]. In order to treat the eigenvalue problem (3) in the vector-valued case by our approach based on Theorem 1, we can follow the same lines as in the case of a single equation. Namely, we use with the associated functional (5), (6). Here a typical example of indefinite functional (i.e., not bounded below or above) on $H^1_0(\Omega;\mathbb{R}^m)$ with $m=m_1+m_2$ is the following

$$I(v)=\int_\Omega \big((-1/q_1)|v_1|^{q_1}+(1/q_2)|v_2|^{q_2}\big)\,\mathrm{d}x+\langle f,v\rangle_{H^1_0} \tag{40}$$

for $v = (v_1, v_2) \in H_0^1(\Omega, \mathbb{R}^{m_1} \times \mathbb{R}^{m_2})$, with a fixed $f \in H^{-1}(\Omega; \mathbb{R}^m) \setminus \{0\}$. In (40), we assume $1 < \max\{q_1, 2\} < q_2 < 2N/(N-2)$ for $N \geq 3$. Clearly, conditions (I_1), (I_2) are satisfied. Concerning condition (I_3) we proceed as follows. Choosing

$$\max\{2, q_1\} < \gamma < q_2\,, \tag{41}$$

formulas (40) and (41) reveal that

$$\begin{aligned}\big(I'(v), v\big) - \gamma I(v) &= \int\limits_{\Omega} \Big(\frac{\gamma - q_1}{q_1}|v_1|^{q_1} + \frac{q_2 - \gamma}{q_2}|v_2|^{q_2}\Big)\, \mathrm{d}x + (1-\gamma)\langle f, v\rangle_{H_0^1} \\ &\geq (1-\gamma)\|f\|_{H^{-1}}\|v\|\end{aligned} \tag{42}$$

for all $v = (v_1, v_2) \in H_0^1(\Omega, \mathbb{R}^m)$. By the same argument as in the final part of the proof of Theorem 10, the inequality (42) shows that (I_3) holds. Therefore we may apply Theorem 6 and its corollaries to the functional in (40).

Remark. Another interesting situation in (40) is when $1 \leq q_2 < 2 < q_1 < 2^*$. Then the preceding argument applies with $\gamma > q_1$.

Remark. By the same method, one can treat nonlinear eigenvalue boundary value problems with discontinuities using the locally Lipschitz versions of the preceding results. The argument relies on the critical point theory for locally Lipschitz functionals in the sense of Chang [2]. Related results regarding elliptic boundary value problems with discontinuities can be found in Chang [2], Lefter and Motreanu [11], Motreanu and Panagiotopoulos [13], Rauch [18], Szulkin [21].

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